

Math 103X.02 Homework 4 Answers & Solutions

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§2.5, 8: (a) $41/686 \approx 0.0598$ points per month; (b) from the solution book: "The rate we found in part (a) is greater than the typical rate by about 49%. I'd monitor the situation monthly so that it doesn't persist for too long, but I wouldn't be very concerned, since the current BMI is roughly 16.84, which is quite low."

§2.5, 11: We are given $w = f(u)$ and $u = xy/(x^2 + y^2)$. By the Chain Rule,

$$\frac{\partial w}{\partial x} = \frac{dw}{du} \frac{\partial u}{\partial x} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} \frac{dw}{du}$$

and

$$\frac{\partial w}{\partial y} = \frac{dw}{du} \frac{\partial u}{\partial y} = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \frac{dw}{du}.$$

Thus

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} = \frac{(y^2 - x^2)(xy - xy)}{(x^2 + y^2)^2} \frac{dw}{du} = 0.$$

§2.5, 26: (a) Write $y = y(x)$; then $F(x, y) = F(x, y(x))$ is a function of x alone. When viewed as such, we have $dF/dx = 0$ since $F(x)$ is constant. On the other hand, by the Chain Rule,

$$\frac{dF}{dx} = \left(\frac{\partial F}{\partial x} \right)_y + \left(\frac{\partial F}{\partial y} \right)_x \frac{dy}{dx}$$

where, e.g., $\left(\frac{\partial F}{\partial x} \right)_y$ denotes the partial derivative of $F(x, y)$ with respect to x while holding y fixed. The result follows. (b) $\frac{dy}{dx} = \frac{3x^2}{2y}$.

§2.5, 28: (a) Here we have z implicitly given as $z = z(x, y)$, and $F(x, y, z(x, y))$ can be viewed as a function of just x and y . As such, we have $(\partial F/\partial x)_y = (\partial F/\partial y)_x = 0$ since $F(x, y, z(x, y)) = 0$. On the other hand, by the Chain Rule,

$$\left(\frac{\partial F}{\partial x} \right)_y = \left(\frac{\partial F}{\partial x} \right)_{y,z} + \left(\frac{\partial F}{\partial z} \right)_{x,y} \left(\frac{\partial z}{\partial x} \right)_y,$$

and so $(\partial z/\partial x)_y = -(\partial F/\partial x)_{y,z}/(\partial F/\partial z)_{x,y}$. A completely analogous calculation shows that $(\partial z/\partial y)_x = -(\partial F/\partial y)_{x,z}/(\partial F/\partial z)_{x,y}$. (b) $\partial z/\partial x = (\partial x/\partial y)_z = -z/x = -2/(x^2y)$ and $\partial z/\partial y = -z/y = -2/(xy^2)$.

§2.5, 35: Use exercise 28: $\left(\frac{\partial z}{\partial x} \right)_y = -\frac{F_x}{F_z}$. Nothing changes if we replace x by y , y by z , and z by x , so it is also true that $\left(\frac{\partial x}{\partial y} \right)_z = -\frac{F_y}{F_x}$; on the other hand, if instead we replace x by z ,

y by x , and z by y , then we obtain $(\frac{\partial y}{\partial z})_x = -\frac{F_z}{F_y}$. It follows that

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = \left(-\frac{F_x}{F_z}\right) \left(-\frac{F_y}{F_x}\right) \left(-\frac{F_z}{F_y}\right) = -1.$$

§2.6, 12: (a) $(3, -2)$ or the unit vector $(3, -2)/\sqrt{13}$; (b) $(-3, 2)$ or the unit vector $(-3, 2)/\sqrt{13}$; (c) $\pm(2, 3)$ or unit vectors $\pm(2, 3)/\sqrt{13}$.

§2.8, 30: (a) If $f(x, y) = x^y = e^{y \ln x}$, then $\partial f/\partial x = ye^{y \ln x}/x = yx^{y-1}$ and $\partial f/\partial y = (\ln x)e^{y \ln x} = x^y \ln x$. Now set $x = x(u) = u$ and $y = y(u) = u$:

$$\frac{df}{du} = \frac{\partial f}{\partial x} \frac{dx}{du} + \frac{\partial f}{\partial y} \frac{dy}{du} = u^u(\ln u + 1).$$

(b) Set $x = x(t) = \sin t$ and $y = y(t) = \cos t$ in the Chain Rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = (\cos t)^2(\sin t)^{\cos t-1} - (\ln \sin t)(\sin t)^{\cos t+1}.$$

Extra problems:

- (a) Let Π_1 and Π_2 denote the tangent planes to S_1 and S_2 , respectively, at the point (x, y, z) ; then $\vec{\nabla}F_1(x, y, z)$ is normal to Π_1 and $\vec{\nabla}F_2(x, y, z)$ is normal to Π_2 . Since the curve C is the intersection of S_1 and S_2 , the tangent line to C at (x, y, z) is the intersection of the tangent planes Π_1 and Π_2 . A vector which is normal to both normals must lie in both tangent planes, and hence must lie in their intersection line. The cross product $\vec{\nabla}F_1(x, y, z) \times \vec{\nabla}F_2(x, y, z)$ thus points in the direction of the tangent line to C .

(b) The line points in the direction of $(16, -10, -1) \times (1, 2, -4) = (42, 63, 42)$ so the line is given by $(x, y, z) = (2, 1, -1) + t(42, 63, 42)$. (Since $(42, 63, 42) = 21(2, 3, 2)$), this can also be written less grossly as $(x, y, z) = (2, 1, -1) + t(2, 3, 2)$.

(c) The angle between S_1 and S_2 at a point of intersection is the angle between the tangent planes to S_1 and S_2 at that point, which is the angle between the normal vectors to (the tangent planes to) S_1 and S_2 at that point. Here the normal vectors are $(16, -10, -1)$ and $(1, 2, -4)$, which are indeed orthogonal.
- (a) $(-1, 1)$ and $(-5, -1)$.

(b) The quadratic formula for y gives

$$y = \frac{-3x - 5 \pm \sqrt{5x^2 + 30x + 25}}{2x} = \frac{-3x - 5 \pm \sqrt{5(x+1)(x+5)}}{2x}.$$

A point (x, y) satisfying the above equation determines a unique choice of sign in the \pm unless the term after the \pm is 0. This latter case happens when $x = -1$ or $x = -5$, leading to the points given in (a).