Stationary Solutions of Stochastic Differential Equation with Memory and Stochastic Partial Differential Equations.

Yuri Bakhtin* Jonathan C. Mattingly †

May 20th, 2003

Abstract

We explore Itô stochastic differential equations where the drift term has possibly infinite dependence on the past. Assuming the existence of a Lyapunov function, we prove the existence of a stationary solution assuming only minimal continuity of the coefficients. Uniqueness of the stationary solution is proved if the dependence on the past decays sufficiently fast. The results of this paper are then applied to stochastically forced dissipative partial differential equations such as the stochastic Navier-Stokes equation and stochastic Ginsburg-Landau equation.

Keywords: stochastic differential equations, memory, Lyapunov functions, ergodicity, stationary solutions, stochastic Navier-Stokes equation, stochastic Ginsburg-Landau equation.

1 Introduction

This note explores the ergodic theory of Itô stochastic differential equations with memory. Specifically we consider equations on $\mathbb{R}^d$ with additive noise of the form

$$dX(t) = a(\pi_t X)dt + dW(t). \tag{1}$$

Here $W(t), t \in \mathbb{R}$ is standard $d-$dimensional Wiener process (i.e. a Gaussian $\mathbb{R}^d$-valued stochastic process with continuous trajectories defined on the

*School of Math, Institute for Advanced Study, Princeton, NJ 08540, USA and International Institute for Earthquake Prediction Theory and Mathematical Geophysics, Warshavskoye sh. 79, kor. 2, Moscow 113556, Russia

†School of Math, Institute for Advanced Study, Princeton, NJ 08540, USA and Department of Math, Duke University, Durham, NC 27708, USA

whole real line $\mathbb{R}$ with independent and stationary increments, $W(0) = 0$, $\mathsf{E}W(t) = 0$, $\mathsf{E}W_i(t)W_j(t) = |t|\delta_{ij}, t \in \mathbb{R}$, $i, j = 1, \ldots, d$). The projection and shift π_t is a map from the space C of $\mathbb{R}^d$-valued continuous functions defined on $\mathbb{R}$ to the space C^- of continuous functions defined on $\mathbb{R}_- = (-\infty, 0]$:

$$\pi_t X(s) = X(s+t), \quad s \in \mathbb{R}_-.$$

This map gives the past of a continuous process up to time $t \in \mathbb{R}$. Lastly the map $a : C^- \to \mathbb{R}^d$ gives the effect of the past on the present moment of time.

First results concerning stationary solutions of SDEs of this type appeared in [IN64]. In this paper we give new sufficient conditions for existence and uniqueness of stationary solution for equation (1) which may be considered as non-Markovian ("Gibbsian") counterparts of those in [Ver97]. Here we do not address the question of mixing. However [Mat01] provides the needed framework to address this question in the non-Markovian setting.

We hope that our results will be useful in many different contexts. However, one of the main guiding examples has been recent progress in the ergodic theory of stochastic partial differential equations driven by white noise found in [BKL01, EMS01, EL02, Mat02]. In [KS00] and subsequent papers related ideas were developed independently in kicked noise setting. In the [MY02] the kicked setting is furtherer developed. In particular, we follow the ideas as laid out in the second reference. There using ideas of determining modes and inertial manifolds (see [FP67, Tem88]), the infinite dimensional diffusion is reduced to an Itô process with memory on a finite dimensional phase space. If the resulting Itô process is elliptic then the methods of this note can be used to establish uniqueness of stationary measure. In [Mat02, Hai02], similar ideas were used to control a convergence rate to the invariant measure; there the packaging was more Markovian. With the exception of [EMS01, Bak02] the memory is less explicit in the preceding works, here we bring it to the foreground and give general conditions under which the ideas are applicable. Similar ideas can be used to treat equations with state dependent diffusion matrices, provided that the system is uniformly elliptic.

After developing the general theory, we examine a number of pedagogical examples. Then we show how one can use the results of this paper to reduce a dynamical system to one of smaller dimension but with memory by removing stable dimensions. This reduced system can then be analyzed to understand the asymptotic properties of the system. The construction reflects the simple fact that the long term dynamics is dictated by the dynamics in the unstable directions. This reduction is particularly useful in the context of dissipative partial differential equations where the reduced system is finite dimensional; and hence, it has a simpler topological structure than the original equation. It is our imperfect understanding of the fine scale topological structure in the Markovian setting which limits our progress there. By switching to the finite dimensional setting where all relevant topologies are equivalent and Lebesgue measure exists, we can make progress.

