Math 262 HW#5

1. (a) Easy computation from $dr = \sum_{i=0}^{n} \frac{x_i\,dx_i}{r}$ and $\sum x_i^2 = r^2$.

(b) $2i^*(x_0dx_0 + \cdots + x_ndx_n) = i^*(d(r^2)) = d(i^*r^2) = d(1) = 0$.

(c) $x_0 = \frac{\sum y_i^2 - 1}{\sum y_i^2 + 1}$, $x_j = \frac{2y_j}{\sum y_i^2+1}$ $\Rightarrow$ $x_0dx_0 = \sum_j \frac{4y_j(\sum y_i^2 - 1)}{(\sum y_i^2+1)^3}dy_j$

$$x_jdx_j = \frac{4y_j}{(\sum y_i^2+1)^2}dy_j - \sum_k \frac{8y_j^2y_k}{(\sum y_i^2+1)^3}dy_k$$

$\Rightarrow$ the coefficient of dy_j in $x_0dx_0 + \sum x_jdx_j$ is

$$\frac{4y_j(\sum y_i^2-1)}{(\sum y_i^2+1)^3} + \frac{4y_j}{(\sum y_i^2+1)^2} - \sum_k \frac{8y_k^2y_j}{(\sum y_i^2+1)^3} = 0$$

$\Rightarrow$ $i^*(x_0dx_0 + \cdots + x_ndx_n) = 0$.

(d) Using $x_0dx_0 + \cdots + x_ndx_n = 0$, we can write both ω and $dy_1\wedge\cdots\wedge dy_n$ in terms of $dx_1, \ldots, dx_n$:

$$\omega = x_0\,dx_1\wedge\cdots\wedge dx_n + \sum_{i=1}^{n}(-1)^i x_i\big(\overbrace{\cdots - \tfrac{x_idx_i}{x_0}\cdots}^{dx_0}\big)\,dx_1\wedge\cdots\widehat{dx_i}\cdots\wedge dx_n$$

$$= \left(x_0 + \frac{\sum_i x_i^2}{x_0}\right)dx_1\wedge\cdots\wedge dx_n = \frac{1}{x_0}dx_1\wedge\cdots\wedge dx_n;$$

$$dy_1\wedge\cdots\wedge dy_n = \frac{1}{(1-x_0)^n}dx_1\wedge\cdots\wedge dx_n + \sum_{i=1}^{n}\frac{x_i}{(1-x_0)^{n+1}}\,dx_1\wedge\cdots\underbrace{dx_0}_{i\text{th spot}}\cdots\wedge dx_n$$

$\uparrow$

$$dy_i = \frac{dx_i}{1-x_0} + \frac{x_i\,dx_0}{(1-x_0)^2}$$

$$= \frac{1}{(1-x_0)^n}dx_1\wedge\cdots\wedge dx_n + \sum_{i=1}^{n} -\frac{x_i^2}{x_0(1-x_0)^{n+1}}dx_1\wedge\cdots\wedge dx_n$$

$$= -\frac{1}{x_0(1-x_0)^n}dx_1\wedge\cdots\wedge dx_n.$$

Thus
$$f = -(1-x_0)^n = -\left(\frac{2}{1+\sum y_i^2}\right)^n.$$

2. (a) Write $\tilde{x}, \tilde{y}$ for the coordinates on U_1. The transition function between U_0 and U_1 sends $z = x+iy$ to $1/z = \frac{1}{\tilde{x}+i\tilde{y}} = \frac{\tilde{x}-i\tilde{y}}{\tilde{x}^2+\tilde{y}^2}$,

so
$$x = \frac{\tilde{x}}{\tilde{x}^2+\tilde{y}^2}, \quad y = -\frac{\tilde{y}}{\tilde{x}^2+\tilde{y}^2}.$$

