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(a)
$$\begin{cases} u_{tt}(x,t) = c^2(x)\,\Delta_x u(x,t), & c(x) = \begin{cases} c_\ell, & x \in [-L,0] \\ c_r, & x \in (0,L] \end{cases} \\ u(-L,t) = 0 = u(L,t) \quad \forall t & \leftarrow \partial \text{ cond's.} \\ \lim_{x\to 0^-} u(x,t) = \lim_{x \to 0^+} u(x,t) \quad \forall t & \\ \lim_{x\to 0^-} \partial_x u(x,t) = \lim_{x\to 0^+} \partial_x u(x,t) \quad \forall t & \leftarrow \text{joining conditions at } 0 \\ u(x,0) = f(x) & \\ u_t(x,0) = g(x) & \leftarrow \text{initial values} \end{cases}$$

(b) $u_n(x,t) = \psi_n(t)\varphi_n(x)$.

$$\psi_n''(t)\varphi_n(x) = c^2(x)\,\varphi_n''(x)\,\psi_n(t)$$

$$\frac{\psi_n''(t)}{\psi_n(t)} = c^2(x)\,\frac{\varphi_n''(x)}{\varphi_n(x)} = -\mu \quad , \quad \mu > 0 \text{ (real exp. sol's would not satisfy } \partial \text{ cond's)}$$

Let's look at the spatial ODE and split it into two, one on $[-L,0]$ and the other on $[0,L]$:

$$\begin{cases} \varphi_n''(x) + \mu c_\ell^2 \varphi_n(x) = 0 & , x \in [-L,0] \\ \varphi_n''(x) + \mu c_r^2 \varphi_n(x) = 0 & , x \in (0,L] \\ + \text{ joining cond's at } x=0 & \\ + \ \partial \text{ cond's} & \end{cases}$$

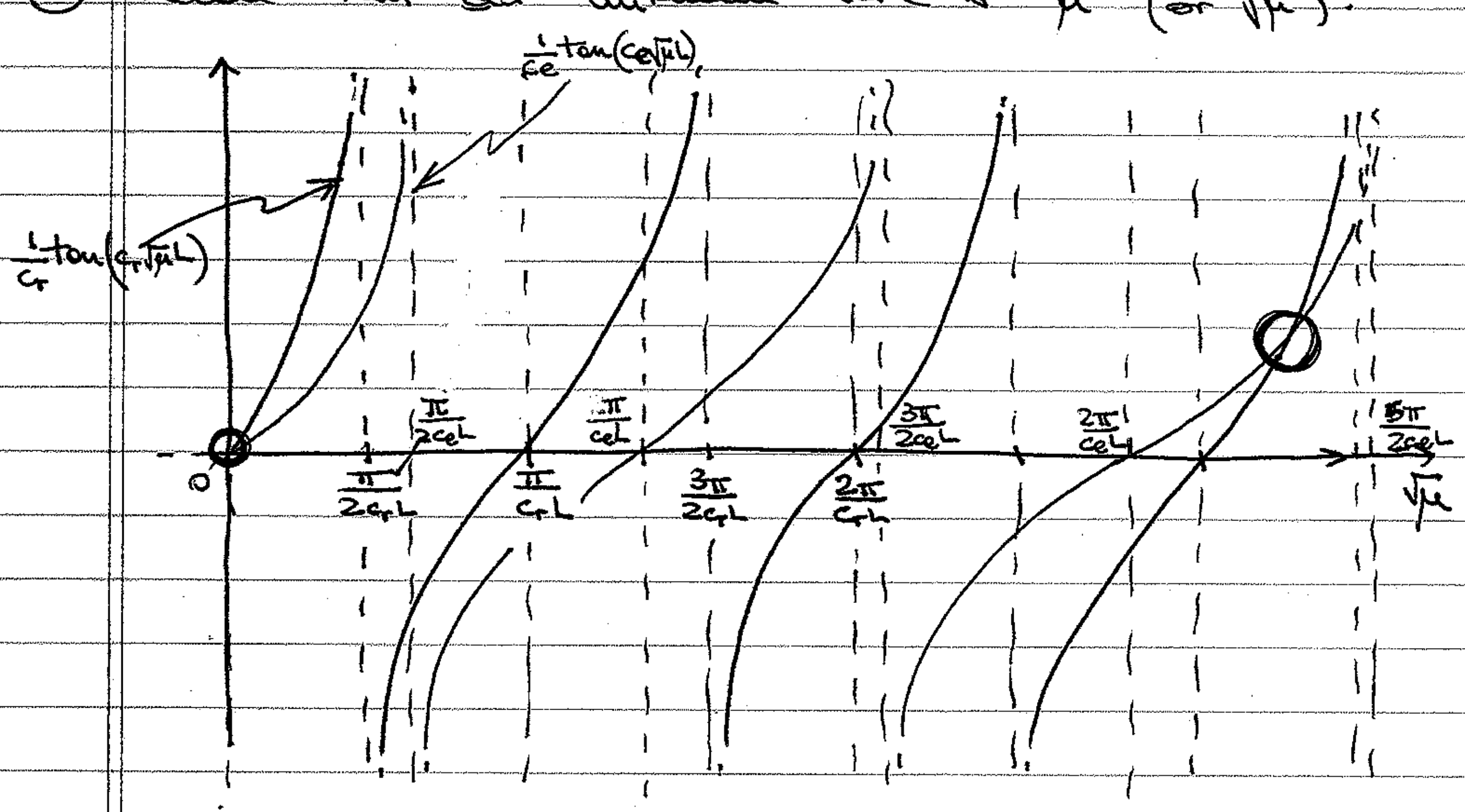
$$\begin{cases} \varphi_n(x) = A_n \cos(c_\ell \sqrt{\mu}\, x) + B_n \sin(c_\ell\sqrt{\mu}\, x) & , x \in [-L,0] \\ \varphi_n(x) = C_n \cos(c_r\sqrt{\mu}\, x) + D_n \sin(c_r \sqrt{\mu}\, x) & , x \in (0,L] \end{cases}$$