2 Definitions and Main Results

Along with the set of pasts C^- defined in the introduction, we also define the set of futures C^+ as the space of $\mathbb{R}^d$-valued continuous functions on $[0,\infty)$. We denote by π_+ the natural projection from $C \to C^+$. For a stochastic process X and a set $A \subset \mathbb{R}$ the σ-algebra generated by random variables $X(s), s \in A$ will be denoted by $\sigma_A(X)$ and the σ-algebra generated by random variables $X(s)-X(t), s,t \in A$ will be denoted by $\sigma_A(dX)$. Consider the space $\Omega = C \times C$ with LU-topology defined by the metric

$$\rho(f,g) = \sum_{n=-\infty}^{\infty} 2^{-|n|}(\|f-g\|_n \wedge 1), \quad f,g \in C \times C,$$

where $\|(h_1,h_2)\|_n = \max_{-n\le t\le n}(|h_1(t)| + |h_2(t)|)$ for $(h_1,h_2) \in \Omega$ and $|\cdot|$ denotes the Euclidean norm.

Solutions of the SDE: A probability measure P on the space Ω with Borel σ-algebra $\mathcal{B}$ is said to define a SOLUTION to the equation (1) on some subset $\mathcal{R}$ of $\mathbb{R}$ if the following three conditions are fulfilled with respect to the measure P:

1. The projection $W : C \times C \to C$, $\omega = (\omega_1,\omega_2) \mapsto \omega_2$, is a standard d-dimensional Wiener process.

2. For any $t \in \mathbb{R}$

$$\sigma_{(-\infty,t]}(X) \vee \sigma_{(-\infty,t]}(dW) \quad \text{is independent of} \quad \sigma_{[t,\infty)}(dW). \tag{2}$$

   Here and further $X : C \times C \to C$, $\omega = (\omega_1,\omega_2) \mapsto \omega_1$.

3. If $s < t$, $s,t \in \mathcal{R}$ then

$$X(t) - X(s) \overset{\text{a.s.}}{=} \int_s^t a(\pi_\theta X)d\theta + W(t) - W(s). \tag{3}$$

The process X or the couple (X, W) will be also often referred to as solution for equation (1).

Stationary Solutions: If P defines a solution on $\mathbb{R}$ and in addition the distribution of the process

$$(X, dW) \equiv (X(t), -\infty < t < \infty, W(v) - W(u), -\infty < u < v < \infty)$$

does not change under time shifts then the measure P is said to define a STATIONARY SOLUTION.

Solution to Cauchy Problem: We will always assume strong existence and pathwise uniqueness of solution to Cauchy problem with initial data which grows sufficiently slow at $-\infty$.

More precisely, for $\rho > 0$ denote C_ρ^- the space of trajectories $x \in C^-$ for which

$$\|x\|_\rho = \sup_{t \in \mathbb{R}_-} \frac{|x(t)|}{1 + |t|^\rho} < \infty.$$

We will require that for some $\rho > 0$ and any $x \in C_\rho^-$ which possesses certain averaging property (which will be described after the definition of Lyapunov function below) there exists a measurable map $\Phi : C^+ \to C^+$ such that if W is a standard Wiener process under some measure P then (X, W) is a solution to (1) on $\mathbb{R}_+$ where $X(t) = x(t)\mathbf{1}_{\{t \le 0\}} + \Phi(\pi_+ W)(t)\mathbf{1}_{\{t > 0\}}$ which means that relations (2) and (3) are true on $\mathbb{R}_+$. Moreover, if under some measure P the process (X, W) is a solution to (1) on $\mathbb{R}_+$ and $X(t) = x(t)$ for $t \le 0$ then $X(t) = \Phi(\pi_+ W)(t), t > 0$ $P-$almost surely. We shall denote $P(\,\cdot \mid x)$ the distribution of the solution to Cauchy problem with initial data equal to $x \in C_\rho^-$.

A theorem providing global existence and uniqueness of solutions to Cauchy problem for the case where the drift coefficient is locally Lipschitz with respect to $\| \cdot \|_\rho$ is given in Appendix B. Some other existence and uniqueness results for Cauchy problem can be found in [IN64], [Pro90].