Thus
$$dx = \frac{1}{(\tilde{x}^2+\tilde{y}^2)^2}\left((\tilde{y}^2-\tilde{x}^2)\,d\tilde{x} - 2\tilde{x}\tilde{y}\,d\tilde{y}\right), \quad dy = \frac{1}{(\tilde{x}^2+\tilde{y}^2)^2}\left(-(\tilde{x}^2-\tilde{y}^2)\,d\tilde{y} + 2\tilde{x}\tilde{y}\,d\tilde{x}\right),$$

$$\frac{1}{1+x^2+y^2} = \frac{\tilde{x}^2+\tilde{y}^2}{1+\tilde{x}^2+\tilde{y}^2}$$

$$\Rightarrow \quad \frac{dx\wedge dy}{(1+x^2+y^2)^2} = \frac{(\tilde{x}^2+\tilde{y}^2)^2}{(1+\tilde{x}^2+\tilde{y}^2)^2}\cdot\frac{1}{(\tilde{x}^2+\tilde{y}^2)^4}\left((\tilde{y}^2-\tilde{x}^2)^2+4\tilde{x}^2\tilde{y}^2\right)d\tilde{x}\wedge d\tilde{y}$$

$$= \frac{d\tilde{x}\wedge d\tilde{y}}{(1+\tilde{x}^2+\tilde{y}^2)^2}$$

So if we define $\omega_{U_1} = \frac{d\tilde{x}\wedge d\tilde{y}}{(1+\tilde{x}^2+\tilde{y}^2)^2}$ (note defined on all of U_1)

then $\omega_{U_0} = \omega_{U_1}$ on $U_0\cap U_1$ so they patch together smoothly to $\omega\in\Omega^2(\mathbb{CP}^1)$.

(b) We have
$$\underset{\mathbb{CP}^1 - \{2\text{ points}\}}{\overset{\cong}{f_a : \mathbb{C}\setminus\{-\tfrac{1}{\bar{a}}\}}} \longrightarrow \underset{U_0 = \mathbb{CP}^1-\{1\text{ point}\}}{\overset{\cong}{\mathbb{C}}}$$

and we claim $\underline{f_a^*(\omega_{U_0}) = \omega_{U_0}}$. If this is the case, then $f_a^*(\omega) = \omega$ at all but possibly 2 points in $\mathbb{CP}^1$, so by continuity $f_a^*(\omega) = \omega$ everywhere.

To prove $f_a^*(\omega_{U_0}) = \omega_{U_0}$:

This can be done in x,y coordinates, but it's easier to use complex coordinates. So write $z = x+iy$ and (formally) define

$$dz = dx + i\,dy \qquad \text{(these are just complex}$$
$$d\bar{z} = dx - i\,dy. \qquad \text{linear combinations of } dx, dy).$$

Then $\omega_{U_0} = \frac{dx\wedge dy}{(1+x^2+y^2)^2} = \frac{i}{2}\,\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2}$.

Now consider $f_a : \underbrace{\mathbb{C} \setminus \{-\frac{1}{\bar{a}}\}}_{\substack{\text{Coordinates } x,y \\ z = x+iy}} \to \underbrace{\mathbb{C}}_{\substack{\text{Coordinates } \tilde{x},\tilde{y} \\ w = \tilde{x}+i\tilde{y}}}$

Then $w = \dfrac{z-a}{1+\bar{a}z}$ and $dw (= d\tilde{x} + i d\tilde{y}) = \dfrac{1+a\bar{a}}{(1+\bar{a}z)^2} dz$

$\bar{w} = \dfrac{\bar{z}-\bar{a}}{1+a\bar{z}}$ and $d\bar{w} (= d\tilde{x} - i d\tilde{y}) = \dfrac{1+a\bar{a}}{(1+a\bar{z})^2} d\bar{z}$

(note: over $\mathbb{C}$, z and $\bar{z}$ are just as valid a coordinate system as x and y, so we can differentiate as usual; alternatively, check the expressions for $dw, d\bar{w}$ by breaking into real and imaginary parts).

$$\to \frac{i}{2} \frac{dw \wedge d\bar{w}}{(1+w\bar{w})^2} = \frac{i}{2} \frac{\left(\frac{1+a\bar{a}}{(1+\bar{a}z)^2}\right)\left(\frac{1+a\bar{a}}{(1+a\bar{z})^2}\right) dz \wedge d\bar{z}}{\left(1 + \frac{(z-a)(\bar{z}-\bar{a})}{(1+\bar{a}z)(1+a\bar{z})}\right)^2}$$

$$= \frac{i}{2} \frac{(1+a\bar{a})^2}{((1+z\bar{z})(1+a\bar{a}))^2} dz \wedge d\bar{z}$$

$$= \frac{i}{2} \frac{dz \wedge d\bar{z}}{(1+z\bar{z})^2}$$

and the result follows.