Math 262 HW6

1. $M = U_1 \cup U_2$, $U_1 = (0,1) \times (0,1)$, $U_2 = \left([0,\tfrac{1}{2}) \times (0,1)\right) \cup \left((\tfrac{1}{2},1] \times (0,1)\right)$

$U_1 \cap U_2 = U_3 \cup U_4$, $U_3 = (0,\tfrac{1}{2}) \times (0,1)$, $U_4 = (\tfrac{1}{2},1) \times (0,1)$

and the coordinate chart on U_2 is $\varphi_2 : U_2 \to V_2 = (0,1)\times(0,1)$, $\varphi_2(x,y) = \begin{cases} (x+\tfrac{1}{2}, 1-y) & \text{if } 0 \le x < \tfrac{1}{2} \\ (x-\tfrac{1}{2}, y) & \text{if } \tfrac{1}{2} < x \le 1. \end{cases}$

M-V:

$$0 \longrightarrow H^0(\underbrace{U \cup V}_{M}) \longrightarrow \underset{\mathbb{R}\oplus\mathbb{R}}{\overset{}{H^0(U)\oplus H^0(V)}} \xrightarrow{\delta} \underset{\mathbb{R}\oplus\mathbb{R}}{H^0(U\cap V)} \longrightarrow H^1(\underbrace{U\cup V}_{M}) \longrightarrow 0$$

and the map δ is given by $\delta(a,b) = (b-a, b-a)$, so

$\underline{H^0(M) = \ker\delta = \mathbb{R},\ H^1(M) = \operatorname{coker}\delta = \mathbb{R},\ H^*(M) = 0 \text{ for } * \ne 0,1}$.

Cpt M-V:

$$0 \to H^1_c(U\cup V) \to H^2_c(U\cap V) \xrightarrow{\delta} H^2_c(U)\oplus H^2_c(V) \to H^2_c(U\cup V) \to 0$$

We need to determine δ. Now $H^2_c(U\cap V) = H^2_c(\overset{U_3}{\mathbb{R}^2} \amalg \overset{U_4}{\mathbb{R}^2}) \cong \mathbb{R}\oplus\mathbb{R}$

with the isomorphism $H^2_c(U_3 \amalg U_4) \longrightarrow \mathbb{R}\oplus\mathbb{R}$ given by

$$(\underbrace{f\,dx\wedge dy}_{\text{on } U_3}, \underbrace{g\,dx\wedge dy}_{\text{on } U_4}) \mapsto \left(\int_{U_3} f\,dx\,dy, \int_{U_4} g\,dx\,dy\right).$$

On the other hand, $(f\,dx\wedge dy, g\,dx\wedge dy)$ add together to give an element ω of $\Omega^2_c(U)$, mapping under the isom. $H^2_c(U) \cong \mathbb{R}$ to $\int_U \omega = \int_{U_3} f\,dx\,dy + \int_{U_4} g\,dx\,dy$.

They also give an element $\tilde{\omega}$ of $\Omega^2_c(V)$, which in the standard coordinates on V_2 is

$$\begin{cases} g(x+\tfrac{1}{2},y)\,dx\wedge dy & \text{if } 0 \le x < \tfrac{1}{2} \\ f(x-\tfrac{1}{2},1-y)\,dx\wedge(-dy) & \text{if } \tfrac{1}{2} < x \le 1 \\ (0 & \text{if } x = \tfrac{1}{2}) \end{cases}.$$

Under the isom. $H^2_c(U_2) \cong \mathbb{R}$, this form maps to $\int_{V_2} \tilde{\omega} = \int_{U_4} g\,dx\,dy - \int_{U_3} f\,dx\,dy$.

