Math 262 Homework 3—due Thursday September 22

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1. Let X be a compact, orientable, connected, simply connected 4-manifold.
   (a) Calculate, with justification, $H_k(X)$ and $H^k(X;\mathbb{Z})$ for all $k \neq 2$.
   (b) Prove that $H_2(X)$ and $H^2(X;\mathbb{Z})$ are both free $\mathbb{Z}$-modules.

2. Let X be a compact, orientable manifold of dimension $4n+2$. Show that it cannot be the case that $H_{2n+1}(X) \cong \mathbb{Z}$.

3. (a) Suppose that $f : \mathbb{RP}^n \to \mathbb{RP}^m$ is a continuous map such that

$$f_* : H_1(\mathbb{RP}^n;\mathbb{Z}/2) \to H_1(\mathbb{RP}^m;\mathbb{Z}/2)$$

   is an isomorphism (note that both sides are isomorphic to $\mathbb{Z}/2$). Use the ring structure on cohomology to prove that $n \leq m$.

   (b) Use (a) to prove that there is no continuous map $f : S^n \to S^{n-1}$ such that $f(-x) = -f(x)$ for all $x \in S^n$. (Hint: since the projection map $p : S^n \to \mathbb{RP}^n$ is a covering map, the image under p of any path in S^n from a point $x_0 \in S^n$ to $-x_0$ represents the nonzero element of $\pi_1(\mathbb{RP}^n, p(x_0)) \cong \mathbb{Z}/2$.)

Note (not part of the exercise): this gives a different, and possibly more elementary, proof of the Borsuk–Ulam Theorem than the one presented in Hatcher (pp. 174–176), which uses transfer sequences. See the first three lines of the proof of Corollary 2B.7 on page 176 for the proof that (b) implies the Borsuk–Ulam Theorem.