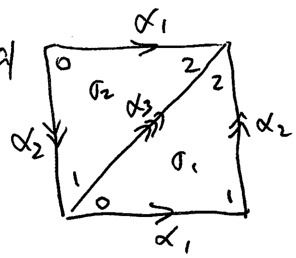
Math 262 HW 2

1. (a)



With vertices ordered as shown,

$$\partial\sigma_1 = \alpha_1 + \alpha_2 - \alpha_3$$
$$\partial\sigma_2 = \alpha_2 + \alpha_3 - \alpha_1 \quad \Rightarrow \quad \partial(\sigma_1+\sigma_2) = 0 \mod 2.$$

Among $\varphi_i \cup \varphi_j$, σ_1 contributes to $\varphi_2 \cup \varphi_1$ only

σ_2 contributes to $\varphi_1\cup\varphi_1$ and $\varphi_1 \cup \varphi_2$.

So
$$\varphi_1\cup\varphi_1([\sigma_1+\sigma_2]) = 1 \qquad \varphi_1\cup\varphi_2([\sigma_1+\sigma_2]) = 1$$
$$\varphi_2\cup\varphi_1([\sigma_1+\sigma_2]) = 1 \qquad \varphi_2\cup\varphi_2([\sigma_1+\sigma_2]) = 0.$$

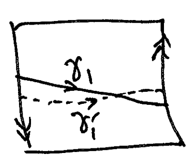
If φ_1, φ_2 aren't a basis for $H^1(K;\mathbb{Z}/2) \cong (\mathbb{Z}/2)^2$, then either $\varphi_1 = 0$ or $\varphi_2 = 0$ (contradicting $\varphi_1\cup\varphi_2([\sigma_1+\sigma_2]) = 1$) or $\varphi_1 = \varphi_2$ (contradicting $\varphi_1\cup\varphi_1([\sigma_1+\sigma_2]) \neq \varphi_2\cup\varphi_2([\sigma_1+\sigma_2])$).

(b)

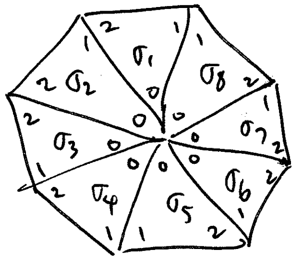
	1	φ_1	φ_2	Θ
1	1	φ_1	φ_2	Θ
φ_1	φ_1	Θ	Θ	0
φ_2	φ_2	Θ	0	0
Θ	Θ	0	0	0

(c) This says that if γ_1' is a perturbation of γ_1, then $\#(\gamma_1 \cap \gamma_1') = 1 \pmod 2$.

Ex:



2. (a)



$\partial(\sigma_1 + \sigma_2 - \sigma_3 - \sigma_4 + \sigma_5 + \sigma_6 - \sigma_7 - \sigma_8) = 0.$

(all the 1-cells cancel out)

Among $\varphi_i \cup \varphi_j$,

σ_1 contributes to (none)	σ_5 contributes to (none)
σ_2 " $+1$ to $\varphi_1 \cup \varphi_2$	σ_6 " $+1$ to $\varphi_3 \cup \varphi_4$
σ_3 " $+1$ to $\varphi_2 \cup \varphi_1$	σ_7 " $+1$ to $\varphi_4 \cup \varphi_3$
σ_4 " to (none)	σ_8 " to (none).

$\cup$	φ_1	φ_2	φ_3	φ_4
φ_1	0	Θ	0	0
φ_2	$-\Theta$	0	0	0
φ_3	0	0	0	Θ
φ_4	0	0	$-\Theta$	0

(b) Suppose $e_1, \dots, e_4$ is the standard basis for $\mathbb{Z}^4 \cong H^1(\Sigma_2; \mathbb{Z})$; then we can write $\varphi_i = \sum_j a_{ij} e_j$ for $a_{ij} \in \mathbb{Z}$. Also write $e_i \cup e_j = b_{ij} \Theta$ for $b_{ij} \in \mathbb{Z}$. If we define 4×4 matrices $A = (a_{ij})$ and $B = (b_{ij})$, then the above multiplication table gives

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = A B A^T.$$

Thus $(\det A)^2 (\det B) = 1 \Rightarrow \det A = \pm 1 \Rightarrow A$ is invertible over $\mathbb{Z}$. It follows that we can express $e_1, \dots, e_4$ as $\mathbb{Z}$-linear combinations of $\varphi_1, \dots, \varphi_4$, so $\{\varphi_1, \dots, \varphi_4\}$ generates $\mathbb{Z}^4$.