Thus the map $\delta : \underset{\mathbb{R}^2}{\overset{}{H^2_c(U\cap V)}} \longrightarrow \underset{\mathbb{R}^2}{H^2_c(U)\oplus H^2_c(V)}$

is given by $(a,b) \mapsto (-a-b, b-a)$

and

$$H^1_c(M) = \ker\delta = 0,\quad H^2_c(M) = \operatorname{coker}\delta = 0,\quad H^*_c(M) = 0 \text{ for } * \ne 1,2.$$

(i.e. $\underline{H^*_c(M) = 0}$).

2. Let $(U_\alpha, V_\alpha, \varphi_\alpha)$, $(U_\beta, V_\beta, \varphi_\beta)$ be coordinate charts in the oriented atlas with coordinates $x_1, \ldots, x_n$ on V_β and $y_1, \ldots, y_n$ on V_α. So

$$(y_1, \ldots, y_n) = g_{\alpha\beta}(x_1, \ldots, x_n).$$

Then

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g\left(\sum_k \frac{\partial y_k}{\partial x_i}\frac{\partial}{\partial y_k}, \sum_\ell \frac{\partial y_\ell}{\partial x_j}\frac{\partial}{\partial y_\ell}\right) = \sum_{k,\ell} (Dg_{\alpha\beta})_{ik} (Dg_{\alpha\beta})_{j\ell}\, g\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_\ell}\right)$$

so if g_β is the matrix $\left(g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)$ and g_α is the matrix $\left(g\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right)\right)$, then

$$g_\beta = (Dg_{\alpha\beta})\, g_\alpha\, (Dg_{\alpha\beta})^T$$

(or maybe the transpose is on the other side; I don't recall precisely how the Jacobian is usually defined)

and on $U_\alpha \cap U_\beta$,

$$\begin{aligned} \omega_\alpha &= (\det g_\alpha)^{1/2}\, dy_1 \wedge \cdots \wedge dy_n \\ &= \frac{(\det g_\beta)^{1/2}}{|\det Dg_{\alpha\beta}|} \cdot (\det Dg_{\alpha\beta})\, dx_1 \wedge \cdots \wedge dx_n \\ &= \omega_\beta \end{aligned}$$

since $\det Dg_{\alpha\beta} > 0$ (here's where we use the fact that the atlas is oriented).

So the ω_α's patch together to give $\omega \in \Omega^n(M)$, which is a volume form since $\det g_\alpha > 0$ at all points in U_α since g is positive definite.

3. By linearity, suffices to check $\omega = f\, dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$ for some $1 \le i \le n$, where $f \in C_c^\infty(\mathbb{H}^n)$. Then

$$\int_{\partial \mathbb{H}^n} \omega = \begin{cases} 0 & \text{if } i \neq n \text{ (since } dx_n = 0 \text{ on } \partial\mathbb{H}^n) \\ (-1)^{n} \int_{\mathbb{R}^{n-1}} f(x_1, \ldots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1} & \text{if } i = n. \end{cases}$$

On the other hand, $d\omega = (-1)^{i-1}\left(\frac{\partial f}{\partial x_i}\right) dx_1 \wedge \cdots \wedge dx_n$, so

$i \neq n$: $$\int_{\mathbb{H}^n} d\omega = (-1)^{i-1} \int \cdots \int \underbrace{\left[\int_{x_i=-\infty}^{\infty} \left(\frac{\partial f}{\partial x_i}\right) dx_i\right]}_{0 \text{ since } f \text{ compactly supported}} \cdots \quad = 0;$$

$i = n$: $$\begin{aligned} \int_{\mathbb{H}^n} d\omega &= (-1)^{n-1} \int \cdots \int \underbrace{\left[\int_{x_n=0}^{\infty} \left(\frac{\partial f}{\partial x_n}\right) dx_n\right]}_{f(x_1, \ldots, x_{n-1}, \infty) - f(x_1, \ldots, x_{n-1}, 0)} dx_1 \cdots dx_{n-1} \\ &= (-1)^n \int_{\mathbb{R}^{n-1}} f(x_1, \ldots, x_{n-1}, 0)\, dx_1 \cdots dx_{n-1}. \end{aligned}$$

□

4.(a) $S^n = \partial B^{n+1}$ where $B^{n+1} = \{x_0^2 + \dots + x_n^2 \le 1\} \subset \mathbb{R}^{n+1}$.

