

Math 103X.02, Test 4—Solutions

Professor: Lenny Ng

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1. (25 points) For (a) and (b): one of the vector fields \vec{F} is conservative, one is not. If \vec{F} is conservative, find a function $f(x, y, z)$ satisfying $\vec{F} = \vec{\nabla} f$; if not, explain why not.

- (a) (10 points) $\vec{F}(x, y, z) = (z/x, z + z/y, \ln(xy))$, with domain $\{(x, y, z) | x > 0, y > 0\}$

One calculates $\vec{\nabla} \times \vec{F} = (-1, 0, 0) \neq \vec{0}$, and hence \vec{F} is not conservative. (Note that the domain of \vec{F} is simply connected, but this isn't relevant!)

- (b) (10 points)

$$\vec{F}(x, y, z) = (2xze^y \cos(z + x^2), ze^y \sin(z + x^2), z + ze^y \cos(z + x^2) + e^y \sin(z + x^2))$$

One calculates $\vec{\nabla} \times \vec{F} = \vec{0}$; since the domain of \vec{F} , \mathbb{R}^3 , is simply connected, \vec{F} is conservative.

We now solve $\vec{F} = \vec{\nabla} f$ for f . Integrate the x component of \vec{F} with respect to x to obtain

$$f(x, y, z) = ze^y \sin(z + x^2) + f_1(y, z);$$

now differentiate f with respect to y to obtain $\partial f_1 / \partial y = 0$, and so $f_1(y, z)$ is just a function of z ; finally, differentiate f with respect to z to obtain $df_1 / dz = z$, so $f_1 = z^2/2$. The final answer is $f(x, y, z) = ze^y \sin(z + x^2) + z^2/2$ (of course, an arbitrary constant can be added to this).

- (c) (5 points) For \vec{F} from (b), evaluate $\int_C \vec{F} \cdot d\vec{s}$, where C is the helix $(\frac{\pi}{2} \sin t, t, \frac{\pi}{2} \cos t)$, $0 \leq t \leq \pi/2$.

Since $\vec{F} = \vec{\nabla} f$,

$$\int_C \vec{F} \cdot d\vec{s} = f(C(\pi/2)) - f(C(0)) = f(\pi/2, \pi/2, 0) - f(0, 0, \pi/2) = \left[-\frac{\pi}{2} - \frac{\pi^2}{8} \right].$$

2. (20 points) Let C be the piecewise smooth closed curve which traverses the boundary of the square $[0, 2] \times [-2, 0] \subset \mathbb{R}^2$ clockwise. Let

$$\vec{F}(x, y) = (5x - 3y, x + y).$$

- (a) (10 points) Evaluate $\oint_C (\vec{F} \cdot \hat{n}) ds$.

Let C' be the same curve as C but oriented counterclockwise, and let $D = [0, 2] \times [-2, 0]$. Then by the Divergence Theorem in the plane,

$$\oint_{C'} (\vec{F} \cdot \hat{n}) ds = \iint_D (\vec{\nabla} \cdot \vec{F}) dA = \iint_D 6 dA = 24.$$

Since the unit normal vectors \hat{n} for C and C' differ by a sign, we conclude that $\oint_C (\vec{F} \cdot \hat{n}) ds = -\oint_{C'} (\vec{F} \cdot \hat{n}) ds = \boxed{-24}$.

Note. For a scalar line integral like $\oint_C (\vec{F} \cdot \hat{n}) ds$, the orientation of C doesn't matter, so we don't get an extra $-$ sign from that. The $-$ sign comes from the fact that \hat{n} switches sign when the orientation of C is reversed.

(b) (10 points) Evaluate $\oint_C (\vec{F} \cdot \vec{T}) ds$.

Note that $\oint_C (\vec{F} \cdot \vec{T}) ds = \oint_C \vec{F} \cdot d\vec{s} = -\oint_{C'} \vec{F} \cdot d\vec{s}$. By Green's Theorem,

$$\oint_{C'} \vec{F} \cdot d\vec{s} = \oint_{C'} (5x - 3y) dx + (x + y) dy = \iint_D 4 dA = 16.$$

It follows that $\oint_C (\vec{F} \cdot \vec{T}) ds = \boxed{-16}$.

3. (10 points) The parametric equations

$$x = \sin^2 t, \quad y = \sin t \cos t$$

for $0 \leq t \leq \pi$ determine a simple closed curve in \mathbb{R}^2 . Find the area of the region bounded by this curve.

(Random fact that you doubtless already know: $\int \sin^2 x dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$.)