3. The isomorphism is of graded rings, where the grading on $\Lambda_{\mathbb{Z}} M_n$ is given by

$$\Lambda_{\mathbb{Z}} M_n \cong \Lambda^0_{\mathbb{Z}} M_n \oplus \Lambda^1_{\mathbb{Z}} M_n \oplus \cdots \oplus \Lambda^n_{\mathbb{Z}} M_n$$

i.e. $\Lambda^k_{\mathbb{Z}} M_n$ is the k-th graded piece of $\Lambda_{\mathbb{Z}} M_n$.

Proof by induction: note for $n=1$ that $H^*(S^1;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^2) \cong \Lambda_{\mathbb{Z}} M_1$.

For the induction step, it suffices to find an isomorphism

(x has grading 1, while 1 has grading 0)

$$\varphi : (\Lambda_{\mathbb{Z}} M_{n-1}) \otimes (\mathbb{Z}[x]/(x^2)) \longrightarrow \Lambda_{\mathbb{Z}} M_n .$$

This is given by $\varphi(v \otimes 1) = v$, $\varphi(v \otimes x) = v \wedge v_n$ for $v \in \Lambda_{\mathbb{Z}} M_{n-1}$.

It's easy to check that this is an isomorphism of graded $\mathbb{Z}$-modules, so we just have to check that it preserves multiplication:

$$\varphi((v_1 \otimes 1)(v_2 \otimes 1)) = \varphi((v_1 \wedge v_2) \otimes 1) = v_1 \wedge v_2 = \varphi(v_1 \otimes 1)\,\varphi(v_2 \otimes 1)$$

$$\varphi((v_1 \otimes 1)(v_2 \otimes x)) = \varphi((v_1 \wedge v_2) \otimes x) = v_1 \wedge v_2 \wedge v_n = \varphi(v_1 \otimes 1)\,\varphi(v_2 \otimes x)$$

$$\varphi((v_1 \otimes x)(v_2 \otimes 1)) = \cancel{\varphi((v_1 \wedge v_2) \otimes x)}$$

~~since~~

$$\hookrightarrow (-1)^{|v_2|} \varphi((v_1 \wedge v_2) \otimes x) = (-1)^{|v_2|} v_1 \wedge v_2 \wedge v_n = v_1 \wedge v_n \wedge v_2 = \varphi(v_1 \otimes x)\,\varphi(v_2 \otimes 1)$$

$|v_2|$ = dimension of v_2

$$\varphi((v_1 \otimes x)(v_2 \otimes x)) = (-1)^{|v_2|} \varphi((v_1 \wedge v_2) \otimes x^2) = 0 = (v_1 \wedge v_n) \wedge (v_2 \wedge v_n) = \varphi(v_1 \otimes x)\,\varphi(v_2 \otimes x).$$

4. If there were a homeomorphism $f : \mathbb{C}P^3 \to S^2 \times S^4$, then we would have a ring isomorphism

$$f^* : H^*(S^2 \times S^4;\mathbb{Z}) \to H^*(\mathbb{C}P^3;\mathbb{Z}).$$

Now say y generates $H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$ and 1 generates $H^0(S^4;\mathbb{Z}) \cong \mathbb{Z}$.

By Künneth, $y \otimes 1$ generates the dimension 2 part of $H^*(S^2) \otimes H^*(S^4)$.

~~Thus $f^*(x \otimes 1)$ generates $H^2(\mathbb{C}P^3;\mathbb{Z})$. But then~~

Thus $f^*(y \otimes 1) = \pm x$ where $H^*(\mathbb{C}P^3;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^4)$. But then

$$x^2 = (\pm x)^2 = f^*(y \otimes 1) \cup f^*(y \otimes 1) = f^*((y \otimes 1)(y \otimes 1)) = f^*(\underbrace{(y \cup y)}_{0} \otimes 1) = 0$$

which isn't the case.