If we give S^n the orientation induced from B^{n+1}, then by Stokes,

$$\int_{S^n} \omega = \int_{B^{n+1}} d\omega = \int_{B^{n+1}} (n+1)\, dx_0 \wedge \dots \wedge dx_n = \boxed{\frac{(n+1)\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+3}{2}\right)}} \left(= \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\right) \text{(up to } \pm\text{)}.$$

Since this is $\neq 0$, and $\int_{S^n} : H^n(S^n) \to \mathbb{R}$ is well-defined, $[\omega] \neq 0$ in $H^n(S^n)$.

(b) For $i > 0$, the transition function between U_0^+ and $U_i^\pm$ is given by

$$(x_0, \dots, \hat{x}_i, \dots, x_n) = \left((1 - x_1^2 - \dots - x_n^2)^{1/2}\, x_1, \dots, \hat{x}_i, \dots, x_n\right).$$

The Jacobian is

(i-1)st column

$$\left(\begin{array}{cccc|c|ccc}
* & * & \cdots & * & -\frac{x_i}{(1-x_1^2-\dots-x_n^2)^{1/2}} & & & \\
1 & & 0 & & 0 & & 0 & \\
 & 1 & & & \vdots & & & \\
0 & & \ddots & & & & & \\
 & & & 1 & 0 & & & \\ \hline
 & 0 & & & 0 & 1 & & 0 \\
 & & & & & & \ddots & \\
 & & & & & 0 & & 1
\end{array}\right)$$

and the determinant is $(-1)^i \frac{x_i}{(1-x_1^2-\dots-x_n^2)^{1/2}}$, which has the same sign as $(-1)^i$ for U_i^+ and the same sign as $(-1)^{i+1}$ for U_i^-.

~~Thus~~ Also note that a similar calculation gives the exact opposite result for U_0^- vs $U_i^\pm$. Thus

- agree with U_0^+: U_i^+, i even; and U_i^-, i odd
- disagree with U_0^+: U_0^-; U_i^+, i odd, and U_i^-, i even.

(c) On B^{n+1} near $(1,0,\dots,0)$, define coords $y_0, \dots, y_n$ by $y_0 = x_1, \dots, y_{n-1} = x_n$, $y_n = 1 - x_0^2 - x_1^2 - \dots - x_n^2$.

Then $B^{n+1} = \{y_n \ge 0\}$ near $(1,0,\dots,0)$.

- On B^{n+1} near $(1,0,\dots,0)$: $dy_0 \wedge \dots \wedge dy_n = 2 \cdot (-1)^{n+1} x_0\, dx_0 \wedge \dots \wedge dx_n$

so orientation induced by $y_0, \dots, y_n$ differs from orientation induced by $x_0, \dots, x_n$ by $\boxed{(-1)^{n+1}}$

- On S^n near $(1,0,\dots,0)$ (i.e. on U_0^+): $dy_0 \wedge \dots \wedge dy_{n-1} = dx_1 \wedge \dots \wedge dx_n$

so orientation induced by $y_0, \dots, y_{n-1}$ agrees with orientation induced by $x_1, \dots, x_n$.

- Orientation given by $y_0, \dots, y_n$ on B^{n+1} induces $\boxed{(-1)^{n+1}}$ times orientation given by $y_0, \dots, y_{n-1}$ on S^n.

$\Rightarrow$ Orientation on $U_0^+ \subset S^n$ induced by orientation $x_0, \dots, x_n$ on B^{n+1} agrees with usual orientation on U_0^+.

$\Rightarrow$ The answer to (a) is correct without any sign correction.