HOMEWORK 1

(1) (a) $u_t + 2u_x = 4t$, $u(x,0) = \sin\left(\frac{x}{3}\right) =: f(x)$

The characteristics of the homogeneous eqn. are $x - 2t = x_0$. If we let $v(t) = u(x_0+2t, t)$, we obtain $\frac{dv}{dt}(t) = 2u_x(x_0+2t, t) + u_t(x_0+2t, t)$ by the chain rule, and therefore, by assuming u is a solution of the hom. eqn., we deduce

$$\frac{dv}{dt}(t) = 4t \Rightarrow v(t) = 2t^2 + \alpha \underset{\substack{\text{def.}\\ \text{of } v}}{\Rightarrow} u(x_0+2t, t) = 2t^2 + \alpha$$

from which we obtain $\alpha \underset{t=0}{=} u(x_0, 0) = f(x_0)$ and therefore by letting $x = x_0 + 2t$, i.e. $x_0 = x - 2t$

$u(x,t) = 2t^2 + f(x-2t) = 2t^2 + \sin\left(\frac{x-2t}{3}\right)$

Check: it does satisfy the PDE + initial condition ✓

The solution is a sinusoidal wave translating to the right with propagation speed 2, and with growing amplitude, since at time t the wave of constant amplitude (in x!) is added to the original sinusoidal wave.

(b) The technique is exactly the same as in (a), and we use the same notation.

We obtain

$\frac{dv}{dt}(t) = x_0 + 3t \Rightarrow v(t) = x_0 t + \frac{3t^2}{2} + \alpha \Rightarrow u(x_0+3t, t) = x_0 t + \frac{3}{2}t^2$

$\Rightarrow \alpha = u(x_0, 0) = f(x_0)$, and if $x = x_0 + 3t$,

$u(x,t) = xt - \frac{3t^2}{2} + f(x-3t) = xt - \frac{3t^2}{2} + \cos(x-3t)$

Check: it satisfies PDE + initial condition!

The solution is, at each time t, the superposition of a cosine wave (moving to the right in time with speed 3) and a linear wave with slope t and moving to the right.

(c) $u_t + 4x u_x = x \qquad , u(x,0) = f(x) = \sin x$

For the characteristics $t \longmapsto x(t)$ we get

$$\frac{dx}{dt} = 4x \Rightarrow x(t) = e^{4t} x_0 \Rightarrow \underbrace{x_0(x,t)}_{p(x,t) \text{ in class!}} = x e^{-4t}$$

Then, as above, we obtain

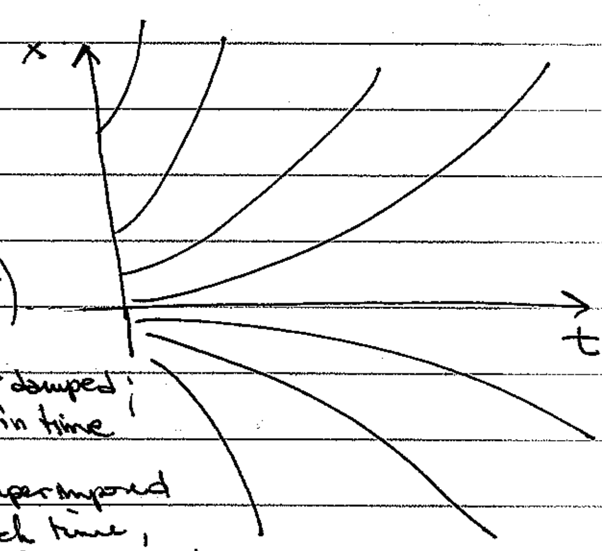
$$\frac{dv}{dt} = x = e^{4t} x_0 \Rightarrow v(t) = \frac{1}{4} e^{4t} x_0 + \alpha$$

and for $t = 0$ we obtain $\alpha = v(0) - \frac{1}{4} x_0 = f(x_0) - \frac{1}{4} x_0$

so that

$$u(x,t) = f(xe^{-4t}) + \frac{1}{4} \cancel{e^{4t}} x \cancel{e^{-4t}} - \frac{1}{4} x e^{-4t}$$
$$= \sin(xe^{-4t}) + \frac{1}{4} x (1 - e^{-4t})$$