We clearly cannot proceed without some control on the growth of solutions in time. We obtain the control by assuming a Lyapunov–Foster structure in the problem. As we are concerned with equations with memory, we allow the Lyapunov functions to have memory.

Lyapunov Function: We will call a function $V : C^- \to \mathbb{R} \cup \{+\infty\}$ a LYAPUNOV FUNCTION for equation (1) if

1. $V(x) \ge C_0 |x(0)|^l$ for some $C_0, l > 0$ and all $x \in C^-$.

2. If a measure P defines a solution (X, W) of (1) on $[T_1, T_2] \subset \mathbb{R}$ and $P\{V(\pi_{T_1} X) < +\infty\} = 1$ then $P\{V(\pi_t X) < +\infty, s \in [T_1, T_2]\} = 1$ and $V(\pi_t X)$ satisfies the following Itô equation on this interval:

$$dV(\pi_t X) = h(\pi_t X)dt + f(\pi_t X)d\widetilde{W}(t).$$

Here $h : C^- \to \mathbb{R}$ is a function satisfying

$$h(x) < C_1 - C_2 V(x)^\gamma$$

for some constants $C_1, C_2, \gamma > 0$ and $f : C^- \to \mathbb{R}$ is a function satisfying

$$|f(x)| \leq C_3 V(x)^\delta,$$

for some $\delta \in [0, (1+\gamma)/2)$ and $C_3 > 0$. Finally $\widetilde{W}$ is a standard one-dimensional Wiener process adapted to the flow generated by (X, dW).

To show that this definition is natural, consider the Markov case where the drift coefficient $a(x) = a(x(0))$ depends only on the present of a trajectory $x \in C^-$. The results of [Ver97] imply that if

$$\langle a(x), x(0)\rangle \leq C_1 - C_2|x(0)|^\alpha \tag{4}$$

for some positive C_1, C_2 and α then there exists a stationary solution which is unique. But condition (4) means that $V(x) = x(0)^2$ is a Lyapunov function which immediately follows from the Itô formula:

$$\begin{aligned} d|X(t)|^2 =& 2\left[\langle a(\pi_t(X)), X(t)\rangle + \frac{d}{2}\right] dt + 2\langle X(t), dW(t)\rangle \\ \leq& 2\left[C_1 + \frac{d}{2} - C_2|X(t)|^\alpha\right] dt + 2|X(t)| \left\langle \frac{X(t)}{|X(t)|}, dW(t)\right\rangle. \end{aligned}$$

If a system possesses a Lyapunov function then it does not fluctuate very strongly along typical trajectories. In addition $V(\pi_t X)^\gamma$ averages with respect to time t to a value less than $\frac{C_1}{C_2}$ and the fluctuations of the average are not strong. To make this more precise we define for $x \in C^-$ the fluctuations of the Lyapunov function, denoted $\mathcal{F}V(x,t)$, by

$$\mathcal{F}V(x,t) = \left|\int_0^t V(\pi_s x)^\gamma ds\right| - \frac{C_1}{C_2}|t|.$$

For any $\rho > 0$ we define the set of NICE PATHS $\mathcal{N}_\rho$ by

$$\mathcal{N}_\rho = \left\{x \in C^- : \limsup_{t \leq 0} \frac{|V(\pi_t x)| + |\mathcal{F}V(x,t)|}{1 + |t|^\rho} < \infty\right\}$$

By the first property of a Lyapunov function $\mathcal{N}_\rho \subset C^-_{\rho/l}$. We shall say that $x \in C^-$ AVERAGES WELL if

$$\limsup_{T\to\infty} \frac{1}{T}\int_{-T}^{0} V(\pi_t x)^\gamma dt \le \frac{C_1}{C_2} .$$

Notice that if $\rho < 1$ then all of the path in $\mathcal{N}_\rho$ average well. Lastly we will say that a function $g : C^- \to \mathbb{R}^d$ is LOCALLY LIPSCHITZ ON $C^-_\rho \cap \mathcal{N}_r$ if it satisfies the Lipschitz condition with respect to the norm $\|\cdot\|_\rho$ on the set

$$\left\{ x \in C^- : \limsup_{t\le 0} \frac{|x(s)|}{1+|t|^\rho} + \frac{|V(\pi_t x)| + |\mathcal{F}V(x,t)|}{1+|t|^r} < K \right\} .$$

for any $K > 0$.

