

## Math 103X.02—Review problems for Test 3

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Test 3, on November 17, will cover Section 1.7 (except standard bases for cylindrical and spherical coordinates), Section 3.3, Section 3.4 (except “Other coordinate formulations”), and Chapter 5 (except moments of inertia from Section 5.6); in other words, the material covered by HW 6 through 8. There will be an optional review session on Wednesday, November 15, at 4pm, room to be announced.

Some of these problems are problems from past tests by Professor Witelski.

1. Review the homework problems and make sure you understand how to solve each of them.
2. Prove that curl fields (i.e., vector fields of the form  $\vec{\nabla} \times \vec{F}$ ) are incompressible.
3. Find a flow line  $\vec{x}(t)$  for the vector field  $\vec{F}(x, y, z) = (y, -x, z)$  with  $\vec{x}(0) = (0, 1, 1)$ .
4. Calculate  $\int_0^1 \int_{\sin^{-1} y}^{\pi y/2} x^2 dx dy$ .
5. For the domain  $D$  bounded by the curves  $y = x$ ,  $y = 2x$ ,  $y = 3x^2$ , and  $y = 4x^2$ , evaluate  $\iint_D 72e^{(y/x)^3} dA$ .
6. Calculate the volume of the region inside the sphere  $x^2 + y^2 + z^2 = 4$ , between the cones  $z = \sqrt{x^2 + y^2}$  and  $z = \sqrt{3(x^2 + y^2)}$ , and above the plane  $z = 1$ .
7. (Colley §5.5 # 31) Find  $\iiint_W (2 + x^2 + y^2) dV$ , where  $W$  is the region inside the sphere  $x^2 + y^2 + z^2 = 25$  and above the plane  $z = 3$ .
8. Let  $D$  be the region in the  $xy$  plane inside  $x^2 + (y - 1)^2 = 1$ , above  $y = x$ , and to the right of  $x = 0$ .
  - (a) Express the general double integral  $\iint_D f(x, y) dA$  in terms of iterated integrals in four different ways, so that the integrand contains the expression:  $dy dx$ ;  $dx dy$ ;  $dr d\theta$ ;  $d\theta dr$ .
  - (b) Express  $\iint_D f(x, y) dA$  in terms of the change of variables  $x = u+v$ ,  $y = 1+u-v$ .
  - (c) Calculate the area and the centroid of  $D$ .
9. Let  $T$  be the torus obtained by revolving the disk  $(x - 2)^2 + z^2 \leq 1$  in the  $xz$  plane about the  $z$  axis.
  - (a) Find the volume of  $T$  using cylindrical coordinates.
  - (b) Find the volume of  $T$  using spherical coordinates.

10. In this problem, you will calculate the 4-dimensional volume (“hypervolume”) of a 4-dimensional ball of radius  $r$ . Here the  $n$ -dimensional ball of radius  $r$  is  $B_n(r) = \{(x_1, \dots, x_n) | x_1^2 + \dots + x_n^2 \leq r^2\}$ .

(a) If we slice  $B_n(r)$  by the “hyperplane”  $x_n = c$ , we obtain an  $(n - 1)$ -dimensional ball. Find the radius of this ball in terms of  $r$  and  $c$ . Conclude that

$$\text{vol}(B_n(r)) = \int_{-r}^r \text{vol}(B_{n-1}(\sqrt{r^2 - x_n^2})) dx_n.$$

(b) Show that  $\text{vol}(B_4(r)) = \frac{\pi^2}{2} r^4$ .

(c) We can find similar formulas for  $\text{vol}(B_n(r))$  for arbitrary  $r$ , as follows. For scaling reasons, if we fix  $n$ , the volume of  $B_n(r)$  should be proportional to  $r^n$ :  $\text{vol}(B_n(r)) = c_n r^n$  for some constant  $c_n$  independent of  $r$  (but depending on  $n$ ). Show that

$$c_n = c_{n-1} \int_{-\pi/2}^{\pi/2} (\cos \theta)^n d\theta.$$

There’s a relatively simple form for  $c_n$ , but I’m too lazy right now to figure out what it is; if you’re interested, you can find it online by Googling “volume  $n$ -dimensional ball”.

Selected answers:

3.  $\vec{x}(t) = (\sin t, \cos t, e^t)$ .

4. Change order of integration to get  $\pi - 2 - \frac{\pi^3}{32}$ .

5. Use change of variables  $x = u/v, y = u^2/v$  to get  $\frac{7}{12}(e^8 - e)$ .

6.  $\frac{2\pi(4\sqrt{2}-5)}{3}$ . It’s a good exercise to do this both in spherical and in cylindrical coordinates.

7.  $656\pi/5$ . This one is actually easier in cylindrical than in spherical coordinates.

8. (a)  $\int_0^1 \int_x^{1+\sqrt{1-x^2}} f(x, y) dy dx; \int_0^1 \int_0^y f(x, y) dx dy + \int_1^2 \int_0^{\sqrt{1-(y-1)^2}} f(x, y) dx dy;$

$\int_{\pi/4}^{\pi/2} \int_0^{2 \sin \theta} f(r \cos \theta, r \sin \theta) r dr d\theta;$

$\int_0^{\sqrt{2}} \int_{\pi/4}^{\pi/2} f(r \cos \theta, r \sin \theta) r dr d\theta + \int_{\sqrt{2}}^2 \int_{\sin^{-1}(r/2)}^{\pi/2} f(r \cos \theta, r \sin \theta) r dr d\theta;$

(b)  $2 \int_{-1/2}^{1/2} \int_{-v}^{\sqrt{(1/2)-v^2}} f(u+v, 1+u-v) du dv;$  (c) Area  $(2 + \pi)/4$ , centroid  $(\frac{2}{2+\pi}, \frac{8+3\pi}{6+3\pi})$ .

9. (a)  $\int_{-1}^1 \int_0^{2\pi} \int_{2-\sqrt{1-z^2}}^{2+\sqrt{1-z^2}} r dr d\theta dz = 4\pi^2;$  (b) It’s probably easiest if you integrate  $\rho$  before  $\varphi$  to get

$$\int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_{2 \sin \varphi - \sqrt{16 \sin^2 \varphi - 3}}^{2 \sin \varphi + \sqrt{16 \sin^2 \varphi - 3}} \rho^2 \sin \varphi d\rho d\varphi d\theta = 4\pi^2.$$

10. (b)  $\text{vol}(B_4(r)) = \frac{4\pi}{3} \int_{-r}^r (r^2 - x_4^2)^{3/2} dx_4 = \pi^2 r^4 / 2$ , where the integral is done using the substitution  $x_4 = r \sin \theta$ , and  $\int_{-\pi/2}^{\pi/2} \cos^4 \theta d\theta = 3\pi/8$  (integrate by parts).