Math 262 HW 1

1. Cellular chain complex $\mathcal{C}$:

(m even) $0 \to \overset{m}{\mathbb{Z}} \xrightarrow{2} \overset{m-1}{\mathbb{Z}} \xrightarrow{0} \cdots \xrightarrow{2} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}} \to 0$

(m odd) $0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

(a) m even: $H_*(X) = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & 0<*<m \text{ odd} \\ 0 & \cancel{*>0 \text{ even}} \text{ otherwise} \end{cases}$ m odd: $H_*(X) = \begin{cases} \mathbb{Z} & *=0 \text{ or } *=m \\ \mathbb{Z}/2 & 0<*<m \text{ odd} \\ 0 & \cancel{*>0 \text{ even}} \text{ otherwise.} \end{cases}$

Dual chain complex: $\mathrm{Hom}(\mathcal{C};\mathbb{Z})$

(m even) $0 \leftarrow \overset{m}{\mathbb{Z}} \xleftarrow{2} \overset{m-1}{\mathbb{Z}} \xleftarrow{0} \cdots \xleftarrow{2} \overset{1}{\mathbb{Z}} \xleftarrow{0} \overset{0}{\mathbb{Z}} \leftarrow 0$

(m odd) $0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \cdots \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$

m ~~odd~~ even: $H^*(X) = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}/2 & 0<*\le m \text{ even} \\ 0 & \text{otherwise} \end{cases}$ m odd: $H^*(X) = \begin{cases} \mathbb{Z} & *=0 \text{ or } *=m \\ \mathbb{Z}/2 & 0<*<m \text{ even} \\ 0 & \text{otherwise.} \end{cases}$

(b) Over $\mathbb{Z}/2$, all chain groups are $\mathbb{Z}/2$ and all maps are 0.

$H_*(X;\mathbb{Z}/2) = H^*(X;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & 0 \le * \le m \\ 0 & \text{otherwise.} \end{cases}$

Over $\mathbb{R}$: all $\times 2$ maps are isomorphisms.

m even: $H_*(X;\mathbb{R}) = \begin{cases} \mathbb{R} & *=0 \\ 0 & \text{otherwise} \end{cases}$ m odd: $H_*(X;\mathbb{R}) = \begin{cases} \mathbb{R} & *=0 \text{ or } *=m \\ 0 & \text{otherwise} \end{cases}$

$H^*(X;\mathbb{R}) = \begin{cases} \mathbb{R} & *=0 \\ 0 & \text{otherwise} \end{cases}$ $H^*(X;\mathbb{R}) = \begin{cases} \mathbb{R} & *=0 \text{ or } *=m \\ 0 & \text{otherwise.} \end{cases}$

(c) Clear.

(d) Easy to check. (Note $\mathrm{Ext}(H,\mathbb{R})=0$ for any H.)

2. Cellular chain complex: $0 \to \overset{2}{\mathbb{Z}} \xrightarrow{\times(2,0)} \overset{1}{\mathbb{Z}^2} \xrightarrow{0} \overset{0}{\mathbb{Z}} \to 0$

Dual chain complex: $0 \leftarrow \mathbb{Z} \xleftarrow[\times\binom{2}{0}]{} \mathbb{Z}^2 \xleftarrow{0} \mathbb{Z} \leftarrow 0$

i.e. $(1,0) \mapsto 2$, $(0,1) \mapsto 0$

(a) $H_*(X) = \begin{cases} \mathbb{Z} & *=0 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & *=1 \\ 0 & \text{otherwise} \end{cases}$ $H^*(X) = \begin{cases} \mathbb{Z} & *=0 \text{ or } *=1 \\ \mathbb{Z}/2 & *=2 \\ 0 & \text{otherwise.} \end{cases}$

2 Cont'd. (b) $H_*(X;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & *=0 \text{ or } *=2 \\ (\mathbb{Z}/2)\oplus(\mathbb{Z}/2) & *=1 \\ 0 & \text{otherwise} \end{cases} = H^*(X;\mathbb{Z}/2)$

$H_*(X;\mathbb{R}) = \begin{cases} \mathbb{R} & *=0 \text{ or } *=1 \\ 0 & \text{otherwise} \end{cases} = H^*(X;\mathbb{R})$

(c), (d) easy to check.

3. Cellular chain complex $\quad 0 \longrightarrow \overset{3}{\mathbb{Z}} \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{a} \overset{1}{\mathbb{Z}} \xrightarrow{0} \overset{0}{\mathbb{Z}} \longrightarrow 0$

Dual chain complex $\quad 0 \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z} \xleftarrow{a} \mathbb{Z} \longleftarrow \mathbb{Z} \longleftarrow 0$

(a) $H_*(X) = \begin{cases} \mathbb{Z} & *=0 \text{ or } *=3 \\ \mathbb{Z}/a & *=1 \\ 0 & \text{otherwise} \end{cases} \qquad H^*(X) = \begin{cases} \mathbb{Z} & *=0 \text{ or } *=3 \\ \mathbb{Z}/a & *=2 \\ 0 & \text{otherwise.} \end{cases}$

(b) Over $\mathbb{Z}/m$: $\quad C\otimes\mathbb{Z}/m \quad 0 \to \mathbb{Z}/m \xrightarrow{0} \mathbb{Z}/m \xrightarrow{a} \mathbb{Z}/m \xrightarrow{0} \mathbb{Z}/m \to 0$

$\mathrm{Hom}(C,\mathbb{Z}/m) \quad 0 \leftarrow \mathbb{Z}/m \xleftarrow{0} \mathbb{Z}/m \xleftarrow{a} \mathbb{Z}/m \xleftarrow{0} \mathbb{Z}/m \leftarrow 0.$

$H_*(X;\mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & *=0 \text{ or } *=3 \\ \mathbb{Z}/(a,m) & *=1 \text{ or } *=2 \\ 0 & \text{otherwise} \end{cases} = H^*(X;\mathbb{Z}/m)$ where $(a,m)=\gcd(a,m)$.

Over $\mathbb{R}$: $H_*(X;\mathbb{R}) = \begin{cases} \mathbb{R} & *=0 \text{ or } *=3 \\ 0 & \text{otherwise} \end{cases} = H^*(X;\mathbb{R})$

(c) Clear.

(d) Straightforward to check. Note

$\mathrm{Hom}(\mathbb{Z}/a, \mathbb{Z}/m) = \mathbb{Z}/(a,m)$

$\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/m) = 0$

$\mathrm{Ext}(\mathbb{Z}/a, \mathbb{Z}/m) = \mathbb{Z}/(a,m).$