Exercise 1 (35 pts). [Inspired by Ex. 5.3 in Trefethen’s book] Consider the matrix
\[ A = \begin{bmatrix} 3 & -10 \\ -2 & 2 \end{bmatrix} \]
- Compute, on paper, a real SVD of \( A \) in the form \( A = U\Sigma V^T \). The SVD is not unique, so find the one with the minimal number of minus signs in \( U \) and \( V \).
- List the singular values, left singular vectors, right singular vectors.
- Draw an accurate picture of the unit ball in \( \mathbb{R}^2 \) and its image under \( A \), together with the singular vectors, with their coordinates marked.
- What are the 1-, 2-, \( \infty \)- and Frobenius norms of \( A \)?
- What is the rank of \( A \)? How can it be read from the SVD?
- Find \( A^{-1} \) not directly, but via the SVD
- Find the eigenvalues \( \lambda_1, \lambda_2 \) of \( A \)
- Verify that \( \det A = \lambda_1 \lambda_2 \) and \( |\det A| = \sigma_1 \sigma_2 \).
- What is the area of the ellipsoid onto which \( A \) maps the unit ball of \( \mathbb{R}^2 \)?
- What is the best rank-1 approximation to \( A \)?

Exercise 2 (15 pts). [Ex. 3.16 from Heath’s book] Let \( A \) be a \( m \times n \) matrix. Under what conditions on \( A \) is the matrix \( A^T A \):
- Symmetric?
- Nonsingular?
- Positive definite? (Recall that a matrix is positive definite if \( \langle Av, v \rangle > 0 \) for all \( v \neq 0 \).)

In each case fully justify your answer, proving your statement and/or providing counterexamples as needed.

Exercise 3 (15 pts). Which of the following matrices are orthogonal? \( A_1 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ A_2 := \begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \ A_3 := \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \ A_5 := A_1 A_3, \ A_6 := A_3 A_2, \ A_7 := A_2 A_3 A_2^{-1}. \)