Math 262 HW #10

1. $H(K) \cong \mathbb{Z}/4$ (generated by y_0) $\qquad H(K/\mathcal{F}^1K) \cong \mathbb{Z}/2$ (generated by y_0)

$H(\mathcal{F}^1K) \cong \mathbb{Z}/2$ (generated by y_1) $\qquad H(\mathcal{F}^1K/\mathcal{F}^2K) = H(\mathcal{F}^1K) \cong \mathbb{Z}/2$.

The map $i: H(\mathcal{F}^1K) \to H(K)$ sends y_1 to $y_1 = -2y_0$, so it's the map $\mathbb{Z}/2 \to \mathbb{Z}/4$, $1 \mapsto 2$.

Thus $\operatorname{Gr} H(K) = \big(H(K)/iH(\mathcal{F}^1K)\big) \oplus \big(iH(\mathcal{F}^1K)\big) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

$E_1 = H(\operatorname{Gr} K) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Since $E_1 = \operatorname{Gr} H(K)$ and the spectral sequence converges to $\operatorname{Gr} H(K)$, it must be the case that $d_1 = d_2 = \cdots = 0$ since otherwise E_∞ would have fewer elements than E_1.

Thus $\boxed{E_1 = E_2 = \cdots = E_\infty = \mathbb{Z}/2 \oplus \mathbb{Z}/2}$ and $E_\infty \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is not isom. to $H(K) \cong \mathbb{Z}/4$.

2. $H(K) = 0$ (all cycles are boundaries: linear combinations of $y_0, y_1, y_2, \ldots$).

Suppose $E_1, \ldots$ converged to $H(K)$. Then for some r, $E_r = 0$ (and $d_r = 0$).

Then from the exact couple
$$\begin{tikzcd} A_r \arrow[rr, "i_r"] & & A_r \arrow[dl] \\ & E_r \arrow[ul] & \end{tikzcd}$$
, $i_r: A_r \to A_r$ would have to be an isomorphism. Now

$$A_r = \quad \cdots \xleftarrow{i} H(K) \xleftarrow{i} iH(\mathcal{F}^1K) \xleftarrow{i} \cdots \xleftarrow{i} i^{r-1}H(\mathcal{F}^{r-1}K) \xleftarrow{i} i^{r-1}H(\mathcal{F}^rK) \leftarrow \cdots$$

and $i^{r-1}H(\mathcal{F}^{r-1}K) = 0$ since it sits in $H(K) = 0$. But

$$H(\mathcal{F}^1K) \cong \mathbb{R}\langle y_0\, y_1\, y_2 \ldots \rangle$$
$$H(\mathcal{F}^rK) \cong \mathbb{R}\langle y_{r-1}\, y_r\, y_{r+1} \ldots \rangle$$
$$i^{r-1}H(\mathcal{F}^rK) \cong \mathbb{R}\langle y_{r-1}\, y_r\, y_{r+1} \ldots \rangle \subset H(\mathcal{F}^1K)$$

so $i: i^{r-1}H(\mathcal{F}^rK) \to i^{r-1}H(\mathcal{F}^{r-1}K)$ is not injective. Thus $i_r: A_r \to A_r$ is not injective and thus not an isomorphism.

3. $H(K)=0$ $\qquad\qquad$ $H(K/\mathcal{J}^1K)=\mathbb{R}\langle x_0\rangle$

$H(\mathcal{J}^mK)=\mathbb{R}\langle y_n\rangle$ for $1\leq m\leq n$ $\qquad$ $H(\mathcal{J}^mK/\mathcal{J}^{m+1}K)=0$ for $1\leq m\leq n-1$ or $m\geq n+1$

$H(\mathcal{J}^mK)=0$ for $m\geq n+1$ $\qquad$ $H(\mathcal{J}^nK/\mathcal{J}^{n+1}K)=\mathbb{R}\langle y_n\rangle$

The first exact couple is:

$$
\begin{tikzcd}
 & (\mathcal{J}^0) & (\mathcal{J}^1) & & & (\mathcal{J}^n) & \\
A_1: & 0 & \langle y_n\rangle \arrow[l] & \langle y_n\rangle \arrow[l, "i_1" above, "=" below] & \cdots \arrow[l, "i_1" above, "=" below] & \langle y_n\rangle \arrow[l, "i_1" above, "=" below] \arrow[dd, "j_1"] & 0 \arrow[l] \\
 & & & & & & \\
E_1: & \langle x_0\rangle \arrow[uur, "k_1"] & 0 & 0 & \cdots & \langle y_n\rangle & \\
 & (\mathcal{J}^0/\mathcal{J}^1) & & & & &
\end{tikzcd}
$$

where $j_1(y_n)=y_n$ and $k_1(x_0)=y_1=y_n$ since $Dx_0=y_1$. Note $d_1=0$.

