Math 262 Homework 1—due Thursday September 8

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For each of the following spaces and groups:

(a) use a cellular chain complex $\mathcal{C}$ to calculate $H_*(X)$ and $H^*(X)$ for all $*$;

(b) use $\mathcal{C}$ to calculate $H_*(X;G)$ and $H^*(X;G)$ for all $*$;

(c) in the cases where G is a field, for all n, verify the Universal Coefficient Theorem for cohomology with field coefficients:

$$H^n(\mathcal{C};G) \cong \mathrm{Hom}(H_n(\mathcal{C};G),G);$$

(d) in all cases, check the general Universal Coefficient Theorem for cohomology. More precisely, verify that

$$H^n(\mathcal{C};\mathbb{Z}) \cong F_n \oplus T_{n-1}$$

where F_k and T_k are the free and torsion subgroups of $H_k(\mathcal{C})$, and verify that

$$H^n(\mathcal{C};G) \cong \mathrm{Hom}(H_n(\mathcal{C}),G) \oplus \mathrm{Ext}(H_{n-1}(\mathcal{C}),G).$$

1. $X = \mathbb{RP}^m$, $G = \mathbb{Z}/2$ and $G = \mathbb{R}$
2. $X =$ Klein bottle, $G = \mathbb{Z}/2$ and $G = \mathbb{R}$
3. $X = L(a,b)$, $G = \mathbb{Z}/m$ and $G = \mathbb{R}$; here a, b, m are integers with $a > 1$, $m > 1$, and b relatively prime to a.

For #3, $L(a,b)$ is the three-dimensional lens space given by quotienting S^3, viewed as the unit sphere in $\mathbb{C}^2$, by the equivalence $(z,w) \sim (\zeta z, \zeta^b w)$ with $\zeta = e^{2\pi i/a}$. See Hatcher, example 2.43, p. 144 for the cellular chain complex in this example (note: he writes $L_a(1,b)$ instead of $L(a,b)$). (As a consistency check with #1, note that $L(2,1) = \mathbb{RP}^3$.)