Homework 5 - Due Fri. Oct. 17th
Math 465 - Fall 2014

Dr. Mauro Maggioni
Office: 309 Gross Hall
Phone: 660-2825
Web page: www.math.duke.edu/~mauro
E-mail: mauro.maggioni at duke.edu

Homework policies: as in previous homework

Assignment

Exercise 1 (50 pts). Consider the space \( L^2([\pi, \pi]) = \{g : \int_{\pi}^{\pi} |g(x)|^2dx < +\infty\} \) with the norm \( ||g|| := \sqrt{\int_{\pi}^{\pi} |g(x)|^2dx} \), associated to the inner product \( \langle g, h \rangle := \int_{\pi}^{\pi} g(x)h(x)dx \).

- Show that \( \{\frac{1}{\sqrt{2\pi}}1, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(mx)\}_{m,n\geq 1} \) is an orthonormal set (here 1 denotes the constant function 1). To simplify the notation in what follows let

\[ (\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5 \ldots ) := \{ \frac{1}{\sqrt{2\pi}}1, \frac{1}{\sqrt{\pi}} \cos(x), \frac{1}{\sqrt{\pi}} \sin(x), \frac{1}{\sqrt{\pi}} \cos(2x), \frac{1}{\sqrt{\pi}} \sin(2x), \ldots \}. \]

This is in fact an orthonormal basis (this is a nontrivial fact that you are not required to prove), i.e. for any \( f \in L^2([\pi, \pi]) \) we have

\[ \left\| f - \sum_{i=1}^{L} < f, \varphi_i > \varphi_i \right\| \rightarrow 0 \quad \text{as} \quad L \rightarrow +\infty \]

- We let \( P_L f = \sum_{i=1}^{L} < f, \varphi_i > \varphi_i \) be the \( L \)-th partial sum used above. Interpret this as an orthogonal projection of \( f \) onto the span of the first \( L \) Fourier modes.

- Sample the interval \([\pi, \pi]\) at \( p \) equispaced points \( x_1, \ldots, x_p \) (for example using the linspace function in Matlab), and implement a function that computes \( P_L f \) for any given \( f \) (sampled at the points \( x_i \)) and given \( L > 0 \). Check your function is reasonable by using to (approximately) check that \( \{\varphi_i\}^{L}_{i=1} \), for moderate values of \( L \), is an orthonormal basis. Discuss how this may be implemented in matrix form, and how many basic computations this function takes, as a function of \( p \) and \( L \).

- Choose various functions (for example \( f(x) = e^{\sin(2\pi x)} \) or \( f(x) = 1_{-[\pi,0]}(x) + 1_{[0,\pi]}(x) \) and/or others) and a suitable \( p \) (e.g. \( p = 1000 \)), and study \( ||f - P_L f|| \) as a function of \( L \): this is monotonic decreasing (why?) and tending to 0 as \( L \rightarrow +\infty \), and if plotted in log, 0 scale you may see a trend in the decay rate. Try this for smooth periodic functions (recall: a function is period if \( f(x+2\pi) = f(x) \) for all \( x \), i.e. \( f \) “repeats itself” from each interval \( I \) of width \( 2\pi \) to any interval \( I+2k\pi \), for any \( k \) positive or negative integer), and for periodic functions that not smooth (e.g. they have a jump discontinuity).

- The same as the previous point, but by adding Gaussian noise of size \( \sigma \) (e.g. for \( \sigma = 0.01, 0.1 \)) to each of the functions \( f \) you considered, obtaining a sampled function \( \hat{f} \). Can you tell what \( ||f - P_L \hat{f}|| \) tends to as \( L \rightarrow +\infty \)? [Hint: it depends on \( \sigma \)]. This experiment needs to be done a bit carefully: make sure you keep \( L \) much smaller than the number of sample points \( p \), e.g. \( L \ll \sqrt{p} \) should suffice. How do the approximations \( P_L f \) look like compared to the noisy \( f \)? More or less noisy? If you look instead at \( ||f - P_L \hat{f}|| \), how does this “error” behave as a function of \( L \)? Is it still monotone? Make sure you try both a smooth \( 2\pi \)-periodic function such as \( e^{2\pi \sin(x)} \) and a
non-smooth 2π-periodic function such as (the periodization of) \( f(x) = 1_{[-\pi, 0)}(x) + 1_{[0, \pi)}(x) \), as the answer to the last question may depend on the smoothness of \( f \), at least in the regimes of values of \( p \) and \( L \) that you are exploring.