

Math 103X.02, Test 2—Solutions

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1. (10 points) Evaluate the following limit, or show that it does not exist:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - xy + y^3}{x^2 + y^2}$$

Solution 1. Use polar coordinates $x = r \cos \theta$, $y = r \sin \theta$. Then

$$\frac{x^3 - xy + y^3}{x^2 + y^2} = r(\cos \theta + \sin \theta) - \cos \theta \sin \theta.$$

As $r \rightarrow 0$, this expression approaches $-\cos \theta \sin \theta$, which depends on θ . Hence the limit does not exist.

Solution 2. Let $f(x, y) = \frac{x^3 - xy + y^3}{x^2 + y^2}$. Since $f(x, 0) = x$, $f(x, y) \rightarrow 0$ as (x, y) approaches $(0, 0)$ along the x axis. Since $f(x, x) = x - \frac{1}{2}$, $f(x, y) \rightarrow -\frac{1}{2}$ as (x, y) approaches $(0, 0)$ along the line $y = x$. It follows that the limit does not exist.

2. (15 points) Find an approximate value for $e^{(0.9)^2-1} \cos(0.2)$ by using the second order Taylor polynomial for $e^{x^2-1} \cos y$ at $(1, 0)$.

Let $f(x, y) = e^{x^2-1} \cos y$. One calculates $f_x = 2xe^{x^2-1} \cos y$, $f_y = -e^{x^2-1} \sin y$, $f_{xx} = (2 + 4x^2)e^{x^2-1} \cos y$, $f_{xy} = -2xe^{x^2-1} \sin y$, $f_{yy} = -e^{x^2-1} \cos y$, and so $f(1, 0) = 1$, $f_x(1, 0) = 2$, $f_y(1, 0) = 0$, $f_{xx}(1, 0) = 6$, $f_{xy}(1, 0) = 0$, $f_{yy}(1, 0) = -1$. The second order Taylor polynomial for $f(x, y)$ at $(1, 0)$ is

$$p_2(x, y) = 1 + 2(x - 1) + 3(x - 1)^2 - \frac{y^2}{2}.$$

Plugging in $(x, y) = (0.9, 0.2)$ yields $f(0.9, 0.2) \approx p_2(0.9, 0.2) = \span style="border: 1px solid black; padding: 2px;">0.81$

. For comparison, the actual value of $e^{(0.9)^2-1} \cos(0.2)$ is 0.810475...

3. (25 points) Consider the function

$$f(x, y) = xy - 2x - y + 1.$$

- (a) (5 points) Find the total differential df and the derivative matrix Df .

Since $f_x = y - 2$ and $f_y = x - 1$, $df = (y - 2)dx + (x - 1)dy$ and $Df = [y - 2 \quad x - 1]$.

- (b) (5 points) Find the maximum possible value for $D_{\vec{v}}f(5, 5)$, where \vec{v} ranges over all unit vectors in \mathbb{R}^2 .

This is $\|\vec{\nabla} f(5, 5)\| = \|(4, 3)\| = \span style="border: 1px solid black; padding: 2px;">5.$

- (c) (5 points) Find all critical point(s) of $f(x, y)$ in \mathbb{R}^2 , if any. For each, determine whether it is a local maximum, local minimum, or saddle.

Any critical point of f satisfies $0 = f_x = y - 2$ and $0 = f_y = x - 1$, so the unique critical point is $(1, 2)$. The Hessian at $(1, 2)$ is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has determinant -1 , and hence $(1, 2)$ is a saddle.

- (d) (10 points) Find the maximum and minimum values for $f(x, y)$ over the closed region in \mathbb{R}^2 bounded by the x and y axes and the line $x + y = 7$.