For any function F on C^-, denote $\mathcal{D}(F) = \{x \in C^- : F(x) < \infty\}$. We will often speak of a $\mathcal{D}(V)$-valued solution P on a set $\mathcal{R} \subset \mathbb{R}$. By this we mean, $P\{\pi_t X \in \mathcal{D}(V), t \in \mathcal{R}\} = 1$.

We now state our main existence and uniqueness theorems for stationary solution. In section 6, we give a few concrete systems where the central theorems of the paper apply. Almost all results in this paper deal only with $\mathcal{D}(V)$-valued solutions; with the exception of Theorem 2, we say nothing about the existence and uniqueness of other solutions.

Theorem 1 *Let the SDE* (1) *admit a Lyapunov function V and suppose there is $x_0 \in C^-$ such that $\bar{V}(x_0) = \sup_{t\in\mathbb{R}_-} V(\pi_t x_0) < \infty$. If there exists a finite Borel measure ν defined on subsets of $\mathbb{R}_-$ and constants $\beta, K > 0$ such that*

$$|a(x)| \le K + \int_{\mathbb{R}_-} |V(\pi_s x)|^\beta \nu(ds), \quad x \in C^-, \tag{5}$$

the drift coefficient $a(\cdot)$ is locally Lipschitz on $C^-_\rho \cap \mathcal{N}_r$ for some $\rho > 0$ and $r > \frac{1}{2}$ then there exist a probability measure P on the space Ω which defines a stationary $\mathcal{D}(V)$-valued solution of equation (1).

Theorem 2 *Consider a sequence of sets $(\mathcal{A}_n)_{n\in\mathbb{N}}$ such that $\mathcal{A}_n \subset \mathcal{A}_{n+1} \subset C^-$ and a sequence of sets $(\mathcal{B}_n)_{n\in\mathbb{N}}$ such that $\mathcal{B}_n \subset \mathcal{B}_{n+1} \subset C^+$. Denote $\mathcal{A}_\infty = \bigcup_{n=1}^\infty A_n$, $\mathcal{B}_\infty = \bigcup_{n=1}^\infty B_n$ and suppose that these sequences and a set $\mathcal{G} \subset C$ satisfy the following properties:*

1. *For any measure Q which defines a $\mathcal{G}$-valued stationary solution to equation* (1)

$$Q\{\pi_0 X \in \mathcal{A}_\infty\} = 1 \text{ and } Q\{\pi_+ X \in \mathcal{B}_\infty\} > 0.$$

2. *For any $n \in \mathbb{N}$ there exists a positive function $\mathcal{K}_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_{\mathbb{R}_+} \mathcal{K}_n(t)dt < \infty$ and for any pair of trajectories $x_1, x_2 \in \mathcal{A}_n$ and a trajectory $y \in \mathcal{B}_n$ with $x_1(0) = x_2(0) = y(0)$*

$$\left|a\big(\pi_t(x_1{:}y)\big) - a\big(\pi_t(x_2{:}y)\big)\right|^2 \le \mathcal{K}_n(t).$$

Then there is at most one $\mathcal{G}$-valued stationary solution.

Remark: The sequence of sets A_n and B_n can be replaced with a single pair of sets A and B where one only requires $Q\{\pi_0 X \in \mathcal{A}, \pi_+ X \in \mathcal{B}\} > 0$. However one must add the additional requirement that for any two stationary measure Q_1 and Q_2, $Q_1\{X(0) \in \cdot$ and $\pi_0 X \in \mathcal{A}, \pi_+ X \in \mathcal{B}\}$ is not singular relative to $Q_2\{X(0) \in \cdot$ and $\pi_0 X \in \mathcal{A}, \pi_+ X \in \mathcal{B}\}$.

The following theorem is a corollary of the proceeding one, but provides simpler to verify conditions which cover many settings.

Theorem 3 *Let the SDE* (1) *admit a Lyapunov function V. Fixing $\rho > 0$ and $r > \frac{1}{2}$, for $n \in \mathbb{N}$ define the sets*

$$\begin{aligned}
\mathcal{A}_n(\rho, r) &= \left\{ x \in C^- : \limsup_{t \le 0} \frac{|x(t)|}{1+|t|^\rho} + \frac{|V(\pi_t x)| + |\mathcal{F}V(x,t)|}{1+|t|^r} < n \right\} \\
\mathcal{B}_n(\rho, r) &= \big\{ y \in C^+ : \text{ for all } x \in \mathcal{A}_n(\rho, r) \\
&\qquad \left. \limsup_{t \ge 0} \frac{|y(t)|}{1+|t|^\rho} + \frac{|V(\pi_t(x{:}y))| + |\mathcal{F}V(x{:}y,t)|}{1+|t|^r} < n \right\} .
\end{aligned}$$

If condition 2) of Theorem 2 holds with these families of sets then there exists at most one $\mathcal{D}(V)$-valued stationary solution.

