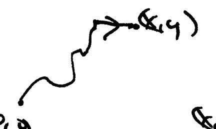
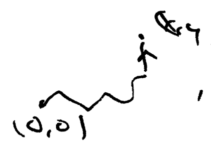
# Math 262 HW#4

1. $M$ connected $\Rightarrow$ $H^0_{DR}(M) \cong \mathbb{R}$, $H^0_c(M) = \begin{cases} \mathbb{R}, & M \text{ compact} \\ 0, & \text{otherwise.} \end{cases}$

$\underline{H^k_{DR}}$: For $k=1$: if $\omega = f\,dx + g\,dy$ is closed ($d\omega = 0 \Leftrightarrow \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$), then by Green's Thm, if $\gamma, \tilde{\gamma}$ are homotopic paths in $\mathbb{R}^2$ with fixed endpoints, then $\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$. So then we can define $h: \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x,y) = \int_\gamma \omega \quad \text{where } \gamma \text{ is any path from } (0,0) \text{ to } (x,y).$$

By choosing $\gamma$ of the form  for all $(x,y)$, we see that $\frac{\partial h}{\partial x} = f$; by choosing $\gamma$ of the form , we see that $\frac{\partial h}{\partial y} = g$.

Thus $\omega = dh$ and $H^1_{DR}(\mathbb{R}^2) = 0$.

For $k=2$: if $\omega = k(x,y)\,dx \wedge dy$, then $\omega = d(\eta)$ where $\eta(x,y) = \left(\int_0^x k(t,y)\,dt\right) dy$.
Thus $H^2_{DR}(\mathbb{R}^2) = 0$.

$\underline{H^k_c}$: For $k=1$: proceed as above, but fix $(x_0,y_0)$ outside $B_R(0) \supset \operatorname{supp}\omega$, and define $h(x,y) = \int_\gamma \omega$ where $\gamma$ is any path from $(x_0,y_0)$ to $(x,y)$. Then $\omega = dh$ as before, and $h = 0$ outside $B_R(0)$ since any $(x,y) \notin B_R(0)$ can be connected to $(x_0,y_0)$ by a path outside $B_R(0)$.

For $k=2$: Suppose $\omega$ is a 2-form, ~~supported in $B_R(0)$~~ compactly supported. If $\omega = d(f\,dx + g\,dy)$ with $f,g$ compactly supported, say $\operatorname{supp} f, \operatorname{supp} g \subset B_R(0)$, then by Green's Theorem,

$$0 = \int_\gamma (f\,dx + g\,dy) = \iint_{B_R(0)} \omega = \iint_{\mathbb{R}^2} \omega \quad \text{where } \gamma = \partial B_R(0).$$

$$\left(:= \iint_{\mathbb{R}^2} k\,dx\,dy \text{ if } \omega = k\,dx \wedge dy\right).$$

Thus if we consider the map

$$\int : \Omega^2_c(\mathbb{R}^2) \to \mathbb{R} \quad \text{defined by } \int(\omega) = \iint_{\mathbb{R}^2} \omega,$$

~~then~~ and we write $B = \operatorname{im}(d: \Omega^1_c(\mathbb{R}^2) \to \Omega^2_c(\mathbb{R}^2)) \subset \Omega^2_c(\mathbb{R}^2)$,
then $B \subseteq \ker \int$.

Claim $\ker \int \subseteq B$.

Then since $\int$ is surjective (consider a bump function times $(dx \wedge dy)$), it will follow that $H^2_c(\mathbb{R}^2) \cong \Omega^2_c(\mathbb{R}^2)/B \cong \Omega^2_c(\mathbb{R}^2)/(\ker \int) \cong \mathbb{R}$.

1. Cont'd. Proof of claim: Suppose $\omega = k(x,y)\,dx\wedge dy$ is compactly supported with $\iint_{\mathbb{R}^2} k\,dx\,dy = 0$.

