Cut Times for Brownian Motion and Random Walk

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Abstract

A cut time for a Brownian motion or a random walk is a time at which the past and the future of the process to not intersect. In this paper we review the work of Erdős on cut times and discuss more recent work on in this area.

1 Erdős and cut times

Let B_t be a Brownian motion taking values in $\mathbb{R}^d$, and let S_n be a simple random walk taking values in $\mathbb{Z}^d$. A cut time is a time such that the paths of the process before that time and after that time do not intersect. To be precise, we say that $t \in (0,1)$ is a cut time for Brownian motion on $[0,1]$ if

$$B_r \neq B_s, \quad 0 \leq r < t < s \leq 1,$$

i.e., if

$$B[0,t) \cap B(t,1] = \emptyset.$$

Similarly m is a cut time for simple random walk on $[0,n]$ if

$$S_j \neq S_k, \quad 0 \leq j \leq m < k \leq n,$$

i.e., if

$$S[0,m] \cap S[m+1,n] = \emptyset.$$

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The question as to whether a "typical" t is a cut time for Brownian motion was answered by Dvoretzky, Erdős, and Kakutani (referred to below as DEK) in [8]. They showed that if $0 < a_1 < b_1 < a_2 < b_2$, then

$$\mathbf{P}\{B[a_1, b_1] \cap B[a_2, b_2] \neq \emptyset\} \begin{cases} = 0, & d \geq 4, \\ > 0, & d = 1, 2, 3. \end{cases} \tag{1}$$

By considering rational values of a_1, b_1, a_2, b_2, it is easy to see that this result implies that if $d \geq 4$ with probability one all $t \in (0, 1)$ are cut times for $[0, 1]$, For $d = 1, 2, 3$, a little more work will convince one that (1) implies that for any fixed $t \in (0, 1)$,

$$\mathbf{P}\{t \text{ is a cut time on } [0, 1]\} = 0. \tag{2}$$

However, it does not follow from this that with probability one there are no cut times. A simple application of Fubini's Theorem does tell us that (2) implies that with probability one the Lebesgue measure of the set of cut times is zero.

The hard cases in (1) were dimensions 3 and 4. If $d = 1, 2$, it is easy to show that there is a positive probability of intersection, and if $d \geq 5$ a straightforward estimate using the Green's function for Brownian motion (fundamental solution of the Laplacian) had already been used to show there was no intersection. The proof of (1) used capacity; in particular, it was shown that with probability one a Brownain path has zero capacity for $d \geq 4$ but positive capacity for $d \leq 3$. The close relationship between Brownian motion and harmonic functions was already known from work of Kakutani; in particular, Brownian paths have a positive probability of hitting a set if and only if the set has positive capacity. A Brownian path is a set of Hausdorff dimension two with zero 2-dimensional measure. Four is the "critical dimension" for Brownian intersections since this is the critical dimension for two "two dimensional" sets to intersect.

What happens when three intervals are considered? DEK and Taylor [12] showed that if $0 < a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < 1$,

$$\mathbf{P}\{B[a_1, b_1] \cap B[a_2, b_2] \cap B[a_3, b_3] \neq \emptyset\} \quad \begin{cases} = 0, & d = 3 \\ > 0, & d = 2. \end{cases}$$

The argument was again based on capacity. If $d = 3$, $B[a_1, b_1] \cap B[a_2, b_2]$, the intersection of two sets of dimension 2, has Hausdorff dimension one but has zero capacity for a similar reason that a single path has zero capacity in four dimensons. In three dimensions, "double points" exist but "triple points" do not. In two dimensions, the heuristic argument shows that the intersection of k Brownian paths should still be a two dimensional set, assuming that the intersection is nonempty. DEK [9, 10] showed that planar

Brownian paths have multiple points of all orders, and, in fact, that there are multiple points of infinite order, both countable and uncountable.

