# Math 103X.02—Review problems for Test 2

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Test 2, on October 23, will cover Chapter 2 (no Newton's Method) and Sections 4.1–4.3 (excepting the "optional" parts and Taylor polynomials of order $3$ or more); in other words, the material covered by HW 3 through 5. There will be an optional review session on Friday, October 20, at 4 pm in **Physics 146**.

1. Review the homework problems and make sure you understand how to solve each of them.

2. For what values of a constant $c$ does $\frac{\cos(x^2+y^2)+c}{x^2+y^2+c}$ have a limit as $(x, y) \to (0, 0)$? What about $\frac{\sin(x^2+y^2)+c}{x^2+y^2+c}$?

3. Let $f(x, y) = x^2 + y^2 - 3x + 2y + 1$.
   (a) Find the derivative matrix for $f$ at $(0, 0)$.
   (b) Use the definition of differentiability to prove that $f$ is differentiable at $(0, 0)$.

4. (Colley §2.5 # 17) Calculate the derivative matrix $D(f \circ g)$ in terms of $s$ and $t$, where $f(x, y) = (xy - y/x, x/y + y^3)$ and $g(s, t) = (s/t, s^2t)$.

5. (a) Find an equation for the tangent plane to the surface $xy + z + 3xz^5 = 4$ at the point $(1, 0, 1)$.
   (b) Show that $xy + z + 3xz^5 = 4$ can be implicitly solved for $z$ in terms of $x$ and $y$ near $(1, 0, 1)$. Find $\partial z/\partial x$ and $\partial z/\partial y$ at $(1, 0)$.

6. (a) Consider the surface $z = x^3 + y^3 - 6xy + 4$. At the point $(1, 2, 1)$, what is the unit vector in the direction of steepest ascent of $z$? If one walks in the $xy$ plane with speed $3$ in this direction with speed $3$, what is the rate of change of $z$ at the moment that one passes through $(1, 2)$?
   (b) Find a unit vector tangent to the level curve $x^3 + y^3 - 6xy = -3$ at the point $(1, 2)$.

7. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is some function, and $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by $g(x, y) = f(f(x, y), x)$. Write $g_x(x, y)$ and $g_y(x, y)$ in terms of partials of $f$. Verify your answer for $f(x, y) = x + y^2$.

8. (From an old test in Professor Witelski's section) Consider the function $f(x, y) = 8x^3 - 4x - 4xy + 2y + y^2$.
   (a) This function has two critical points. Find their coordinates and use the second derivative test to determine if each is a saddle, max, or min.

(b) Write the second order Taylor polynomial of $f(x, y)$ centered at the point $(1, 2)$.

(c) Find the $(x, y)$ points whose $f$-values need to be compared to determine the absolute max and min of $f(x, y)$ in the closed region bounded by the curves $y = x^2 - 1$ and $y = 2x + 7$. You do not need to calculate the $f$-values.

9. Let $S_1$ denote the surface $z^2 = x^2 + y^2$ and $S_2$ denote the plane $4x + 3y - 7z = 15$. Let $C$ denote the curve of intersection of $S_1$ and $S_2$. Finally, let $f(x, y, z)$ be the function $x^2 + y^2 + z^2$.

   (a) For an arbitrary point $(x, y, z)$ on $C$, find a tangent vector $\vec{v}$ to $C$ at $(x, y, z)$.

   (b) Find the absolute value of the directional derivative of the function $f(x, y, z)$ at the point $(-5, 0, -5) \in C$ in either direction tangent to $C$. Remember that a directional derivative only makes sense for a unit vector.

   (c) Consider the function $f(x, y, z)$ along $C$. Suppose that $(x_0, y_0, z_0)$ is a point on $C$ for which $f$ is either maximized or minimized. Write an equation relating $x_0$, $y_0$, and $z_0$ involving the gradient of $f$ and the answer from (a). Use this to find the maximum and minimum distance between the origin and points on $C$. Why do these extrema necessarily exist?

   (d) Check your answer to (c) using Lagrange multipliers.

Selected answers:

2. $c \neq 0$ and any $c$, respectively.

3. (a) $[-3 \;\; 2]$.

4. $\begin{bmatrix} 3s^2 - t^2 & -2st \\ 6s^5t^3 - s^{-2}t^{-2} & 3s^6t^2 - 2s^{-1}t^{-3} \end{bmatrix}$.

5. (a) $3x + y + 16z = 19$, (b) $-3/16$ and $-1/16$.

6. (a) $(-3/\sqrt{13}, 2/\sqrt{13}$ and $9\sqrt{13}$, (b) $\pm(2/\sqrt{13}, 3/\sqrt{13})$.

7. $g_x(x, y) = f_x(f(x, y), x)f_x(x, y) + f_y(f(x, y), x)$ and $g_y(x, y) = f_x(f(x, y), x)f_y(x, y)$.

8. (a) saddle at $(0, -1)$ and local minimum at $(1/3, -1/3)$, (b) $4 + 12(x - 1) + 2(y - 2) + 24(x - 1)^2 - 4(x - 1)(y - 2) + (y - 2)^2$, (c) $(0, -1)$, $(1/3, -1/3)$, $(-2, 3)$, $(4, 15)$, $(0, 7)$, and $(1/3, 23/3)$.

9. (a) $(-7y + 3z, 7x - 4z, 3x - 4y)$, (b) $20/\sqrt{3}$, (c) $\vec{\nabla} f(x_0, y_0, z_0) \cdot (-7y_0 + 3z_0, 7x_0 - 4z_0, 3x_0 - 4y_0) = 0$; the maximum distance is $15/\sqrt{2}$ and the minimum distance is $5/(2\sqrt{2})$.