Mathematics 287
Fall 1998
Problem Set 2

1. Let $X_1, X_2, \ldots,$ be independent, identically distributed random variables with $\mathbf{E}(X_i) = 0$ and $\mathbf{E}(|X_i|^n) < \infty$ for every $n < \infty$.

a. Show that for each positive integer k there is a $C_k < \infty$, such that

$$\mathbf{E}[(X_1 + \cdots + X_n)^{2k}] \leq C_k n^k.$$

b. Show for every $\epsilon > 0$, with probability one,

$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n^{(1/2)+\epsilon}} = 0.$$

2. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Call a set B a null set if $B \subset C$ for some $C \in \mathcal{F}$ with $\mathbf{P}(C) = 0$. Let $\bar{\mathcal{F}}$ denote the collection of all sets of the form $A \cup B$ where $A \in \mathcal{F}$ and B is a null set. Show that $\bar{\mathcal{F}}$ is a σ-field. Show also that $\bar{\mathcal{F}}$ is complete (i.e., every null set of $\mathcal{F}$ is in $\bar{\mathcal{F}}$) and that $\bar{\mathcal{F}}$ is the smallest complete σ-field that contains $\mathcal{F}$.

3. Suppose X, Y are random variables with

$$E[X \mid Y] = Y, \quad E[Y \mid X] = X.$$

Show that with probability one $X = Y$.