Math 103X.02 Homework 8 Answers & Solutions

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§5.5, 10: $1/3$; 14: π; 26. $2\pi(b^2 - a^2)$.

§5.6, 8: Let $\min(x, y)$ denote the minimum of x and y. Then the average waiting time is

$$\frac{1}{36}\int_0^6\int_0^6 \min(x,y)\,dx\,dy = \frac{1}{36}\left(\int_0^6\int_0^y x\,dy\,dx + \int_0^6\int_0^x y\,dy\,dx\right) = 2.$$

§5.6, 15: there's a typo in the answer in the back of the book; $\overline{y}$ should be $15/(3\sqrt{3} + 4\pi)$.

§5.6, 32, 33, 34: see attached solution.

§3.3, 24: (a) $\vec{F} = \vec{\nabla} f$ where $f(x, y, z) = x^2 + y^2 - 3z$; (b) the equipotential surfaces are of the form $z = x^2/3 + y^2/3 + c$, paraboloids pointing upwards and rotationally symmetric about the z axis.

§3.4, 17:

$$\vec{\nabla} r^n = \vec{\nabla}(x^2 + y^2 + z^2)^{n/2} = (n/2)(x^2 + y^2 + z^2)^{(n-2)/2}(2x, 2y, 2z) = nr^{n-2}\vec{r}.$$

§3.4, 28: (a) $\vec{\nabla} \cdot \vec{\nabla} = (\partial/\partial x, \partial/\partial y, \partial/\partial z) \cdot (\partial/\partial x, \partial/\partial y, \partial/\partial z) = (\partial^2/\partial x^2, \partial^2/\partial y^2, \partial^2/\partial z^2) = \nabla^2$; (b) just follows from calculating the second order partial derivatives of fg; (c) is straightforward once you write it out in components.

Extra problem:

1. (a) The conditions $T(0, 0) = (a_1, a_2)$, $T(1, 0) = (b_1, b_2)$, $T(0, 1) = (c_1, c_2)$ uniquely determine $a = b_1 - a_1$, $b = c_1 - a_1$, $c = a_1$, $d = b_2 - a_2$, $e = c_2 - a_2$, $f = a_2$.

   (b) $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & b \\ d & e \end{vmatrix} = (b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2)$ is independent of u, v.

   (c) Write C for the constant $\frac{\partial(x,y)}{\partial(u,v)}$ from (b). The average x coordinate for a point in D is

$$\begin{aligned}\overline{x} &= \frac{\iint_D x\,dx\,dy}{\iint_D dx\,dy} = \frac{\iint_{D^*}((b_1 - a_1)u + (c_1 - a_1)v + a_1)C\,du\,dv}{\iint_{D^*} C\,du\,dv} \\ &= \frac{\int_0^1\int_0^{1-v}((b_1 - a_1)u + (c_1 - a_1)v + a_1)\,du\,dv}{Area(D^*)} = \frac{a_1 + b_1 + c_1}{3}.\end{aligned}$$

   A similar computation shows that $\overline{y} = \frac{a_2+b_2+c_2}{3}$.