Let C be the closed curve given by the parametric equations, and let D be the region that it bounds. (It turns out that C is oriented the wrong way to calculate area, but that won't matter.) Then

$$\oint_C (-y dx + x dy) = \int_0^\pi (-\sin^2 t \cos^2 t - \sin^4 t) dt = -\int_0^\pi \sin^2 t dt = -\frac{\pi}{2}.$$

The area of D is $|\frac{1}{2} \oint_C (-y dx + x dy)| = \boxed{\frac{\pi}{4}}$.

Remark: the curve is actually just the circle $(x - 1/2)^2 + y^2 = 1/4$, which has radius $1/2$ and thus area $\pi/4$.

4. (30 points) Let S denote the surface (a frustum of a cone) given by the part of $z^2 = x^2 + y^2$ satisfying $1 \leq z \leq 2$. Orient S upwards (in the positive z direction).

(a) (10 points) Find the surface area of S .

Parametrize S by $x = s \cos t, y = s \sin t, z = s$, with $1 \leq s \leq 2$ and $0 \leq t \leq 2\pi$.

The normal vector is $\vec{N} = (-s \cos t, -s \sin t, s)$. Hence the surface area of S is

$$\iint_S dS = \int_1^2 \int_0^{2\pi} \|\vec{N}\| dt ds = \int_1^2 \int_0^{2\pi} s\sqrt{2} dt ds = \boxed{3\pi\sqrt{2}}.$$

(b) (10 points) Find the average z -coordinate of points in S .

We calculate

$$\iint_S z \, dS = \int_1^2 \int_0^{2\pi} s \|\vec{N}\| \, dt \, ds = \int_1^2 \int_0^{2\pi} s^2 \sqrt{2} \, dt \, ds = \frac{14\pi\sqrt{2}}{3}.$$

Thus $\bar{z} = (\iint_S z \, dS) / (\iint_S dS) = \boxed{\frac{14}{9}}$.

(c) (10 points) Let S' be the surface $z = 2$, $x^2 + y^2 \leq 4$; note that S' shares a circle boundary with S . Orient S' so as to be consistent with the orientation of S . Finally, let S'' be the union of S and S' . Find the flux through S'' of

$$\vec{F} = x\hat{i} + y\hat{j} + \hat{k}.$$

Note that our parametrization of S has the correct orientation; hence

$$\iint_S \vec{F} \cdot d\vec{S} = \int_1^2 \int_0^{2\pi} (s \cos t, s \sin t, 1) \cdot \vec{N} \, dt \, ds = \int_1^2 \int_0^{2\pi} (s - s^2) \, dt \, ds = -\frac{5\pi}{3}.$$

Viewed from above, the orientation on S induces a counterclockwise orientation on the boundary circle $z = 2$, $x^2 + y^2 = 4$. The compatible orientation on S' must induce the opposite orientation on this circle; it follows that S' is oriented downwards.

A parametrization \mathbf{X} for S' is given by $x = s$, $y = t$, $z = 2$ for (s, t) in the disk $s^2 + t^2 \leq 4$. The normal vector for this parametrization is \hat{k} , which points the wrong way. Nevertheless, for this parametrization, we get

$$\iint_{\mathbf{X}} \vec{F} \cdot d\vec{S} = \iint_{\mathbf{X}} \vec{F} \cdot \hat{n} \, dS = \iint_{\mathbf{X}} dS = \text{Area}(S') = 4\pi.$$

With S' oriented downwards, we thus have $\iint_{S'} \vec{F} \cdot d\vec{S} = -4\pi$. Adding everything together gives

$$\iint_{S''} \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S'} \vec{F} \cdot d\vec{S} = \boxed{-\frac{17\pi}{3}}.$$

5. (15 points) Let C be the “castle-shaped” path in \mathbb{R}^2 consisting of straight line segments connecting the following points, in order: $(0, 0)$, $(0, 2)$, $(1, 2)$, $(1, 1)$, $(2, 1)$, $(2, 2)$, $(3, 2)$, $(3, 0)$. (Thus C begins at $(0, 0)$ and ends at $(3, 0)$.) Compute

$$\int_C y \sin(xy) \, dx + (x \sin(xy) + 5x) \, dy.$$

(Hint: there's an easier solution than computing seven line integrals.)

Let C' be the straight-line path from $(0, 0)$ to $(3, 0)$. The union of C and C' bounds a rectilinear region D with area 5, and the counterclockwise-oriented boundary of D consists of C' along with the orientation reverse of C .

Write $M = y \sin(xy)$ and $N = x \sin(xy) + 5x$. By Green's Theorem,

$$\int_{C'} M dx + N dy - \int_C M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_D 5 dA = 25.$$

Parametrize C' as $\vec{x}(t) = (t, 0)$, $0 \leq t \leq 3$; then

$$\int_{C'} M dx + N dy = \int_0^3 ((0)(1) + (5t)(0)) dt = 0.$$

It follows that $\int_C M dx + N dy = \boxed{-25}$.