Picture for the characteristics



$$u(x,t) = \underbrace{\sin(xe^{-4t})}_{\text{"wave" transported along diverging characteristics}} + \frac{x}{4}(1 - e^{-4t})$$

damped in time

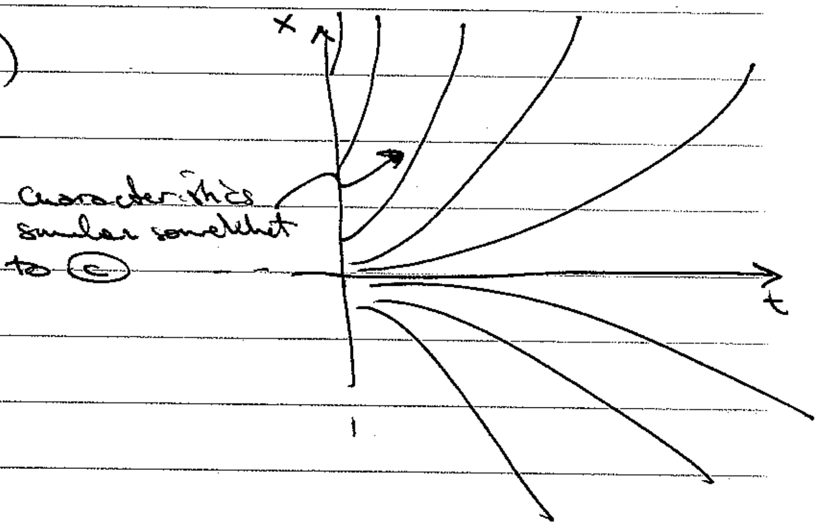
line superimposed at each time, with slope $\xrightarrow[t \to \infty]{} 1$

(d) The characteristics are

$$\frac{dx}{dt} = 2tx \Rightarrow x = e^{t^2}x_0 \Rightarrow \underbrace{x_0(t,x)}_{p(t,x)} = xe^{-t^2}$$

Therefore

$$u(t,x) = f(e^{-t^2}x)$$



Ex 1
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$u_t + cu_x + u = 0$, $u(x,0) = f(x)$

$v(t) = u(x_0+ct, t)$

$\frac{dv}{dt}(t) = cu_x + u_t = -v(t) \Rightarrow v(t) = e^{-t}\alpha$

Since $\alpha = v(0) = u(x_0,0) = f(x_0)$, we have $v(t) = e^{-t}f(x_0)$

Therefore, since $x = x_0+ct \Rightarrow x_0 = x-ct$

$u(x,t) = e^{-t}f(x-ct)$

Ex 2
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As above, $v(t) = u(x_0+ct, t)$,

$\frac{dv}{dt}(t) = (x_0+ct)t \Rightarrow v(t) = x_0\frac{t^2}{2} + c\frac{t^3}{3} + \alpha$

and from $\alpha = v(0) = f(x_0)$, we deduce

$u(x,t) = (x-ct)\frac{t^2}{2} + \frac{ct^3}{3} + f(x-ct)$

Ex3 page 27

Exactly as above,

$$\frac{dv}{dt} = v^2 \Rightarrow \frac{dv}{v^2} = dt \Rightarrow -d\left(\frac{1}{v}\right) = dt$$

$$\Rightarrow -\frac{1}{v}\Big|_0^t = t \Rightarrow \frac{1}{v(0)} - \frac{1}{v} = t \Rightarrow \frac{1}{f(x_0)} - \frac{1}{v} = t$$

$$\Rightarrow v(t) = \frac{f(x_0)}{1 - t f(x_0)} \Rightarrow u(x,t) = \frac{f(x-ct)}{1 - t f(x-ct)}$$