3 Existence of Stationary Solutions

The proof of the existence of stationary solutions will proceed through a weak-limit point argument applied to the Krylov–Bogoljubov measures (see [Sin94],[IN64]).

Proof of Theorem 1: Let P_0 denote a law defining a solution for the Cauchy problem for the initial data x_0 and P_s denote the time s-shift of this distribution i.e. a solution of the Cauchy problem subject to initial data $x_0(t-s)$ defined for $t \in (-\infty, s], s \in \mathbb{R}$. Formally $P_s = P_0\theta_s^{-1}$ where $\theta_s(f,g) = (\widetilde{f}, \widetilde{g})$, $\widetilde{f}(t) = f(t-s)$, $\widetilde{g}(t) = g(t-s) - g(-s)$.

Since the function $P_s(E)$ is measurable with respect to s for all $E \in \mathcal{B}$ (see [IN64]), for $T > 0$ one can define a probability measure

$$Q_T(\,\cdot\,) = \frac{1}{T}\int_{-T}^{0} P_s(\,\cdot\,)ds$$

on the space $(\Omega, \mathcal{B})$. We will show that the family of measures $\{Q_T\}$ is tight on a subset of Ω in an appropriate topology, which we shall describe now.

Denote C_ρ the set of trajectories x in C such that $\pi_0 x \in C_\rho^-$. Define the following metric on C_ρ:

$$d_\rho(x,y) = \sum_{n=1}^{\infty} 2^{-n}(1 \wedge \|\pi_n x - \pi_n y\|_\rho).$$

Finally equip the space $\Omega_\rho = C_\rho \times C$ with product topology of $d_\rho-$topology in C_ρ and LU-topology in C.

The general idea of proof of tightness in a space of continuous trajectories is to obtain uniform bounds for marginal distributions and for distributions of increments of the trajectories, see [Bil68].

Lemma 3.1 *For any $\kappa \geq 0$, $T \geq 1$ and $S \geq 0$ the moments $\mathsf{E}_{Q_T} V(\pi_t X)^\kappa$ are uniformly bounded for $t \in (-\infty, S]$.*

Proof of Lemma 3.1: In the sequel, we will write V_t instead of $V(\pi_t X)$ for brevity. We may here assume without loss of generality that $V(\,\cdot\,) \geq 1$ (indeed, if V is a Lyapunov function so is $V + 1$ with possibly different choice of constants C_1 and C_2) and hence the moments of negative order are uniformly bounded.

We prove first that for each positive κ there exists a finite constant N_κ such that for any $T > 1$ and any $t \in [-T, 0]$

$$\frac{1}{T}\mathsf{E}_{P_0}\int_0^{T+t} V_s^\kappa \leq N_\kappa. \tag{6}$$

Define $\tau_R(t) = t \wedge \tau_R$ where $\tau_R = \inf\{t : V_t \geq R\}$. Let $m \geq 1$. The Itô formula and the assumptions on Lyapunov function V imply that for $T > 0$

$$V^m_{\tau_R(T+t)} - V_0^m \leq m\int_0^{\tau_R(T+t)}\Big\{V_s^{m-1}[C_1 - C_2 V_s^\gamma] + \frac{(m-1)}{2}C_3^2 V_s^{m-2+2\delta}\Big\}ds + m\int_0^{\tau_R(T+t)} V_s^{m-1} f(\pi_t X)d\widetilde{W}(s).$$

holds P_0-a.s. Taking expectations of both sides, passing to limit $R \to \infty$, using the regularity of the solution ($\tau_R \to \infty$ as $R \to \infty$) established in the Appendix B we get

$$\mathsf{E}_{P_0}\left[V_{T+t}^m - V_0^m\right] \le mC_1 \mathsf{E}_{P_0} \int_0^{T+t} V_s^{m-1} ds$$
$$- mC_2 \mathsf{E}_{P_0} \int_0^{T+t} V_s^{m-1+\gamma} ds + \frac{m(m-1)}{2} C_3^2 \mathsf{E}_{P_0} \int_0^{T+t} V_s^{m-2+2\delta} ds.$$

