HOMEWORK 5 - [illegible]

HOMEWORK 3

Pb.1 pg 80 This simply requires checking that the given fcn's satisfy the heat equation $u_t = k\Delta_x u$:

(a) $u(x,t) = kt + \frac{1}{2}x^2 + C$

$\partial_t u = k$, $\partial_{xx}u = 1$, so indeed $u_t = k\Delta_x u$

(b) $u(x,t) = e^{-\gamma^2 kt}\sin(\gamma x)$

$\partial_t u = -\gamma^2 k e^{-\gamma^2 kt}\sin(\gamma x)$, $\partial_{xx}u = e^{-\gamma^2 kt}(-\gamma^2)\sin(\gamma x) \Rightarrow$ ok!

(c) Exactly as (b)

(d) $u(x,t) = e^{kt \pm x}$

$\partial_t u = k e^{kt\pm x}$, $\partial_{xx}u = \partial_x(\pm e^{kt\pm x}) = \pm(\pm e^{kt\pm x}) = e^{kt\pm x} \Rightarrow$ o

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(a) The maximum of the functions above, in $[-a,a]\times[0,T]$ is attained at:

ⓐ $D_{(x,t)}u = (2x, k) = (0,0)$ for no $(x,t) \Rightarrow$ maximum is / minimum at the boundary of $[-a,a]\times[0,T]$.

The maximum is at $(\pm a, T)$, since clearly $u(x,t) \le kT + \frac{1}{2}(\pm a)^2 + C$, since u is monotone increas. in t, and in $|x|$.

(The minimum is at $(0,0)$ ~~same~~ for the same reason)

(b,c) Since u is a product of a fcn of x and a fcn of t, and u is monotonically decreasing in t, the maximum is at $t=0$, and it is attained when $\sin(\gamma x)$ (resp. $\cos(\gamma x)$) is maximum.

(The minimum is attained at $t=0$ and x: $\sin\gamma x = -$ (resp. $\cos\gamma x = -$

ⓓ As in (b,c), u is also a product of a function increasing in t and monotonically increasing (if +) or decreasing (if −) in x. Therefore the maximum is at (a,T) (if +) or $(-a,T)$ (if −).

We may apply Theorem 3.2 only if the initial condition is bounded on $\mathbb{R}$: this is the case only for functions (b,c), since in that case $|u(x,t)| \leq \underbrace{1}_{t=0} \cdot \underbrace{1}_{\substack{\sin (\text{or } \cos) \\ \text{are bounded by } 1}} = 1$.

(3*) The homogeneous equation is linear in u.
If u, v both satisfy the nonhomogeneous eqn, then $u-v$ satisfies the homogeneous eqn:

$$\partial_t(u-v) - k\partial_{xx}(u-v) = \underbrace{(\partial_t u - k\partial_{xx}u)}_{=q} - \underbrace{(\partial_t v - k\partial_{xx}v)}_{=q} = 0$$

So $u-v$ satisfies the heat equation (homogeneous) and by assumption $u-v \leq 0$ for $|x| \leq L$ and $t=0$ and for $|x|=L$, $t\in[0,T]$. By the maximum principle, $u \leq v$ in $|x|\leq L$, $t\in[0,T]$.

(4) $$\begin{cases} \partial_t u = k\partial_{xx}u \\ u(0,t) = 2, \quad u(1,t) = 1 \end{cases}$$

At equilibrium, by definition of equilibrium, u_∞ does not vary in time, i.e. $\partial_t u_\infty = 0$.
By the heat equation, $\cancel{k}\partial_{xx}u_\infty = 0$
$\Rightarrow u_\infty = \alpha x + \beta \Rightarrow u_\infty = -x+2$.

