RESEARCH STATEMENT

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1. Introduction

My research has been in the field of symplectic geometry, especially the relation between symplectic geometry and low dimensional topology or other areas of mathematics. More precisely, I am interested in applying the theory of holomorphic curves to study not only symplectic manifolds, but also general smooth manifolds and their submanifolds.

The starting point for much of my research is the powerful new Symplectic Field Theory of Eliashberg, Givental, and Hofer [EGH], which uses holomorphic curves in Floer theory to construct invariants of symplectic and contact manifolds, and which is closely related to string theory and enumerative algebraic geometry. My doctoral dissertation (2001) applied Symplectic Field Theory to answer some outstanding questions in the theory of Legendrian knots, and much of my current research is at least somewhat tied to Legendrian knot theory.

I have several ongoing projects which lie entirely within the realm of contact geometry and Legendrian knots; please see the following sections for details. Recently, however, I have concentrated on relating symplectic geometry and low dimensional topology. A construction introduced in physics by Ooguri and Vafa [OV] associates to a topological knot (or other submanifold) a Legendrian manifold. By studying this manifold using holomorphic curves and Floer theory, I introduced a new knot invariant called knot contact homology, which is quite strong and has ties to invariants such as the Alexander polynomial and the A-polynomial. (An analogous invariant for braids has interesting connections with braid representation theory.)

Knot contact homology is an invariant with a rich algebraic structure which seems to be unique to Legendrian knot theory. My previous work in understanding this algebraic structure allows one to deduce invariants from knot contact homology which are readily calculable by computer, both numerical and algebro-geometric.

Because of the relative youth of holomorphic-curve techniques in symplectic geometry, the field is wide open. I hope in the near future to develop my results in several ways, including: extension of the techniques used to construct knot contact homology to allow computation of Symplectic Field Theory for more general Legendrian submanifolds; development of the means to compute knot contact homology in general; and generalization of knot contact homology to arbitrary submanifolds in arbitrary manifolds. These are all small steps in a much larger research project which uses holomorphic curves to study the topology of smooth three- and four-manifolds and their submanifolds. This project links such disparate fields as string theory, algebraic geometry, and low dimensional topology, and has already been shown (e.g., through the work of Ozsváth and Szabó) to be a significant and powerful tool in these subjects.

2. Research To Date

2.1. **Legendrian knot theory and Symplectic Field Theory.** Legendrian knots play a role in contact geometry analogous to that of knots in geometric topology. They are interesting objects to consider in their own right, and are also crucial to understanding the geometry of contact manifolds, especially since the advent of symplectic topology in the 1980's. In the 1990's, Eliashberg and Hofer sparked a revolution in the subject by introducing an invariant called contact homology, which was later expanded to Symplectic Field Theory [EGH]. Chekanov followed with an influential paper [Ch] giving a combinatorial invariant which was believed to coincide with the contact homology of Legendrian knots in $\mathbb{R}^3$ with the standard contact structure.

The precise form of the Legendrian invariant is a differential graded algebra (DGA); the homology of this complex yields the contact homology of the Legendrian knot. The relevant equivalence relation on DGAs, stable tame isomorphism, seems to be a new and intractable construct in homological algebra related to derived categories, and my Ph.D. thesis work [Ng1] derived computable algebraic invariants from the Chekanov–Eliashberg DGA, which can be used to deduce a large amount of information about a Legendrian knot.

Although Symplectic Field Theory gives an abstract definition for Legendrian contact homology, it is difficult to compute contact homology without a combinatorial version à la Chekanov. Legendrian contact homology has since been combinatorially defined and refined for a fair number of contact manifolds. In [ENS], Etnyre, Sabloff, and I extended Chekanov's DGA, which was defined over $\mathbb{Z}/2$, to an invariant over $\mathbb{Z}$, by considering orientations on the relevant Floer-theoretic moduli spaces, and showed analytically that Chekanov's combinatorial invariant does in fact produce contact homology. In joint work with Traynor [NT], I also extended the DGA to an invariant of knots in the 1-jet space of S^1. A more involved computation of contact homology led to the invariant described in the following section.

