Mathematics 241
Fall 1997
Problem Set 7

1. If f is a measurable function from a measure space $(X, \mathcal{F}, \mu)$ to $\mathbb{C}$ define the $\mathcal{L}^\infty$ norm of f, $\|f\|_\infty$ to be the infimum of all positive numbers t such that

$$\mu\{x : |f(x)| > t\} = 0.$$

a. Show that $\|f - g\|_\infty = 0$ if and only if

$$f = g \text{ a.e.}$$

b. Show that $\mathcal{L}^\infty$ is a normed vector space (under the assumption that two functions are considered the same if they are equal almost everywhere).

c. Is $\mathcal{L}^\infty$ a Banach space?

2. Let $1 \leq p < \infty$. Give an example of a measure space and a measurable function f such that $f \in \mathcal{L}^p$ but $f \notin \mathcal{L}^r$ for any $r \neq p$. Give an example of a sequence of functions that converges in $\mathcal{L}^p$ but not in $\mathcal{L}^r$ for any $r \neq p$.

3. Let $C[0, 1]$ be the set of continuous functions $f : [0.1] \to \mathbb{C}$. Show that f is a Banach space under the norm

$$\|f\| = \sup\{|f(x)| : 0 \leq x \leq 1\}.$$

(Note: $\|f\| = \|f\|_\infty$ for $f \in C[0, 1]$ and hence this is a subspace of $\mathcal{L}^\infty[0, 1]$.)

4. Show that if f_n converges to f in $\mathcal{L}^p$ for some $1 \leq p \leq \infty$, then f_n converges to f in measure.

5. Suppose f_n is a sequence of measurable functions of a measure space $(X, \mathcal{F}, \mu)$ with $\mu(X) < \infty$. Show that

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

Give an example to show this is not necessarily true if $\mu(X) = \infty$.

6. A function $\phi : \mathbf{R} \to \mathbf{R}$ is called convex on $[a, b]$ if for every $0 \leq t \leq 1$ and every $a \leq x, y \leq b$,

$$\phi(tx + (1 - t)y) \leq t\phi(x) + (1 - t)\phi(y).$$

Suppose f is a measurable function on a probability space $(X, \mathcal{F}, \mu)$ taking values in $[a, b]$ and ϕ is convex on $[a, b]$. Show that

$$\phi(\int_X f \, d\mu) \leq \int_X \phi(f) \, d\mu.$$

This is called *Jensen's inequality.*