Since the one critical point of $f(x, y)$, which lies inside the region, is a saddle, it cannot be either an absolute minimum or an absolute maximum. Thus the absolute extrema must occur on the boundary of the triangular region. On the x axis, $f(x, 0) = -2x + 1$ has no critical points; on the y axis, $f(0, y) = -y + 1$ has no critical points; on the third side of the triangle, $f(x, 7 - x) = -x^2 + 6x - 6$ has one critical point at $x = 3$ (and $y = 4$). So the possible points where $f(x, y)$ could achieve an absolute extremum are $(3, 4)$ along with the vertices of the triangle, $(0, 0)$, $(7, 0)$, and $(0, 7)$. Since $f(3, 4) = 3$, $f(0, 0) = 1$, $f(7, 0) = -13$, and $f(0, 7) = -6$, the maximum and minimum values are 3 and -13 , respectively.

4. (25 points) Consider the surface S in \mathbb{R}^3 defined by

$$x^2 + y^2 = z + z^3.$$

- (a) (5 points) Find the equation for the tangent plane to S at the point $(1, -1, 1)$.

Write $F(x, y, z) = x^2 + y^2 - z - z^3$. The normal to S at $(1, -1, 1)$ is $\vec{\nabla}F(1, -1, 1) = (2, -2, -4)$. It follows that the tangent plane is given by $2(x - 1) - 2(y + 1) - 4(z - 1) = 0$, or $x - y - 2z = 0$.

- (b) (5 points) Explain why one can implicitly write z as a function of x and y for points (x, y, z) on S near $(1, -1, 1)$.

By the Implicit Function Theorem, z is implicitly a function of x and y near $(1, -1, 1)$ if $F_z(1, -1, 1) \neq 0$. Since $F_z(1, -1, 1) = -4$, the result follows.

For the remainder of this problem, consider z as an implicit function $z = z(x, y)$ for (x, y, z) on S near $(1, -1, 1)$.

- (c) (5 points) At $(1, -1, 1)$, calculate $\frac{\partial z}{\partial y}$.

We have

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{2y}{1 + 3z^2}.$$

At $(1, -1, 1)$, $\frac{\partial z}{\partial y} = -\frac{1}{2}$.

- (d) (10 points) At $(1, -1, 1)$, calculate $\frac{\partial^2 z}{\partial y^2}$.

Write $u = u(x, y, z) = \frac{\partial z}{\partial y} = \frac{2y}{1+3z^2}$; we can view u as a function of only x and y by using the implicit expression $z = z(x, y)$. We wish to calculate $(\frac{\partial u}{\partial y})_x$. By the Chain Rule,

$$\left(\frac{\partial u}{\partial y}\right)_x = \left(\frac{\partial u}{\partial y}\right)_{x,z} + \left(\frac{\partial u}{\partial z}\right)_{x,y} \left(\frac{\partial z}{\partial y}\right)_x = \frac{2}{1+3z^2} - \frac{12yz}{(1+3z^2)^2} \frac{2y}{1+3z^2}.$$

At $(1, -1, 1)$, we have $\frac{\partial^2 z}{\partial y^2} = (\frac{\partial u}{\partial y})_x = \boxed{\frac{1}{8}}$.

Note. An alternative solution (equivalent, but possibly easier to remember) for (c) and (d) would proceed as follows: We are given $x^2 + y^2 = z + z^3$. Take the partial derivative of both sides with respect to y to obtain

$$2y = z_y + 3z^2 z_y = (1 + 3z^2)z_y.$$

Plugging in $y = -1, z = 1$ yields $z_y = -\frac{1}{2}$ at $(1, -1, 1)$. If we take a further partial derivative with respect to y , we get

$$2 = \frac{\partial(1 + 3z^2)}{\partial y} z_y + (1 + 3z^2) z_{yy} = 6z(z_y)^2 + (1 + 3z^2) z_{yy}.$$

Plugging in $y = -1, z = 1$ now yields $z_{yy} = \frac{1}{8}$ at $(1, -1, 1)$.

5. (25 points) Let C denote the intersection of the surfaces $x + y + z = 12$ and $y = x^2 + z^2$.

(a) (15 points) Find the points on C with the maximal and minimal y coordinates. (You may assume that an absolute maximum and minimum exist.)

