

# Math 103X.02 Homework 1 Answers & Solutions

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Here are the answers to the problems which do not have answers in the back of the book, and more explanation where appropriate.

§1.1: 16.  $(14, 2, 1)$ ; 24. (a) 4 miles per hour, (b) 1/10 hour, or 6 minutes, (c) the plane is 862 feet above the top of the skyscraper.

§1.2: 10.  $(\pi, -1, 0)$ ; 16.  $x = 2 + t, y = 1 - 2t, z = 2 + 3t$  (other answers are possible); 22.  $x = 2 + 5t, y = 3 - 2t, z = -1 + 4t$  (other answers are possible but this is almost certainly the one you'd come up with); 24. They are the same. Both point in the direction  $(2, 3, -6)$  and both contain the point  $(-5, 2, 1)$ .

§1.3: 14.  $\frac{1}{\sqrt{5}}(1, 0, -2)$ ; 22. If  $\vec{a} = (a_1, a_2, a_3)$ , then  $\cos \alpha = a_1/\sqrt{a_1^2 + a_2^2 + a_3^2}$ ,  $\cos \beta = a_2/\sqrt{a_1^2 + a_2^2 + a_3^2}$ , and  $\cos \gamma = a_3/\sqrt{a_1^2 + a_2^2 + a_3^2}$ .

§1.3, 24: It suffices to show that  $M_1\vec{M}_2 = M_3\vec{M}_4$ , since then  $M_1M_2M_3M_4$  has opposite sides which are equal and parallel. Now  $\vec{M}_1 = (\vec{A} + \vec{B})/2$  since  $\vec{M}_1 - \vec{A} = A\vec{M}_1 = \frac{1}{2}\vec{AB} = \frac{1}{2}\vec{B} - \frac{1}{2}\vec{A}$ . Similarly  $\vec{M}_2 = (\vec{B} + \vec{C})/2$ ,  $\vec{M}_3 = (\vec{C} + \vec{D})/2$ , and  $\vec{M}_4 = (\vec{A} + \vec{D})/2$ . Thus  $M_1\vec{M}_2 = \vec{M}_2 - \vec{M}_1 = (\vec{A} + \vec{C})/2$  while  $M_3\vec{M}_4 = \vec{M}_4 - \vec{M}_3 = (\vec{A} + \vec{C})/2$ , and hence  $M_1\vec{M}_2 = M_3\vec{M}_4$ , as desired.

Note that this result holds for arbitrary quadrilaterals in space, including skew ones (those which don't lie in a plane) and planar quadrilaterals that aren't convex!

§1.3, 28: (a)  $(\|\vec{b}\|\vec{a} + \|\vec{a}\|\vec{b}) \cdot (\|\vec{b}\|\vec{a} - \|\vec{a}\|\vec{b}) = \|\vec{b}\|^2(\vec{a} \cdot \vec{a}) - \|a\|^2(\vec{b} \cdot \vec{b}) = 0$ ; (b) Let  $\theta_1$  be the angle between  $\vec{a}$  and  $\|\vec{b}\|\vec{a} + \|\vec{a}\|\vec{b}$ , and let  $\theta_2$  be the angle between  $\vec{b}$  and  $\|\vec{b}\|\vec{a} + \|\vec{a}\|\vec{b}$ . Then

$$\left\|(\|\vec{b}\|\vec{a} + \|\vec{a}\|\vec{b})\right\| \cos^{-1} \theta_1 = \frac{\vec{a} \cdot (\|\vec{b}\|\vec{a} + \|\vec{a}\|\vec{b})}{\|\vec{a}\|} = \frac{\|\vec{a}\|^2\|\vec{b}\| + \|\vec{a}\|(\vec{a} \cdot \vec{b})}{\|\vec{a}\|} = \|\vec{a}\|\|\vec{b}\| + \vec{a} \cdot \vec{b}$$

and a similar calculation shows that  $\left\|(\|\vec{b}\|\vec{a} + \|\vec{a}\|\vec{b})\right\| \cos^{-1} \theta_2 = \|\vec{a}\|\|\vec{b}\| + \vec{a} \cdot \vec{b}$ . It follows that  $\theta_1 = \theta_2$ .

§1.4: 6.  $-3\vec{i} - 2\vec{j} + 5\vec{k}$ ; 18. 3; 26. (a) vector, (b) nonsense, (c) nonsense, (d) scalar, (e) nonsense, (f) vector, (g) scalar, (h) vector.

§1.5: 8.  $7x + y - 2z = 7$ ; 24.  $292/\sqrt{2141}$ .

§1.9, 19(a): The distance is the length of the vector  $\text{proj}_{\vec{n}}\vec{P}_1\vec{P}_0$ , where  $\vec{n}$  is the normal vector  $(A, B, C)$ . Since  $P_1$  lies on  $\Pi$ ,  $\vec{n} \cdot P_1 = D$ , and so  $\vec{n} \cdot P_1\vec{P}_0 = Ax_0 + By_0 + Cz_0 - D$ . It follows

that

$$\|\text{proj}_{\vec{n}} P_1 \vec{P}_0\| = \frac{|\vec{n} \cdot P_1 \vec{P}_0|}{\|\vec{n}\|} = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Extra problems:

- (a)  $\vec{a} = \pm \frac{\vec{b} \times \vec{c}}{\|\vec{b} \times \vec{c}\|}$ .  
(b)  $(2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6})$  or  $(-2/\sqrt{6}, -1/\sqrt{6}, -1/\sqrt{6})$ .
- This is a straightforward, if messy, computation. If  $\vec{a} = (a_1, a_2, a_3)$  and  $\vec{b} = (b_1, b_2, b_3)$ , then

$$\begin{aligned}\|\vec{a} \times \vec{b}\|^2 &= (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - b_1 a_2)^2 \\ &= a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 \\ &\quad - 2a_1 a_2 b_1 b_2 - 2a_1 a_3 b_1 b_3 - 2a_2 a_3 b_2 b_3\end{aligned}$$

while

$$\begin{aligned}\|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\ &= a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 \\ &\quad - 2a_1 a_2 b_1 b_2 - 2a_1 a_3 b_1 b_3 - 2a_2 a_3 b_2 b_3.\end{aligned}$$

It follows that  $\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2$ . Since  $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$ , we conclude that

$$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 (1 - \cos^2 \theta) = \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2 \theta.$$

Now  $\sin \theta \geq 0$ , and so  $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$ , as desired.