An interesting facet of Legendrian knot theory, and the subject of much current research, is the relationship between contact homology and invariants of Legendrian submanifolds derived through a markedly different approach, using generating functions. We have formulated a meta-conjecture that the information from generating functions is precisely the information contained in a linearization of contact homology (but in general contact homology is a much stronger invariant). Traynor and I demonstrated in [NT] that this is true for a large family of links in the solid torus. It would be interesting to establish a similar result for standard contact $\mathbb{R}^3$. In joint work with Sabloff [NS], I have built on results of Fuchs [Fu], Fuchs–Ishkhanov, and Sabloff to deduce a correspondence for knots in $\mathbb{R}^3$ between the decomposition invariant of Chekanov–Pushkar' [ChP], derived from generating functions, and augmentations of the contact homology DGA. However, the general meta-conjecture remains open in this case.

In another direction, I have been working on the classification of Legendrian knots in standard contact $\mathbb{R}^3$. An important part of this problem is determining a topological knot invariant known as the maximal Thurston–Bennequin number $\overline{tb}$. This invariant is related to many familiar knot invariants such as genus, Kauffman polynomial, and HOMFLYPT polynomial, and can be used in some cases to calculate slice genus or other invariants. In [Ng7], I established a relation between $\overline{tb}$ and Khovanov homology [Kh] (more precisely, an upper bound on $\overline{tb}$), and applied it to calculate $\overline{tb}$ for alternating knots. The sharpness of the Khovanov bound in many cases, and the simplicity of its proof (many previously established bounds require rather involved proofs), suggest a possibly deep connection, still being explored, between Khovanov homology and contact topology.

2.2. **Knot contact homology.** A link between symplectic topology and knot theory comes from a simple construction of Ooguri and Vafa [OV]: given a knot (or any submanifold) in a smooth manifold, its conormal bundle is a Lagrangian submanifold of the cotangent bundle of the manifold. Ooguri and Vafa then derived a knot invariant which has only been computed for a few knots so far, and is currently the subject of intense study. One can, however, proceed in a slightly different direction and note that the unit conormal bundle is a Legendrian submanifold of the cosphere bundle of the manifold, and hence that the contact homology of this Legendrian submanifold yields an invariant of the original knot. In a series of papers [Ng2, Ng3, Ng4], I presented a combinatorial form for this knot contact homology, for knots in $\mathbb{R}^3$, and studied its properties; see [Ng5] for a summary of these papers.

As in Chekanov's work, the new knot invariant is a DGA, and the algebraic methods developed in [Ng1] derive computable invariants from it. In particular, knot contact homology encodes the Alexander polynomial, contains the A-polynomial, and, as a consequence of work of Kronheimer and Mrowka [KM] related to the proof of Property P for knots, distinguishes the unknot from all other knots [Ng4]. It is possible that it encodes the Jones polynomial or other classical knot invariants, and even conceivable that it is a complete knot invariant.

This work is closely related to the theory of braids. The definition of knot contact homology depends on a braid whose closure is the desired knot, and uses a representation of the braid group which has been studied extensively in the past, beginning with work of Magnus. In [Ng3], I presented a geometric interpretation of this (algebraic) representation, which resembles Bigelow's version of the Lawrence–Krammer representation [Big].

Using this interpretation, one can give a purely topological definition of a portion of knot contact homology, namely its 0-dimensional part HC_0, in terms of paths, which I termed "cords", linking points on the knot. The resulting "cord algebra", which is isomorphic to HC_0, has a homotopy-theoretic definition which extends to a highly nontrivial invariant of arbitrary knots in any manifold, and is conjectured to be the corresponding contact homology algebra in the general case [Ng4]. These results with cords are a relative analogue of Viterbo's celebrated result [Vit] that the Floer homology of a cotangent bundle is the cohomology of the loop space of the underlying manifold. Cords also have an interpretation in terms of quandles and racks.

The symplectic geometry underlying the combinatorial formulation of knot contact homology involves the technique of gradient trees, which links Lagrangian Floer theory and Morse theory. Gradient trees were introduced by Fukaya and Oh in [FO] and are related, by the work of Betz and Cohen [BC], to string topology and Chas–Sullivan homology [CS]. In [EENS], Ekholm, Etnyre, Sullivan, and I extend and generalize the results of Fukaya and Oh to the setting of contact homology, and use this to show that my combinatorial form for knot contact homology agrees with the analytical definition of contact homology.

Knot contact homology extends to an invariant in other contexts, including embedded graphs in $\mathbb{R}^3$, closed embedded surfaces in $\mathbb{R}^3$, virtual and welded knots in $\mathbb{R}^3$, and 2-knots in $\mathbb{R}^4$. I introduced the invariant in these cases in [Ng4] and demonstrated its nontriviality in each case. The conormal construction also leads to holomorphic-curve invariants of immersed curves in the plane without dangerous self-tangencies, à la Arnold; see [Ng6] for details.

