Math 262 Homework 2—due Thursday September 15

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1. For the Klein bottle K drawn below:

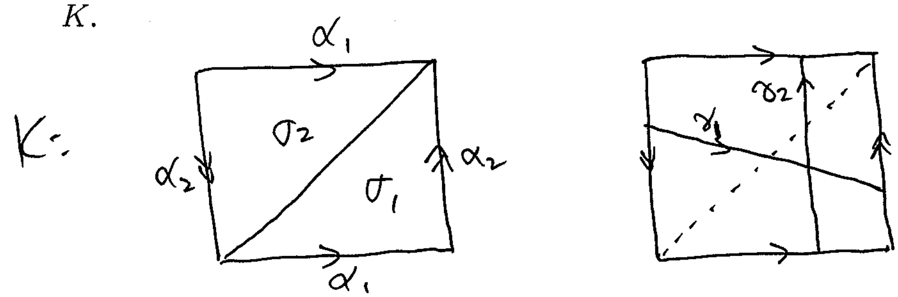
   (a) Order the vertices of the 2-simplices σ_1 and σ_2 so that $\sigma_1 + \sigma_2$ is a cycle mod 2. Let $\phi_1, \phi_2 \in H^1(K; \mathbb{Z}/2)$ be the cohomology classes given by the Poincaré dual cocycles to the closed loops γ_1, γ_2. Calculate

   $$(\phi_i \smile \phi_j)([\sigma_1 + \sigma_2]) \in \mathbb{Z}/2$$

   for each (i, j) with $1 \leq i, j \leq 2$. Use this to deduce that ϕ_1 and ϕ_2 form a basis for $H^1(K; \mathbb{Z}/2)$.

   (b) Let Θ be the generator for $H^2(K; \mathbb{Z}/2)$, so that $H^*(K; \mathbb{Z}/2)$ is generated by $1, \phi_1, \phi_2, \Theta$. Write the multiplication table for $H^*(K; \mathbb{Z}/2)$.

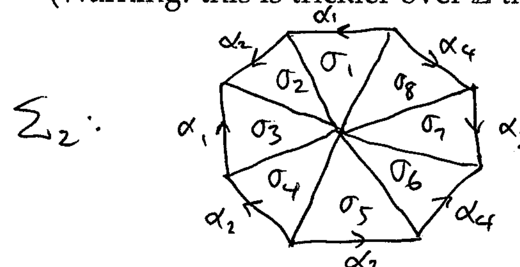
   (c) You should find that $\phi_i \smile \phi_i \neq 0$ for one of $i = 1, 2$. Interpret this fact geometrically in terms of an intersection number between two closed curves in K.

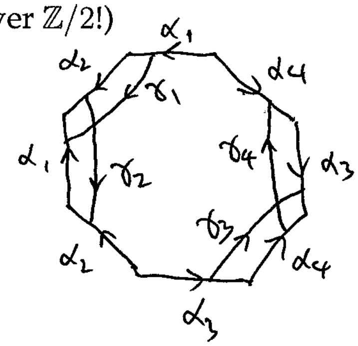


2. Consider the genus 2 surface Σ_2 drawn below.

   (a) Order the vertices of the 2-simplices $\sigma_1, \ldots, \sigma_8$ so that some nontrivial linear combination is a cycle (over $\mathbb{Z}$). Let $\phi_1, \ldots, \phi_4 \in H^1(\Sigma_2; \mathbb{Z})$ be the cohomology classes given by the Poincaré dual cocycles to $\gamma_1, \ldots, \gamma_4$, and let Θ be a generator for $H^2(\Sigma_2)$. Write the 4×4 multiplication (cup product) table for $\phi_1, \ldots, \phi_4$.

   (b) Explain why it follows that $\{\phi_1, \ldots, \phi_4\}$ generates $H^1(\Sigma_2; \mathbb{Z}) \cong \mathbb{Z}^4$ as a $\mathbb{Z}$-module. (Warning: this is trickier over $\mathbb{Z}$ than over $\mathbb{Z}/2$!)





(Please turn over for problems 3 and 4.)

3. Let the n-torus T^n denote the product of n copies of S^1. Use the Künneth Theorem to prove the ring isomorphism

$$H^*(T^n; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}} M_n$$

where $M_n = \mathbb{Z}\langle v_1, \ldots, v_n \rangle$ is the free $\mathbb{Z}$-module with n generators.

4. We will soon see that $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$ as rings, where the isomorphism identifies x with the generator of $H^2(\mathbb{CP}^n; \mathbb{Z})$. Use this fact and the Künneth Theorem to show that $S^2 \times S^4$ is not homeomorphic to $\mathbb{CP}^3$ (even though they share the same cohomology groups).