As before, define

$$g_0(x,y) = \int_{-\infty}^{x} k(t,y)\,dt$$

$$\eta_0(x,y) = g_0(x,y)\,dy.$$

Then $\omega = d\eta_0$. The problem is that $g_0$ isn't necessarily compactly supported: for fixed $y$, if $x \gg 0$, then $g_0(x,y) = \int_{-\infty}^{\infty} k(t,y)\,dt$. So define $l: \mathbb{R}\to\mathbb{R}$ by

$$l(y) = \int_{-\infty}^{\infty} k(t,y)\,dt$$

and note $\iint k\,dx\,dy = 0 \Rightarrow \int_{-\infty}^{\infty} l(y)\,dy = 0$, and $l$ is compactly supported since $k$ is.

Also define $\sigma: \mathbb{R}\to\mathbb{R}$ to be any smooth function such that

$$\sigma(x) = \begin{cases} 0, & x \ll 0 \\ 1, & x \gg 0. \end{cases}$$

Then ~~$\sigma(x)l(y)$~~ is a compactly supported function:

$g(x,y) = g_0(x,y) - \sigma(x)l(y)$

if $|y| \gg 0$ then both $g_0(x,y) = \sigma(x)l(y) = 0$, and for fixed $y$, if $|x| \gg 0$ then $g_0(x,y) = \sigma(x)\,l(y)$.

Furthermore,

$$d(g(x,y)\,dy) = d(\underbrace{g_0(x,y)\,dy}^{\eta_0}) - d(\sigma(x)l(y)\,dy) = \omega - \sigma'(x)\,l(y)\,dx\wedge dy.$$

Now define $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \sigma'(x)\left(\int_{-\infty}^{y} l(t)\,dt\right);$$

Since $\sigma = \begin{cases} 0 & x \ll 0 \\ 1 & x \gg 0 \end{cases}$ and $\int_{-\infty}^{\infty} l(t)\,dt = 0$, $f$ is compactly supported.

Finally,

$$d(f(x,y)\,dx) = \sigma'(x)\,l(y)\,dy\wedge dx = -\sigma'(x)\,l(y)\,dx\wedge dy$$

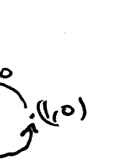
So

$$\omega = d(g(x,y)\,dy - f(x,y)\,dx)$$

and $\omega \in B$, as desired. $\square$

2. $H^k_{DR}(\mathbb{R}^2 \setminus 0)$ is clear except for $k=1$ and $k=2^*$. (*Sorry, I forgot $k=2$; see next page for this case)

**For $k=1$:** Define $Z, B \subset \Omega^1(\mathbb{R}^2 \setminus 0)$ by $Z = \{\omega \in \Omega^1 \mid d\omega = 0\}$ and $B = \{df \mid f: (\mathbb{R}^2\setminus 0) \to \mathbb{R}\}$. We want $Z/B \cong \mathbb{R}$.




Consider the linear map $\Phi: Z \to \mathbb{R}$ by

$$\Phi(\omega) = \int_{\gamma_0} \omega \quad \text{where } \gamma_0 = \text{unit circle in } \mathbb{R}^2, \text{ oriented counterclockwise.}$$

Then $\underline{\Phi \text{ is surjective}}$: if $\omega_0 = \text{"}d\theta\text{"} = \frac{x\,dy - y\,dx}{x^2+y^2}$, then $\Phi(\omega_0) = 2\pi$ and $d\omega_0 = 0$ (straightforward computation).

Also $\underline{B \subset \ker \Phi}$: $\int_{\gamma_0} df = 0$ by the fundamental theorem of line integrals.

$\underline{\text{Claim}}$: $\underline{\ker \Phi \subset B}$. Then $\ker\Phi = B$, so $\mathbb{R} \cong Z/\ker\Phi = Z/B$, as desired.

$\underline{\text{Proof of claim}}$: Suppose $\omega = f\,dx + g\,dy$ is closed, so $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Then as in problem 1, we want to define $h: \mathbb{R}^2\setminus 0 \to \mathbb{R}$ by

$$h(x,y) = \int_\gamma \omega \quad \text{where } \gamma \text{ is any path from } (1,0) \text{ to } (x,y).$$

If this is well-defined, then $\omega = dh$ so $\omega \in B$.

But if $\Phi(\omega) = 0$, then this $\underline{\text{is}}$ well-defined: if $\gamma, \tilde\gamma$ are two paths from $(1,0)$ to $(x,y)$, then there is some $k \in \mathbb{Z}$ such that $\tilde\gamma$ is homotopic to $k\gamma_0 + \gamma$ (i.e. trace $\gamma_0$ $k$ times, then follow with $\gamma$).