$$\begin{cases} \varphi_n(L) = 0 \Rightarrow A_n \cos(c_\ell\sqrt{\mu}L) + B_n \sin(c_\ell \sqrt{\mu}L) = 0 \\ \varphi_n(-L) = 0 \Rightarrow C_n \cos(c_r\sqrt{\mu}L) - D_n \sin(c_r\sqrt{\mu}L) = 0 \\ \varphi_n(0^-) = \varphi_n(0^+) \Rightarrow A_n = C_n \\ \varphi_n'(0^-) = \varphi_n'(0^+) \Rightarrow B_n c_\ell \sqrt{\mu} = -D_n c_r \sqrt{\mu} \Rightarrow D_n = -B_n \frac{c_\ell}{c_r} \end{cases}$$

$$\Rightarrow \tan(c_\ell\sqrt{\mu}L) = -\frac{A_n}{B_n}$$

$$\tan(c_r\sqrt{\mu}L) = \frac{C_n}{D_n} = \frac{A_n}{-B_n\frac{c_\ell}{c_r}} = -\frac{A_n}{B_n}\frac{c_r}{c_\ell} = \frac{c_r}{c_\ell}\tan(c_\ell\sqrt{\mu}L)$$

(c) Recall that our unknown here is $\mu$ (or $\sqrt{\mu}$).




Here we assumed (without loss of generality) $c_r > c_e$ and (with some loss of generality) $c_r < 2c_e$.
Depending on the relative size of $c_r, c_e$, we'll find the intersections between the left and right-hand side of

$$\frac{1}{c_r}\tan(c_r\sqrt{\mu}L) = \frac{1}{c_e}\tan(c_e\sqrt{\mu}L)$$

thereby determining an $\infty$ sequence of solutions $\mu_n$, $\mu_n \nearrow +\infty$.
Observe that $\mu = 0$ sol. is to be discarded.
The corresponding eigenfunctions are given by

$$\begin{cases} \varphi_n(x) = A_n\cos(c_e\sqrt{\mu_n}x) + B_n\sin(c_e\sqrt{\mu_n}x), & x\in[-L,0] \\ \varphi_n(x) = A_n\cos(c_r\sqrt{\mu_n}x) - B_n\frac{c_e}{c_r}\sin(c_r\sqrt{\mu_n}x) & x\in(0,L] \end{cases}$$

and are guaranteed to be $\mathcal{C}^1$ in $[-L,L]$.