Dividing both sides of this inequality by T and using positivity of V_{T+t} we obtain that

$$\frac{mC_2}{T} \mathsf{E}_{P_0} \int_0^{T+t} V_s^{m-1+\gamma} ds \le \frac{mC_1}{T} \mathsf{E}_{P_0} \int_0^{T+t} V_s^{m-1} ds$$
$$+ \frac{m(m-1)}{2T} C_3^2 \mathsf{E}_{P_0} \int_0^{T+t} V_s^{m-2+2\delta} ds + \frac{V(x_0)^m}{T}.$$

Since $\varepsilon = [1+\gamma-2\delta] \wedge \gamma > 0$ we can apply the last inequality iteratively thus extending the domain of applicability of (6) at each iteration by ε.

There are three cases to consider, namely $t \in (-\infty, -T)$, $t \in [-T, 0)$ and $t \in [0, S)$. Suppose now that $t \in [-T, 0]$. Then

$$\mathsf{E}_{Q_T} V_t^\kappa = \frac{1}{T} \mathsf{E}_{P_0} \int_t^{t+T} V_s^\kappa ds = \frac{1}{T} \mathsf{E}_{P_0} \int_0^{t+T} V_s^\kappa ds + \frac{t}{T} \bar{V}(x_0)^\kappa \le N_\kappa + \bar{V}(x_0)^\kappa$$

and the same estimate is obviously true if $t < -T$.

Suppose now $t \in [0, S]$. Then

$$\mathsf{E}_{Q_T} V_t^\kappa = \frac{1}{T} \mathsf{E}_{P_0} \int_t^{t+T} V_s^\kappa ds \le \frac{T+S}{T} \mathsf{E}_{Q_T} V_0^\kappa \le \frac{T+S}{T} [N_\kappa + \bar{V}(x_0)^\kappa].$$

So for any κ we have proved that $\mathsf{E}_{Q_T} V_t^\kappa$ is bounded uniformly with respect to $T > 1$ and $t \in (-\infty, S]$. □

Property 1 of Lyapunov function V and Lemma 3.1 imply that all the moments of X are also uniformly bounded.

Now let us estimate increments of the process X.

Lemma 3.2 *There exists a constant $C > 0$ so that for any t_1, t_2 and $T \ge 1$, one has $Q_T\{|X(t_2) - X(t_1)| > z\} \le C(z^{-4} + z^{-2})|t_2 - t_1|^2$.*

Proof of Lemma 3.2:

$$
\begin{aligned}
Q_T\{|X(t_2)-X(t_1)|>z\} \\
&\le Q_T\{|W(t_2)-W(t_1)|>z/2\}+Q_T\Big\{\int_{t_1}^{t_2}a(\pi_\theta X)d\theta>z/2\Big\} \\
&\le \frac{16}{z^4}\mathsf{E}_{Q_T}|W(t_2)-W(t_1)|^4+\frac{4}{z^2}\mathsf{E}_{Q_T}\Big(\int_{t_1}^{t_2}a(\pi_\theta X)d\theta\Big)^2. \qquad (7)
\end{aligned}
$$

The first term can be estimated through the well-known expression for moments of Gaussian distribution. To estimate the second term we use the Fubini theorem, elementary inequality $|xy|\le(x^2+y^2)/2$ and relation (5):

$$
\begin{aligned}
\mathsf{E}\int_{t_1}^{t_2}a\big(\pi_\theta(x)\big)d\theta &= \int_{t_1}^{t_2}\int_{t_1}^{t_2}\mathsf{E}a(\pi_{\theta_1}(x))a\big(\pi_{\theta_2}(x)\big)d\theta_1 d\theta_2 \\
&\le (t_2-t_1)^2\sup\mathsf{E}a\big(\pi_\theta(x)\big)^2 \\
&\le (t_2-t_1)^2\sup\mathsf{E}\Big(K+\int_{\mathbb{R}_-}V(t+s)^\beta\nu(ds)\Big)^2 \\
&\le (t_2-t_1)^2\Big(K^2+2K\int_{\mathbb{R}_-}\sup\mathsf{E}V(t+s)^\beta\nu(ds) \\
&\qquad +\int_{\mathbb{R}_-}\int_{\mathbb{R}_-}\sup\mathsf{E}V(t+s_1)^\beta V(t+s_2)^\beta\nu(ds_1)\nu(ds_2)\Big) \\
&\le (t_2-t_1)^2\left(K^2+2K(M+1)^\beta\nu(\mathbb{R}_-)+(M+1)^{2\beta}\nu(\mathbb{R}_-)^2\right). \qquad (8)
\end{aligned}
$$