But then by Green's Theorem,

$$\int_{\tilde\gamma} \omega = \int_{k\gamma_0 + \gamma} \omega = k\int_{\gamma_0} \omega + \int_\gamma \omega = \int_\gamma \omega. \quad \square$$

→ Note: $[\omega_0]$ generates $H^1_{DR}(\mathbb{R}^2\setminus 0)$.

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3. Suppose $k: \mathbb{R}^3 \setminus 0 \to \mathbb{R}$ is compactly supported. ~~with~~

~~$\operatorname{supp} k \subset \{(x,y,z) \mid r \le \|(x,y,z)\| \le R\}$ for some $r, R$~~

Choose any such $k$ with $\iiint_{\mathbb{R}^3\setminus 0} k\,dx\,dy\,dz \neq 0$.

We claim $k\,dx\wedge dy \wedge dz \notin \operatorname{im}(d: \Omega^2_c(\mathbb{R}^3\setminus 0) \to \Omega^3_c(\mathbb{R}^3 \setminus 0))$.

Indeed, if $\omega \in \Omega^2_c(\mathbb{R}^3\setminus 0)$ is given by $\omega = f_1\, dy\wedge dz - f_2\, dx\wedge dz + f_3\, dx\wedge dy$ and $\vec F$ is the vector field on $\mathbb{R}^3\setminus 0$ given by $\vec F = (f_1, f_2, f_3)$, then $d\omega = (\vec\nabla\cdot\vec F)\, dx\wedge dy\wedge dz$. Now suppose $r, R$ are such that

$$(\operatorname{supp} f_1) \cup (\operatorname{supp} f_2) \cup (\operatorname{supp} f_3) \subset \{(x,y,z) \mid r \le \|(x,y,z)\| \le R\} =: K.$$

Then by the Divergence Theorem, $\iiint_{\mathbb{R}^3\setminus 0} (\vec\nabla\cdot\vec F)\,dx\,dy\,dz = \iiint_K (\vec\nabla\cdot\vec F)\,dx\,dy\,dz = \iint_{S^3_R(0)} \vec F\cdot d\vec n - \iint_{S^3_r(0)} \vec F \cdot d\vec n = 0.$

Thus $d\omega \neq k\,dx\wedge dy\wedge dz$. $\square$

2. Cont'd. For $k=2$: The easiest way to do this is via polar coordinates $(r,\theta)$ with $x = r\cos\theta$, $y = r\sin\theta$. Then as usual

$$dr = \cos\theta\, dx + \sin\theta\, dy = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$

$$d\theta = \frac{-\sin\theta\, dx + \cos\theta\, dy}{r} = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

$$r\, dr \wedge d\theta = dx \wedge dy.$$

If $\omega = k(x,y)\, dx \wedge dy$ on $\mathbb{R}^2 \setminus 0$, then define $l : \mathbb{R}^2 \setminus 0 \to \mathbb{R}$ by

$$\cancel{\eta(x,y) = \int_1^r}$$

$$l(r\cos\theta, r\sin\theta) = \int_1^r t\, k(t\cos\theta, t\sin\theta)\, dt$$

$$\Rightarrow \frac{\partial l}{\partial r} = r\, k(r\cos\theta, r\sin\theta).$$

Thus if we define $\eta \in \Omega^1(\mathbb{R}^2 \setminus 0)$ by $\eta = l\, d\theta$, then

$$d\eta = \frac{\partial l}{\partial r}\, dr \wedge d\theta = k\, r\, dr \wedge d\theta = k\, dx \wedge dy = \omega.$$

(If you don't like polar coordinates, then this can be done in Cartesian:

$$l(x,y) = \int_1^{\sqrt{x^2+y^2}} t\, k\left(\frac{tx}{\sqrt{x^2+y^2}}, \frac{ty}{\sqrt{x^2+y^2}}\right) dt$$

$$\eta = l\left(\frac{-y\,dx + x\,dy}{x^2+y^2}\right)$$

and it's an involved but straightforward calculation to show $\omega = d\eta$.)