Math 103X.02—Line and Surface Integrals

Instructor: Lenny Ng

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Line integrals

Path $\vec{x} : [a, b] \to \mathbb{R}^n$, scalar function $f : \mathbb{R}^n \to \mathbb{R}$, vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$.

- Scalar line integral $\displaystyle\int_{\vec{x}} f\, ds = \int_a^b f(\vec{x}(t))\|\vec{x}'(t)\|\, dt$. Special case: $\int_{\vec{x}} ds = \text{Length}(\vec{x})$.
- Vector line integral $\displaystyle\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\, dt$.
- Also write $\displaystyle\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_{\vec{x}} F_1\, dx + F_2\, dy + F_3\, dz$ if $\vec{F} = (F_1, F_2, F_3)$.
- Relation between scalar and vector line integrals: $\displaystyle\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_{\vec{x}} (\vec{F} \cdot \vec{T})\, ds$, where $\vec{T} = \vec{x}'/\|\vec{x}\|$ is the unit tangent vector to the curve.
- Scalar line integrals are independent of parametrization; vector line integrals depend on an orientation of the path. (Reversing the orientation negates the vector line integral.)
- If $\vec{x}$ represents a one-dimensional object in $\mathbb{R}^3$ with density function $\delta(x, y, z)$, then the mass of the object is $\int_{\vec{x}} \delta\, ds$ and its center of mass is $(\overline{x}, \overline{y}, \overline{z})$, where $\overline{x} = \frac{\int_{\vec{x}} x\delta\, ds}{\int_{\vec{x}} \delta\, ds}$, etc.
- If $\vec{F}$ is a force field, then $\int_{\vec{x}} \vec{F} \cdot d\vec{s} =$ work done by $\vec{F}$ on a particle moving along the path $\vec{x}$.
- If $\vec{F}$ is a velocity field, then $\int_{\vec{x}} (\vec{F} \cdot \hat{n})\, ds =$ flux of $\vec{F}$ across the curve, where $\hat{n}$ is the outward-pointing unit normal. If C is a closed curve, then $\int_{\vec{x}} \vec{F} \cdot d\vec{s} =$ circulation of F around C.

Surface integrals

Parametrized surface $\mathbf{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, underlying surface S.

- Tangent vectors $\vec{T}_s = \partial\mathbf{X}/\partial s$, $\vec{T}_t = \partial\mathbf{X}/\partial t$, normal vector $\vec{N} = \vec{T}_s \times \vec{T}_t$. A surface is smooth if $\vec{N} \neq \vec{0}$.
- Scalar surface integral $\displaystyle\iint_S f\, dS = \iint_D f(\mathbf{X}(s, t))\|\vec{N}(s, t)\|\, ds\, dt$. Special case: $\iint_S dS =$ Surface area(S).

- Vector surface integral $\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_D \vec{F}(s,t) \cdot \vec{N}(s,t)\, ds\, dt$.

- Relation between scalar and vector surface integrals: $\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_S (\vec{F} \cdot \hat{n})\, dS$, where $\hat{n} = \vec{N}/\|\vec{N}\|$ is the unit normal vector to S.

- Scalar surface integrals are independent of parametrization; vector surface integrals depend on an orientation of the surface. (Reversing the orientation negates the vector surface integral.) Not all surfaces have a continuously varying orientation.

- If S represents a two-dimensional object with density function $\delta(x,y,z)$, then the mass of S is $\iint_S \delta\, dS$ and its center of mass is $(\overline{x}, \overline{y}, \overline{z})$, where $\overline{x} = \frac{\iint_S x\delta\, dS}{\int_S \delta\, dS}$, etc.

- $\iint_S \vec{F} \cdot d\vec{S} =$ flux of $\vec{F}$ across S.

Green's Theorem

- Green's Theorem: $D \subset \mathbb{R}^2$ region, ∂D oriented leftwise. Then

$$\oint_{\partial D} M\, dx + N\, dy = \iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy.$$

- Vector reformulation of Green's Theorem (and special case of Stokes' Theorem): $\oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint\limits_D (\vec{\nabla} \times \vec{F}) \cdot \hat{k}\, dA$.

- Divergence Theorem in the plane: $\oint_{\partial D} (\vec{F} \cdot \hat{n})\, ds = \iint\limits_D (\vec{\nabla} \cdot \vec{F})\, dA$.

- Special case of Green's Theorem: Area$(D) = \frac{1}{2} \oint_{\partial D} (-y\, dx + x\, dy)$.

Conservative vector fields

The following are equivalent:

- $\vec{F}$ is a conservative vector field.
- $\vec{F} = \vec{\nabla} f$ for some scalar function f.
- $\vec{F}$ has path-independent line integrals.
- $\oint_C \vec{F} \cdot d\vec{s} = 0$ for all simple closed curves C in the domain of $\vec{F}$.

If the domain of $\vec{F}$ is *simply connected*, then these conditions are also equivalent to:

- $\vec{\nabla} \times \vec{F} = \vec{0}$.

The following material will be covered on the final but not on Test 4.

Stokes' Theorem

- Stokes' Theorem: S oriented, piecewise smooth surface, orientation on ∂S induced from orientation on S, $\vec{F}$ vector field. Then
$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}.$$
- Use to convert from surface integral to line integral, from line integral to surface integral, or from one surface to another surface with the same boundary.
- Curl = circulation per unit area: C_r = circle of radius r centered at a point P in the plane normal to $\hat{n}$, oriented by right hand rule; then $(\vec{\nabla} \times \vec{F})(P) \cdot \hat{n} = \lim_{r \to 0^+} \frac{1}{\pi r^2} \oint_{C_r} \vec{F} \cdot d\vec{s}$.

Divergence Theorem (Gauss's Theorem)

- Divergence Theorem: R solid region in $\mathbb{R}^3$, ∂R oriented away from R, $\vec{F}$ vector field. Then
$$\oiint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R (\vec{\nabla} \cdot \vec{F})\, dV.$$
- Use to convert from surface integral to triple integral, from triple integral to surface integral, or to convert from one surface to another surface with the same boundary.
- Divergence = flux per unit volume: S_r = spherical surface of radius r centered at a point P, oriented outwards; then $(\vec{\nabla} \cdot \vec{F})(P) = \lim_{r \to 0^+} \oiint_{S_r} \vec{F} \cdot d\vec{S}$.