The second exact couple is:

$$
\begin{tikzcd}
 & (iH(\mathcal{J}^1)) & (iH(\mathcal{J}^2)) & (iH(\mathcal{J}^3)) & & (iH(\mathcal{J}^n)) \\
A_2: & 0 & \langle y_n\rangle & \langle y_n\rangle \arrow[l, "i_2" above, "=" below] & \cdots \arrow[l, "i_2" above, "=" below] & \langle y_n\rangle \arrow[l, "i_2" above, "=" below] \arrow[d, "j_2"] \\
E_2: & \langle x_0\rangle \arrow[ur, "k_2"] & & & & \langle y_n\rangle
\end{tikzcd}
$$

etc. (note $d_2=0$). The n-th exact couple is

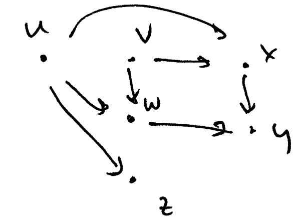
$$
\begin{tikzcd}
 & & (i^{n-1}H(\mathcal{J}^n)) \\
A_n: & & \langle y_n\rangle \arrow[d, "j_n"] \\
E_n: & \langle x_0\rangle \arrow[ur, "k_n"] & \langle y_n\rangle
\end{tikzcd}
$$

and $d_n(x_0)=y_n$, $d_n(y_n)=0$ $\Rightarrow$ $E_{n+1}=H(E_n,d_n)=0$.

So $\boxed{E_r\cong\mathbb{R}\langle x_0,y_n\rangle\cong\mathbb{R}^2 \text{ for } r\leq n}$ and $\boxed{E_r\cong 0\cong H(K) \text{ for } r\geq n+1.}$ $\qquad E_\infty = E_r$

4. (a) $H(K)=0$ (cycles and boundaries are both linear combination of $z, w+x, y$).

(b)



$\underline{E_1 = \langle u, z\rangle}$

$H(K)=0$ $\qquad$ $H(K/J^1K)=\langle u\rangle$

$H(J^1K)=\langle z\rangle$ $\qquad$ $H(J^1K/J^2K)=\langle z\rangle$

$H(J^2K)=0$ $\qquad$ $H(J^2K/J^3K)=0$

$$\begin{tikzcd}
A_1 = & 0 & \langle z\rangle \arrow[l] \arrow[d, "j_1"] & 0 \arrow[l] \\
E_1 = & \langle u\rangle \arrow[ur, "k_1"] & \langle z\rangle &
\end{tikzcd}$$

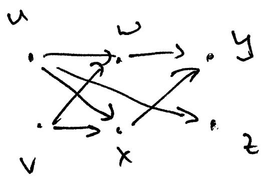
$k_1\langle u\rangle = w+z = z$ in $H(J^1K)$.

$j_1(z)=z$

$\Rightarrow d_1(u)=z,\ d_1(z)=0$

$\Rightarrow E_2 = H(E_1, d_1)=0$ and $\underline{E_2=\cdots=E_\infty=0=H(K)}$.

(c)



$H(K)=0$ $\qquad$ $H(K/J^1K)=\langle u,v\rangle$

$H(J^1K)=\langle w+x, z\rangle$ $\qquad$ $H(J^1K/J^2K)=\langle w,x\rangle$

$H(J^2K)=\langle y,z\rangle$ $\qquad$ $H(J^2K/J^3K)=\langle y,z\rangle$.

$$\begin{tikzcd}
 & & (H(J^1K)) & & (H(J^2K)) & \\
A_1 = & 0 & \langle w+x, z\rangle \arrow[l] \arrow[d, "j_1"] & & \langle y,z\rangle \arrow[ll, "i_1"'] \arrow[d, "j_1"] & 0 \arrow[l] \\
E_1 = & \langle u,v\rangle \arrow[ur, "k_1"] & \langle w,x\rangle \arrow[urr, "k_1"] & & \langle y,z\rangle &
\end{tikzcd}$$

$\underline{E_1 \cong \langle u,v,w,x,y,z\rangle \cong \mathbb{R}^6}$

$i_1(y)=0,\ i_1(z)=z$

in $H(J^1K)$: $j_1(w+x)=w+x,\ j_1(z)=0$; in $H(J^2K)$: $j_1(y)=y,\ j_1(z)=z$

$k_1(u)=w+x,\ k_1(v)=w+x,\ k_1(w)=y,\ k_1(x)=-y$

$\Rightarrow d_1(u)=w+x,\ d_1(v)=w+x,\ d_1(w)=y,\ d_1(x)=-y$

$\Rightarrow \underline{E_2 = H(E_1, d_1) \cong \langle u-v, z\rangle \cong \mathbb{R}^2}$.

$$\begin{tikzcd}
 & & (iH(J^2K)) & \\
A_2 = & & \langle z\rangle \arrow[dr, "j_2"] & \\
E_2 = & \langle u-v\rangle \arrow[ur, "k_2"] & & \langle z\rangle
\end{tikzcd}$$

$j_2(z)=z$; $k_2(u-v)=z$ since $D(u-v)=z$

$\Rightarrow d_2(u-v)=z,\ d_2(z)=0$

$\Rightarrow \underline{E_3=\cdots=E_\infty=0=H(K)}$.