Mathematics 241
Fall 1997
Final Problem Set
Due: Thursday, December 11, 5:00 pm

You may consult any inanimate reference materials but you cannot talk to anyone else about these problems.

For each of the following say if they are true or false. If they are true say why (you may cite facts from Adams and Guillimen, the chapters in Rudin done in class, or previously done homework exercises). If they are false, give a counterexample.

In 1-6, f_n and f are measurable functions on a (positive) measure space $(\Omega, \mathcal{F}, \mu)$.

1. If $f_n \to f$ in measure, then $f_n \to f$ a. e.

2. If $f_n \to f$ in measure, then $f_n \to f$ in $\mathcal{L}^1$.

3. If $f_n \to f$ a.e., then $f_n \to f$ in measure.

4. If $f_n \to f$ in $\mathcal{L}^1$, then $f_n \to f$ in measure.

5. If $f_n \to f$ $\mathcal{L}^1$, then $f_n \to f$ a. e.

6. If $f_n \to f$ a.e., then $f_n \to f$ in $\mathcal{L}^1$.

7. If $f : \mathbb{R} \to \mathbb{R}$ is continuous such that $f'(x) = 0$ except for a set of Lebesgue measure zero, then f is a constant function.

8. If $f : [0,1] \to [0,1]$ is Lebesgue measurable,

$$\int_0^1 f(x)^2 \, dx \geq [\int_0^1 f(x) \, dx]^2.$$

9. If X is a random variable on a probability space Ω with distribution μ,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \, d\mu.$$

10. Let f be the function, periodic with period 2π, with

$$f(x) = \begin{cases} \pi|x|^{3/2}, & -\pi < x \leq 0 \\ |x|^{5/2}, & 0 \leq x \leq \pi \end{cases} .$$

Let c_n be Fourier coefficients of f. Then for all x,

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} c_n e^{inx}.$$

11.

$$\lim_{a \downarrow 0} \int_1^{\infty} x^{-2} e^{-ax|\sin x|} \, dx = 1.$$

12. If $(X, \mathcal{F}, \mu)$ is a (positive) measure space and $A_1 \supset A_2 \supset A_3 \supset \cdots$, then

$$\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

13. Let B be the set of continuous functions on the reals that are periodic with period 2π. Then B is a Banach space under the sup norm

$$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\}.$$

14. Let X be a random variable such that for each positive integer n,

$$E[X^n] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-x^2/2}\, dx.$$

Then X has a standard normal distribution, i.e., X has density

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \ -\infty < x < \infty.$$

15. If $\{f_n\}$ is a sequence of complex-valued measurable functions on $[0,1]$ (with Lebesgue measure) with

$$\int_0^1 |f_n|^2\, dx \leq 1,$$

then there exists a subsequence n_j and a $\mathcal{L}^2$ function f such that

$$f_{n_j} \to f \ \text{ in } \mathcal{L}^2.$$

16. Suppose a fair coin is flipped infinitely often. Let H_n be the number of heads in the first n flips. Let A_n be the event that $H_n \geq 2n/3$. Then

$$P[\limsup A_n] = 0.$$

17. Let μ be a (positive) measure on $\mathbb{R}$ such that $\mu(I) > 0$ for every open interval I. Then Lebesgue measure m is absolutely continuous with respect to μ.

18. The set of continuous functions on $[0,1]$ is dense in $\mathcal{L}^1[0,1]$.

19. If $f : \mathbb{R}^3 \to \mathbb{C}$ is a measurable function such that for each compact set K,

$$\int_K f\, dm = 0,$$

then $f = 0$ a.e. (Here m refers to Lebesgue measure on $\mathbb{R}^3$.)

20. If $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty,$$

then the Fourier transform $\hat{f}$ is well-defined and continuously differentiable.

21. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, i.e., if $x < y$ then $f(x) \leq f(y)$. Then f is a Borel measurable function.

22. Let f_n be a sequence of nonnegative functions on a measure space $(X, \mathcal{F}, \mu)$. Then

$$\int [\sum_{n=1}^{\infty} f_n(x)] \, d\mu = \sum_{n=1}^{\infty} \int f_n(x) \, d\mu.$$

23. If μ, λ are two positive measures on the Borel subsets of $\mathbb{R}$ and $\lambda(E) = 0$ for every Borel set E with $\mu(E) = 0$, then there exists a Borel measurable function f such that for each Borel set E,

$$\lambda(E) = \int_E f \, d\mu.$$

24. If X is a (real-valued) random variable with $E[X^2] < \infty$ then the characteristic function of X is C^2, i.e., the second derivative exists and is continuous.

25. If $f(x, y)$ is a continuous function on $(0,1) \times (0,1)$, then

$$\int_0^1 [\int_0^1 f(x, y) \, dy] \, dx = \int_0^1 [\int_0^1 f(x, y) \, dx] \, dy,$$

provided that both iterated integrals exists.

26. If $\mathcal{F}$ is a σ-field of sets then either $\mathcal{F}$ is finite or it is uncountable.

27. If X and Y are independent non-negative random variables, then

$$E[e^{-(X^2+Y^2)}] = E[e^{-X^2}]E[e^{-Y^2}].$$