Homework policies: as in previous homework

Assignment

Exercise 1 (5 pts). [Ex. 3.16 from Heath’s book] Let $A$ be a $m \times n$ matrix. Under what conditions on $A$ is the matrix $A^T A$:

- Symmetric?
- Nonsingular?
- Positive definite? (Recall that a matrix $C$ is positive definite if $\langle Cv, v \rangle > 0$ for all $v \neq 0$.)

In each case fully justify your answer, proving your statement and/or providing counterexamples as needed.

Exercise 2. For this exercise you may want to reuse/modify the code of the scripts used in class (from the first lectures or from the lecture on 9/22, in SVD_Examples.zip on the wiki).

Construct a data set $X \in \mathbb{R}^{D \times n}$ like the one with images of randomly translated “blobs” (the intensity in those images is in fact a Gaussian), with blobs of circular shape, and then with blobs with square shape. Perform Principal Component Analysis of these data sets. Are the singular values of the two data sets similar or different? Are the left singular vectors similar or different (if there are degenerate or almost degenerate eigenvalues, look at eigenspaces corresponding to degenerate eigenvalues, instead of single eigenvectors)? Attempt to explain why, but do not worry if you cannot as this is not trivial.

Fix one of the two shapes, and consider the matrix $V(:, 1:2)$ consisting of the first two columns of $V$ (each has length $n$). We interpret each row $y_i$ of this matrix as a point in 2 dimensions. Show that these 2-dimensional vectors are in fact the first two coordinates of the data $X$ projected onto the best (in the mean squared error sense) 2-dimensional plane approximating the data. Plot this set $\{y_i\}_{i=1}^n$ of $n$ points in 2 dimensions, and comment on what you obtain. Each of these points is a projection of $x_i := X(:, i)$, and therefore is a circle (or square) with a given center $(c_{x,i}, c_{y,i})$. Color the points $y_i$ by $c_{x,i}$ and then by $c_{y,i}$, and describe (and perhaps explain?) what you see.