Even though a typical time for a Brownian path is not a cut time, one can still ask if a path has some cut times. DEK studied this question for $d = 1$ in [11]. In one dimension, a cut time can be considered as a point of increase or point of decrease of the Brownian motion. There is an intriguing heuristic argument as to why a path should have points of increase. Let τ be the time at which the Brownian motion obtains its maximum value on $[0, 1]$. With probability one, $0 < \tau < 1$. Consider the process "reflected at the maximum":

$$X_t = \begin{cases} B_t, & 0 \leq t \leq \tau \\ B_\tau + (B_\tau - B_t), & \tau \leq t \leq 1 \end{cases}$$

Note that τ is a cut time for X_t on $[0, 1]$. Since B_t and $-B_t$ have the same distribution, one might hope that X_t has the same distribution as a Brownian motion. Unfortunately, this is not true. One easy way to see this is that the distribution of the maximum of X_t on $[0, 1]$ is not the same as the distrbution of the maximum of B_t. However. one might hope to use this basic reflection idea to show the existence of cutpoints. In trying to make a rigorous argument out of this heuristic, DEK discovered in fact that with probability one the Brownian path in one dimension has no cut times. There are now very short proofs of this fact (see, e.g., [3]). If $\epsilon > 0$, let $J_{\epsilon,n}$ be the number of intervals of the form

$$[\frac{j}{n}, \frac{j+1}{n}], \quad \epsilon \leq \frac{j}{n} \leq 1 - \epsilon,$$

with

$$B[0, \frac{j}{n}] \cap B[\frac{j+1}{n}, 1] = \emptyset.$$

The routine estimates for one dimensional Brownian motion (details omitted) can be used to show that for every $\epsilon > 0$

$$\mathbf{E}[J_{\epsilon,n}] \leq c_\epsilon,$$

$$\mathbf{E}[J_{\epsilon,n} \mid J_{2\epsilon,n} > 0] \geq C_\epsilon \log n,$$

and hence

$$\lim_{n \to \infty} \mathbf{P}\{J_{2\epsilon,n} > 0\} = 0.$$

The problem with the heuristic argument is that that random time τ is not what is called now a "stopping time" for Brownian motion. Stopping times are random times where the decision to stop at a certain time must be made by only looking at the information up to that time. For stopping times, the reflection idea described above is valid and does give a process with the same distribution as a Brownian motion. However, the time at which a process reaches its maximum is not a stopping time since one must look into the future to see if a certain time is the maximum.

Erdős and Taylor [13] considered an analogous question about cut times for simple random walk. Let S^1, S^2 be independent simple random walks starting at the origin in $\mathbb{Z}^d$ and let

$$p(n) = p_d(n) = \mathbf{P}\{S^1(j) \neq S^2(k) : 0 \leq j \leq n, 1 \leq k \leq n\}.$$

Note that $p(n)$ is the probability that n is a cut time for a simple random walk on $[0, 2n]$. They showed that

$$\lim_{n\to\infty} p(n) = p(\infty) \begin{cases} > 0, & d \geq 5 \\ = 0, & d \leq 4. \end{cases}$$

As in the case of Brownian motion, four is the critical dimension, but paths do intersect eventually in four dimensions. This may initially be surprising, but a similar thing happens in the well known case of recurrence of random walk where the critical dimension is two. Simple random walk returns infinitely often to the origin in two dimensions while Brownian motion gets arbitrarily close to the origin without reaching the origin. (In four dimensions, the paths of two Brownian motions starting at different points run for infinite time do not intersect but the distance between the paths is zero.) Erdős and Taylor studied the question as to how $p(n)$ tends to zero for $d \leq 4$. They were able to give a bound in one direction. Let K_n be the total number of intersections

$$K_n = \sum_{j=0}^{n} \sum_{k=1}^{n} I\{S^1(j) = S^2(k)\},$$

where I denotes the indicator function. Then it is straightforward to show that

$$\mathbf{E}[K_n] \asymp \begin{cases} \log n, & d = 4, \\ n^{d/2}, & d < 4, \end{cases}$$

where $\asymp$ means that both sides are bounded by a constant times the other side. They showed that

$$p(n) \geq \mathbf{E}[K_n]^{-1}. \tag{3}$$

They were unable to get a bound in the other direction, and it was not clear from their proof of (3) whether this should be giving the right order of decay. As in the case of Brownian cut times in one dimension, there was a heuristic argument using a random time which would indicate this might give the right order of decay, but the random time was not a stopping time. What one would want to do is consider the "first" time that the paths intersected. Unfortunately there were two time scales involved and hence there was not a well defined "first" time.