3. Proposed Research

There has been an explosion of recent progress and promising new results in symplectic topology, and one can reasonably expect significant breakthroughs to occur soon in both symplectic topology itself and its relations to other fields.

3.1. **Legendrian knots and holomorphic curves.** There are several facets of Legendrian knot theory which merit exploration and seem to be tractable at this point. Although contact homology for Legendrian knots is now reasonably well understood, very little is known about how to properly formulate and compute the more general Symplectic Field Theory for Legendrian knots, for which contact homology is a first-order approximation. A correct combinatorial expression for Legendrian SFT would likely yield nontrivial invariants of transverse knots in contact manifolds, and thus make contact with work of Birman and Menasco in braid theory [BM]. It would also yield an improved version of the knot contact homology invariant.

Even in basic contact homology, many questions remain open. It should be possible to extend the work of [EENS] to give a more general recipe for computing the contact homology of Legendrians. Via the conormal construction, one could then generalize knot contact homology to a holomorphic invariant of arbitrary knots in any dimension. I expect that this would recover and possibly enhance invariants from string topology.

As mentioned before, there is an intriguing relationship between the two main sources of Legendrian invariants, contact homology and generating functions, which has yet to be fully explained. It seems likely that the answer lies in studying gradient trees and Morse theory, along the lines of my joint work [EENS].

Within Legendrian knot theory itself, it seems possible that the techniques in [Ng7] could lead to a classification of Legendrian knots in alternating knot types. Through the work of Eliashberg–Fraser and Etnyre–Honda, such a classification has been effected for Legendrian knots in several special knot types, including the unknot and torus knots, but adding alternating knots to this list would be a significant advance in the subject.

It should also be possible to use knot contact homology to obtain a new, likely nontrivial, invariant of transverse knots. One can lift a contact structure on $\mathbb{R}^3$ to its conormal in the cosphere bundle, and counts intersections of holomorphic disks with this conormal. The result is a filtration on knot contact homology, invariant under transverse isotopy.

Most of my work thus far has dealt with the relative version of contact homology, and it would be interesting to connect this with the very similar theory of relative Gromov–Witten invariants. It may also be possible to formulate a combinatorial version of usual (absolute) contact homology for contact manifolds, along the lines of the relative case. Through a study of the contact homology of the cosphere bundle of a manifold, this would lead to new invariants of smooth manifolds.

3.2. **Symplectic Field Theory and other fields.** I am especially interested in the consequences that Symplectic Field Theory promises for other areas of mathematics.

A concrete starting point would be a better understanding of knot contact homology and its ramifications in knot theory. Although there is a topological interpretation for a portion (HC_0) of knot contact homology, we cannot yet see the full invariant purely topologically. Physicists have used a similar construction to deduce a knot invariant which contains the colored Jones polynomial, and it seems likely that knot contact homology encodes the same or similar information. I also expect there to be ties to Khovanov homology, as well as the Heegaard Floer homology of Ozsváth–Szabó [OSz], which is also derived by counting holomorphic curves. In particular, using conormals, one can translate the setup of knot Floer homology into the contact world, resulting in disjoint Legendrians, one for a knot and the other for a Heegaard surface.

Beyond knot theory, the theory leading to knot contact homology seems to have deep links to algebraic topology and low dimensional topology. The connection to string topology has already been mentioned. In addition, Symplectic Field Theory yields topological invariants of manifolds which would be interesting to study, especially in three and four dimensions; as a first step, it seems likely that there is a surgery formula involving knot contact homology which would produce three-manifold invariants. I am also working on a joint project with Lipshitz and Manolescu to use gradient trees to compute Heegaard Floer homology, with the eventual goal of producing a purely combinatorial form for this powerful but sometimes intractable set of invariants.

Presently a large amount of research in string theory and enumerative algebraic geometry is being devoted to relating Chern–Simons knot invariants and relative Gromov–Witten invariants, starting with the conormal construction of Ooguri and Vafa. Most recently, Gukov–Schwarz–Vafa [GSV] have connected conormal bundles to $\mathfrak{sl}(N)$ Khovanov–Rozansky homology. Knot contact homology begins with the same construction but counts a slightly different set of holomorphic curves, resulting in a different but probably related invariant. Needless to say, understanding the relation between the physics picture and the symplectic picture would enrich both sides and possibly lead to new theories on either side.

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