We wish to find the extrema for the function y given the constraints $x + y + z = 12$ and $y - x^2 - z^2 = 0$. Using Lagrange multipliers, we find that at an extremum, there exist scalars λ and μ , not both 0, such that $(0, 1, 0) = \lambda(1, 1, 1) + \mu(-2x, 1, -2z)$. This implies that $\lambda = 2\mu x = 2\mu z$ (and $1 = \lambda + \mu$, but we won't need this relation). It follows that either $\mu = 0$ or $x = z$. If $\mu = 0$, then $\lambda = 2\mu x = 0$, which is not allowed. Thus $x = z$.

Substituting into the constraints $x + y + z = 12$ and $y - x^2 - z^2 = 0$, we find that $y = 12 - 2x$ and $y = 2x^2$, so $2x^2 + 2x - 12 = 0$. The roots of this equation are $x = -3$ and $x = 2$, leading to the points $(x, y, z) = (-3, 18, -3)$ and $(2, 8, 2)$.

It follows that $\boxed{(-3, 18, -3) \text{ and } (2, 8, 2)}$ have the maximal and minimal y coordinates, respectively, on C .

(b) (10 points) A particle is moving along C , with its position at time t given by $(x(t), y(t), z(t))$, in such a way that $x(0) = -4, y(0) = 16$, and $y'(0) = 1$. Let

$$f(t) = (x(t))^2 + (y(t))^2 + (z(t))^2$$

be the square of the distance between the particle and the origin, at time t . Find $f'(0)$.

Solution 1. Write $\vec{x}(t) = (x(t), y(t), z(t))$. Note that since $(x(0), y(0), z(0))$ lies on the plane $x + y + z = 12$, we have $z(0) = 0$.

We next determine $\vec{x}'(0) = (x'(0), y'(0), z'(0))$. This vector is tangent to C . Normal vectors to the surfaces $x + y + z = 12$ and $y - x^2 - z^2 = 0$ at $(-4, 16, 0)$ are given by the respective gradients: $(1, 1, 1)$ and $(-2x, 1, -2z) = (8, 1, 0)$, respectively. The cross product of these two vectors, $(-1, 8, -7)$, is perpendicular to both normals and thus tangent to C . It follows that $(x'(0), y'(0), z'(0))$ is parallel to $(-1, 8, -7)$. Since we are given $y'(0) = 1$, we obtain $\vec{x}'(0) = (-1/8, 1, -7/8)$.

Write $f(x, y, z) = x^2 + y^2 + z^2$, so that $f(t) = f(x(t), y(t), z(t))$. Then by the Chain Rule, $f'(0)$ is the directional derivative of f at $\vec{x}(0)$ in the direction of $\vec{x}'(0)$:

$$f'(0) = \vec{\nabla} f(\vec{x}(0)) \cdot \vec{x}'(0) = (-8, 32, 0) \cdot (-1/8, 1, -7/8) = \boxed{33}.$$

Solution 2. As in Solution 1, write $\vec{x}(t) = (x(t), y(t), z(t))$; then $\vec{x}(0) = (-4, 16, 0)$. We are given

$$(0, 1, 0) \cdot \vec{x}'(0) = y'(0) = 1.$$

In addition, since the path $\vec{x}(t)$ lies entirely in the surface $y - x^2 - z^2 = 0$, it is perpendicular at $\vec{x}(0)$ to the normal vector to $y - x^2 - z^2 = 0$, given by the gradient $(-2x, 1, -2z) = (8, 1, 0)$. It follows that

$$(8, 1, 0) \cdot \vec{x}'(0) = 0.$$

By the Chain Rule, $f'(0) = \vec{\nabla} f(\vec{x}(0)) \cdot \vec{x}'(0) = (-8, 32, 0) \cdot \vec{x}'(0)$. But $(-8, 32, 0) = 33(0, 1, 0) - (8, 1, 0)$; thus

$$f'(0) = (-8, 32, 0) \cdot \vec{x}'(0) = 33(0, 1, 0) \cdot \vec{x}'(0) - (8, 1, 0) \cdot \vec{x}'(0) = 33.$$