2 Random walk intersections

In this section we will describe what is known today about the function $p(n)$ in the last section. As it turns out, the lower bound in (3) does not give the correct order of decay. We first consider a different problem. Let S^1, S^2, S^3 be three independent simple random walks starting at the origin and for notational ease write

$$\omega_n^1 = S^1[0,n],\ \omega_n^2 = S^2[1,n],\ \omega_n^3 = S^3[1,n].$$

Then

$$p(n) = \mathbf{P}\{\omega_n^1 \cap \omega_n^2 = \emptyset\},$$

Let

$$q(n) = \mathbf{P}\{\omega_n^1 \cap [\omega_n^2 \cup \omega_n^3] = \emptyset\}.$$

Note that if we consider the random variable depending only on the first path:

$$Y_n(\omega_1) = \mathbf{P}\{\omega_n^1 \cap \omega_n^2 = \emptyset \mid \omega_n^1\},$$

then

$$p(n) = \mathbf{E}[Y_n], \qquad q(n) = \mathbf{E}[Y_n^2]. \tag{4}$$

It turns out that $q(n)$ is an easier quantity to estimate than $p(n)$ and its rate of decay is given by $\mathbf{E}[K_n]^{-1}$ (see [15]),

$$q(n) \asymp \begin{cases} (\log n)^{-1}, & d = 4 \\ n^{(d-4)/2}, & d < 4. \end{cases} \tag{5}$$

The proof of this estimate starts by taking two random walks a reasonable distance apart and considering the first time (measured on the the time scale of the first walker) that the first walk hits the path of the second walk. The point at which the walk hits the second path tends to be in the "middle" of the second path and hence locally at the hitting point the second path looks like the union of two random walks. Note that (4) implies

$$q(n) \le p(n) \le q(n)^{1/2}.$$

Random walk intersections are analogous to a number of models in statistical physics such as self-avoiding walks (polymers), percolation, Ising model, and loop-erased walks (uniform spanning trees, dimers). These models are characterized by the following conjectured behavior:

- A "critical dimension" above which the behavior is in some sense trivial (mean-field). In the case of random walk intersections this dimension is four.

- "Mean-field" behavior holds at the critical dimension with logarithmic corretions.

- Below the critical dimension, the behavior is not mean-field and there are nontrivial critical exponents.

- In two dimensions, there are conjectured rational values for the exponents derived from nonrigorous conformal field theory.

- In three dimensions (more generally, dimensions above two and less than the critical dimension), the exponents do not appear to be rational and probably can not be given precisely.

It is sometimes vague what is meant by mean-field behavior, but in the case of random walk intersections we can state mean-field behavior as

$$\mathbf{E}[Y_n^2] \approx \mathbf{E}[Y_n]^2,$$

or equivalently,

$$\mathbf{P}\{\omega_n^1 \cap \omega_n^3 = \emptyset \mid \omega_n^1 \cap \omega_n^2 = \emptyset\} \approx \mathbf{P}\{\omega_n^1 \cap \omega_n^3 = \emptyset\}.$$

It has been shown [15, 16] that mean-field behavior does hold at the critical dimenion $d = 4$,

$$p(n) \asymp q(n)^{1/2} \asymp (\log n)^{-1/2}.$$

For $d = 2, 3$, we would expect that

$$p(n) \asymp n^{-\zeta} \tag{6}$$

for some $\zeta = \zeta_d$, and if mean-field behavior does not hold we would conjecture

$$\zeta > \frac{4-d}{2}.$$

In fact, it can be shown [4, 17] that (6) holds where $\zeta = \xi(1,1)/2$ and $\xi(1,1)$ is the intersection exponent for Brownian motion as described in the next section. It is not difficult to show that the Brownian exponent exists using scaling and subadditivity. There seems to be no direct proof of the existence of ζ; the existence is shown by proving that $\zeta = \xi(1,1)/2$.

3 Brownian intersection exponent

Let $B^1, \ldots, B^k, B$ be independent Brownian motions in $\mathbb{R}^d$ ($d = 2, 3$) starting at the origin and let

$$T_r^j = \inf\{t : |B_t^j| = r\}, \quad T_r = \inf\{t : |B_t| = r\}.$$

$$\Gamma_r = \Gamma_{r,k} = B^1[T_1^1, T_r^1] \cup \cdots \cup B^k[T_1^k, T_r^k].$$

