# Math 262 HW 3

1. (a) $H_0(X)=\mathbb{Z}$ since $X$ connected $\Rightarrow H^0(X;\mathbb{Z})=$ free part of $\mathbb{Z}=\mathbb{Z}$ by UCT.
$H_1(X)=0$ since $X$ simply connected $\Rightarrow H^1(X;\mathbb{Z})=0$ by UCT
Poincaré duality $\Rightarrow$ $H^4(X;\mathbb{Z})\cong H_0(X)=\mathbb{Z}$, $H_4(X)\cong H^0(X;\mathbb{Z})=\mathbb{Z}$,
$H^3(X;\mathbb{Z})\cong H_1(X)=0$; $H_3(X)\cong H^1(X;\mathbb{Z})=0$.

(b) UCT: $H^2(X;\mathbb{Z})\cong(\text{free part of } H_2(X))\oplus(\text{torsion part of } H_1(X))=\text{free}$;
PD $\Rightarrow$ $H_2(X)\cong H^2(X;\mathbb{Z})=\text{free}$.

2. Suppose $H_{2n+1}(X)\cong\mathbb{Z}$; then $\overset{\text{UCT}}{\Rightarrow}$ free part of $H^{2n+1}(X;\mathbb{Z})=\mathbb{Z}$.
Then by PD, $\cup:\underbrace{\left(\text{free part of } H^{2n+1}(X;\mathbb{Z})\right)}_{\cong\,\mathbb{Z}}\otimes\underbrace{\left(\text{free part of } H^{2n+1}(X;\mathbb{Z})\right)}_{\cong\,\mathbb{Z}}\to\mathbb{Z}$ is nonsingular.
But if $\alpha$ generates the free part of $H^{2n+1}(X;\mathbb{Z})$, then
$$\alpha\cup\alpha=(-1)^{(2n+1)^2}(\alpha\cup\alpha)\Rightarrow\alpha\cup\alpha=0$$
So $\cup:\mathbb{Z}\otimes\mathbb{Z}\to\mathbb{Z}$ is the zero map, contradiction.

3. (a) Let $\alpha$ generate $\underbrace{H^1(\mathbb{RP}^n;\mathbb{Z}/2)}_{=\mathbb{Z}/2}$ and $\beta$ generate $H^1(\mathbb{RP}^m;\mathbb{Z}/2)=\mathbb{Z}/2$.
~~Let~~
Then by naturality in UCT for fields, $f_*:H_1\to H_1$ isomorphism implies
$f^*:H^1(\mathbb{RP}^m;\mathbb{Z}/2)\to H^1(\mathbb{RP}^n;\mathbb{Z}/2)$ is an isomorphism.
(since it's the dual map to $f_*$). Thus $f^*\beta=\alpha$.
But $H^*(\mathbb{RP}^n;\mathbb{Z}/2)=\mathbb{Z}/2[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{RP}^m;\mathbb{Z}/2)=\mathbb{Z}/2[\beta]/(\beta^{m+1})$
so
$$\beta^{m+1}=0\Rightarrow 0=f^*(\beta^{m+1})=(f^*\beta)^{m+1}=\alpha^{m+1}\Rightarrow n\le m.$$

(b) Suppose there were such a map; this would induce a map $\tilde f:\mathbb{RP}^n\to\mathbb{RP}^{n-1}$. Let $x_0\in S^n$, and let $\gamma$ be any path in $S^n$ from $x_0$ to $-x_0$; then $p\circ\gamma$ generates $\pi_1(\mathbb{RP}^n,p(x_0))\cong\mathbb{Z}/2$ (by covering maps), and $f\circ\gamma$ connects $f(x_0)$ to $f(-x_0)=-f(x_0)$ so $p\circ f\circ\gamma$ generates $\pi_1(\mathbb{RP}^m,p(f(x_0)))\cong\mathbb{Z}/2$. Since $p\circ f=\tilde f\circ p$, it follows that $f_*$ maps the generator of $H_1(\mathbb{RP}^n)$ to the generator of $H_1(\mathbb{RP}^{n-1})$, contradicting (a).

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "p"] & S^{n-1} \arrow[d, "p"] \\ \mathbb{RP}^n \arrow[r, "\tilde f"] & \mathbb{RP}^{n-1} \end{tikzcd}$$