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$$\begin{cases} u_t - k u_{xx} = q & , x, t > 0 \\ u(0,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & x > 0 \end{cases}$$

(a) $H(x,y,t) = S(x-y,t) - S(x+y,t)$

We extend the problem to the whole real line $\mathbb{R}$ by considering the odd (because of Dirichlet bdry conditions) extension. By abuse of notation, let u, q, f be the odd extensions. Then, by Duhamel's principle:

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t) f(y)\, dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s)\, q(y,s)\, dy\, ds =$$

$$= \int_0^{+\infty} H(x,y,t) f(y)\, dy + \int_0^t \int_0^{+\infty} H(x,y,t-s)\, q(y,s)\, dy\, ds$$

(b) Assume $q = 0$ for $x > a$, $t > T$, with $T \gg 0$.

$w = \int_{-\infty} x f(x)$

(b) Let $q = 0$ for $x > a$, $t > T$, $T \gg a$.

$$u(x,t) = \int_0^{+\infty} H(x,y,t) f(y)\,dy + \int_0^t \int_0^a H(x,y,t-s)\, q(y,s)\,dy\,ds$$

$$= \int_0^{+\infty} \Big[H(x,m,t) + \underbrace{\cancel{\partial_2 H(x,m,t)\cdot(y-m)}}_{=0 \text{ by choosing } m = \int_0^{+\infty} y f(y) dy} + \tfrac{1}{2}\partial_{22} H(x,m,t)(y-m)^2 \Big] f(y)\,dy$$

$$+ \int_0^T \int_0^a \Big[H(x,y_0(s),t-s) + \underbrace{\cancel{\partial_2 H(x,y_0(s),t-s)(y-y_0(s))}}_{=0 \text{ by choosing } y_0(s) = \int_0^a y\, q(y,s) dy} + \partial_{22} H(x,y_0(s),t-s)(y-y_0(s)) \Big] \cdot q(y,s)\,dy\,ds$$

$m = \int_0^{+\infty} y f(y)\,dy$

$y_0(s) = \int_0^a y\, q(y,s)\,dy$

"gaussian" approx

$$= H(x,m,t) \int_0^{+\infty} f(y)\,dy + \tfrac{1}{2}\partial_{22} H(x,m,t) \int_0^{+\infty} (y-m)^2 f(y)\,dy +$$

$$+ \int_0^T H(x,y_0(s),t-s) \int_0^a q(y,s)\,dy\,ds + \int_0^T \partial_{22} H(x,y_0(s),t-s) \int_0^a (y-y_0(s))^2 q(y,s)\,dy\,ds$$

$$= \dots + \int_0^T H(x,y_0(s_0),t-s_0) \int_0^a q(y,s)\,dy\,ds +$$

$s_0 = \int_0^T \int_0^a q(y,s)\,dy\,ds$

$$\underbrace{\cancel{- \int_0^T \partial_3 H(x,y_0(s_0),t-s_0)(s-s_0) \int_0^a q(y,s)\,dy\,ds}}_{=0 \text{ by choosing } s_0 = \int_0^T \int_0^a q(y,s)} +$$

$$+ \int_0^T \partial_{33} H(x,y_0(s_0),t-s_0)(s-s_0)^2 \int_0^a q(y,s)\,dy\,ds +$$

All these terms are either $\sim H$ or $\sim \partial_t H$ at some fixed point $y_0(s_0), t-s_0$ and are "Gaussian", and rapidly decreasing in

$$\underbrace{\cancel{+ \int_0^T \partial_2 H(x,y_0(s_0),t-s_0)\, y_0'(s_0)(s-s_0) \int_0^a q(y,s)\,dy\,ds}}_{=0} +$$

$$+ \int_0^T \partial_{22} H(x,y_0(s_0),t-s_0)\, y''(s_0)(s-s_0)^2 \int_0^a q(y,s)\,dy\,ds +$$

$$- 2\int_0^T \partial_{23} H(x,y_0(s_0),t-s_0)\, y'(s_0)(s-s_0)^2 \int_0^a q(y,s)\,dy\,ds +$$

$$+ \int_0^T \partial_{22} H(x,y_0(s),t-s) \int_0^a (y-y_0(s))^2 q(y,s)\,dy\,ds$$