Let $Z_r = Z_{r,k}$ be the random variable

$$Z_r = \mathbf{P}\{B[T_1, T_r] \cap \Gamma_r = \emptyset \mid \Gamma_r\}.$$

If $\lambda > 0$, the intersection exponent $\xi = \xi(k, \lambda) = \xi_d(k, \lambda)$ is defined by

$$\mathbf{E}[Z_r^\lambda] \approx r^{-\xi}, \quad r \to \infty,$$

where $\approx$ denotes logarithmic asymptotics, i.e., the logarithms of boths sides are asymptotic. Subadditivity and scaling can be used to show the exponent exists; with more work [17, 22], one can show that the $\approx$ can be replaced with $\asymp$. If $d = 2$, the disconnection exponent is defined by taking $\lambda = 0$,

$$\mathbf{P}\{Z_r > 0\} \asymp r^{-\xi_2(k,0)}, \quad r \to \infty.$$

While $\xi(k, \lambda)$ is known to exist, the only rigorously known value is the Brownian motion analogue to (5),

$$\xi_d(1, 2) = \xi_d(2, 1) = 4 - d.$$

(The factor of two in the exponent comes from the fact that it takes time of order r^2 to go distance r.) The best rigorous bounds for $\xi_d(1, 1)$ are [5, 22]

$$1 + \frac{1}{4\pi} < \xi_2(1, 1) < \frac{3}{2},$$

$$\frac{1}{2} < \xi_3(1, 1) < 1.$$

The mean-field estimate would be $\xi_d(1, 1) = (4 - d)/2$ so we do know rigorously that the behavior is not mean-field. Duplantier and Kwon [7] used nonrigorous conformal field theory to make the conjecture $\xi_2(1, 1) = 5/4$ and numerical simulations are consistent with this conjecture. As stated before, there is no reason to believe that the exponent is rational in three dimensions; numerical simulation [6] suggest that $\xi_3(1, 1)$ is about .58. The work of Duplantier and Kwon also gives a prediction for the disconnection exponent $\xi_2(1, 0) = 1/4$, and Mandelbrot [25] made a conjecture that is equivalent (see the next section) to the conjecture $\xi_2(2, 0) = 2/3$. Recently

[23] the conjectures of Duplantier and Kwon were combined with rigorous results to give the conjecture

$$\xi_2(k,\lambda) = \frac{[\sqrt{24k+1}+\sqrt{24\lambda+1}][\sqrt{24k+1}+\sqrt{24\lambda+1}-4]}{48}. \qquad (7)$$

Current work on the intersection exponent is focused on finding a way to use rigorous conformal invariance to establish (7). Also, there is good reason to believe (see, e.g., [24]) that exponents for percolation and self-avoiding walks in two dimensions (two models that are very far from being understood rigorously) are very closely related to those of Brownian motion.

4 Dimension of cut times

Burdzy [2] gave the first argument to show that two dimensional (and hence three dimensional) Brownian paths do have cut times. More recently, it was proved [17], with probability one, the set of cut times in two and three dimensions is $1-(\xi_d(1,1)/2)$. While we still do not know the value of this exponent, the rigorous bounds to tell us that the dimension is strictly between 0 and 1. The set of cut points (images of cut times) has dimension $2-\xi_d(1,1)$. This follows from the fact (first observed by Kaufman [14]) that the image of a set under Brownian motion is twice the dimension of the set, even for sets that are defined in terms of the path.

There is a related question first posed by Mandelbrot [25] that at first glance might not seem related. Consider the frontier of the planar Brownian motion $B[0,1]$, which is defined as the boundary of the unbounded component of the complement. Mandelbrot, making an analogy with self-avoiding walks, conjectured that this dimension was $4/3$. It has been shown [19] that the Hausdorff dimension of the boundary is given by $2-\xi_2(2,0)$ and so the Mandelbrot conjecture is a particular case of the conjecture (7). The best rigorous estimates [1, 5, 19, 26] tell us that the dimension lies in the interval $(1.015, 1.48)$. The exponent $\xi(2,\lambda)$ can be used to give another geometric property of the Brownian path, the multifractal spectrum of harmonic measure of the path [20].

One can ask questions about the “dimension” of random walk cut times. One problem, which was first posed in [13], was the behavior for $d \geq 3$ of

$$R_n = \#\{1 \leq j \leq n : S[0,j] \cap S[j+1,\infty) = \emptyset\}.$$

It can be shown [18] that with probability one

$$\frac{R_n}{n} \to p_d(\infty) > 0, \quad d \geq 5,$$

$$\frac{R_n}{n} \approx (\log n)^{1/2}, \quad d = 4,$$
$$R_n \approx n^{\alpha}, \quad d = 3,$$

where $\alpha = 1 - (\xi_3(1,1)/2)$.

Results on the dimension and multifractal spectrum of the boundary of simple random walk paths in the plane can be found in [21].

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