Inequalities (7) and (8) imply

$$
Q_T\{|X(t_2)-X(t_1)|>z\}\le 48z^{-4}|t_2-t_1|^2+Cz^{-2}|t_2-t_1|^2.
$$

for a constant $C>0$. □

Since we have uniform moment estimates and uniform increment estimates in probability tightness of distributions of the process X under measures Q_T in the uniform topology on finite interval $[-S,S]$ for any $S>0$ is immediate consequence of [Bil68, Theorem 12.3].

To finish the proof of tightness in Ω_ρ we need the following lemma.

Lemma 3.3 *For any $\sigma>0$ and any $t\in\mathbb{R}$ the random variable $\|\pi_t X\|_\sigma$ is finite Q_T-a.s. for any $T>0$. Moreover, the family of distributions $Q_T\{\|\pi_t X\|_\sigma\in\ \cdot\ \}$ is tight.*

Proof of Lemma 3.3: We omit the proof which is similar to the proof of Lemma 3.4 below and Theorem 5 from the next section. It relies on uniform estimates of marginals and increments in probability and the Borel–Cantelli lemma. □

Since the distribution of W in C is the same under all measures Q_T, it suffices to demonstrate tightness of the distributions of X. Consider any $S > 0$ and fix $\varepsilon > 0$. Choose any $\sigma < \rho$ and use Lemma 3.3 to find $K_\varepsilon > 0$ such that

$$Q_T\{\|\pi_t X\|_\sigma > K_\varepsilon\} > 1 - \frac{\varepsilon}{2}.$$

For any $n \in N$ due to tightness of distribution of X on $[-n, S]$ one can choose a compact set $E_n \subset C_{[-n,S]}$ such that

$$Q_T\{X[-n, S] \in E_n\} > 1 - 2^{-n-1}\varepsilon.$$

Let $E_\infty = \big\{x \in C_{(-\infty,S]}\, |x[-n, S] \in E_n, n \in \mathbb{N} \text{ and } \|x\|_\sigma \le K_\varepsilon\big\}$.

Since $\sigma < \rho$ it is straightforward to show that E_∞ is compact in the norm $\|\pi_S\ \cdot\ \|_\rho$ and $Q_T\{\pi_S X \in E_\infty\} > 1 - \varepsilon$.

So for any $S \in \mathbb{N}$ and ε we can build a set $E^{(S)}$ which is compact in the corresponding norm and $Q_T\{X(-\infty, S] \in E^{(S)}\} > 1 - \varepsilon 2^{-S}$. Use the same construction to build a compact set $E \subset C_\rho$ with $Q_T\{X \in E\} > 1 - \varepsilon$.

So tightness of distributions of X in C_ρ under Q_T is proved. The classical Prokhorov theorem implies that $Q_{T_n} \overset{Law}{\to} Q_\infty$ when $n \to \infty$ for some infinitely increasing sequence $(T_n)_{n\in\mathbb{N}}$.

To conclude the proof of Theorem 1 we need to establish some properties of the trajectories on which Q_∞ is concentrated.

Lemma 3.4 *For any $\varkappa > \frac{1}{2}$ and $\rho > 0$*

$$Q_\infty\left\{\sup_{t\le 0} \frac{|\mathcal{F}V(X,t)|}{1+|t|^\varkappa} < \infty\right\} = Q_\infty\left\{\sup_{t\le 0} \frac{V(\pi_t X)}{1+|t|^\rho} < \infty\right\} = 1\ .$$

Proof of Lemma 3.4:

First notice that due to the construction of Q_T as an average over initial value problems, if T be chosen so that $n+1 < T2^{-n}$ then for any set $A \in \Omega$ the probability $Q_T(A)$ is bounded by $Q_T(A\cap E)+2^{-n}$ where for any $\omega \in E$ the equation (1) is satisfied on $[-n, 0]$. This will allow us to estimate Q_T using the dynamics up to a small error.

We begin with the first claim controlling $\mathcal{F}V(\pi_t X)$. The claim is implied by

$$\sum_{n=1}^{\infty} Q_\infty\left\{\int_{-S}^{0} V_s^\gamma ds - \frac{C_1}{C_2} S > 3n^\varkappa, S \in [-n-1, -n]\right\} < \infty$$

which in turn follows from

$$\limsup_{m\to\infty} Q_{T_m}\left\{\int_{-S}^{0} V_s^\gamma ds - \frac{C_1}{C_2}S > 3n^{\varkappa}, S\in[-n-1,-n]\right\} \le q_n \tag{9}$$

and $\sum_n q_n < \infty$.

Let T_m be chosen so that $n+1 < T_m 2^{-n}$ then as mentioned above for any set $A \in \Omega$ the probability $Q_{T_m}(A)$ is bounded by $Q_{T_m}(A\cap E) + 2^{-n}$ where for any $\omega \in E$ the equation (1) is satisfied on $[-n,0]$ and hence on E one has

$$V_0 - V_{-S} \le \int_{-S}^{0}(C_1 - C_2 V_t^\gamma)dt + \int_{-S}^{0} f(\pi_t X)d\widetilde{W}(t).$$

So,

$$\int_{-n-1}^{0} V_t^\gamma dt - \frac{C_1}{C_2}(n+1) \le \frac{V_{-n-1}-V_0}{C_2} + \frac{1}{C_2}\int_{-n-1}^{0} f(\pi_t X)d\widetilde{W}(t)$$

and

$$\begin{aligned} Q_{T_m}&\left\{\int_{-n-1}^{0} V_t^\gamma dt - \frac{C_1}{C_2} > 2n^{\varkappa}\right\} \le Q_{T_m}\{V_{-n-1}-V_0 > C_2 n^{\varkappa}\} \\ &+ Q_{T_m}\left\{\int_{-n-1}^{0} f(\pi_t X)d\widetilde{W}(t) > C_2 n^{\varkappa}\right\} \le C(p)n^{p\left(\frac{1}{2}-\varkappa\right)} \le C(p)n^{-2} \end{aligned} \tag{10}$$

for $\varkappa > 1/2$ and large enough p where the last line follows from uniform boundedness of all moments of V and Burkholder's inequality (see [Pro90, Theorem 54]).

$$\begin{aligned} Q_{T_m}&\left\{\sup_{S\in[-n-1,-n]}\left[\int_{-n-1}^{S} V_s^\gamma ds - \frac{C_1}{C_2}(S+n+1)\right] > n^{\varkappa}\right\} \\ &\le Q_{T_m}\left\{\int_{-n-1}^{-n} V_s^\gamma ds > n^{\varkappa}\right\} \le Cn^{-2}. \end{aligned} \tag{11}$$

where we used boundedness of all moments of V in the last estimate.

Now (9) with $q_n = Cn^{-2} + 2^{-n}$ follows from (10) and (11) and control of the time average fluctuations claimed by Lemma 3.4 is proved.

We now turn to the remaining claim controlling $V(\pi_t X)$. Choosing T_m as above, for $-n \in \mathbb{N}$

$$Q_{T_m}\left\{\sup_{t\in[n,n+1]} V_t > K(|n|^\rho+1)\right\} \le Q_{T_m}\left\{V(n) > \frac{K}{2}(|n|^\rho+1)\right\} + Q_{T_m}\left\{\sup_{t\in[n,n+1]} V_t - V(n) > \frac{K}{2}(|n|^\rho+1)\right\} = I_1 + I_2. \quad (12)$$

Next we estimate both terms using Chebyshev's inequality and the uniform bounds from Lemma 3.1.

$$I_1 \le \frac{\mathsf{E}_{Q_{T_m}} V(n)^\kappa}{\frac{K}{2}(|n|^\rho+1)^\kappa} \le \frac{C}{n^2+1}$$

for some constant $C > 0$ if $\kappa > 2/\rho$.

$$I_2 \le Q_{T_m}\left\{\sup_{t\in[n,n+1]} \int_n^t f(\pi_s X)d\widetilde{W}(s) > \frac{K}{2}(|n|^\rho+1) - C_1\right\} \le \frac{C\mathsf{E}_{Q_{T_m}}\left[\sup_{t\in[n,n+1]}\int_n^t f(\pi_s X)d\widetilde{W}(s)\right]^{2p}}{n^2+1} \quad (13)$$

for some constant $C > 0$ and $p \in \mathbb{N}$ such that $2p\rho > 2$.

To prove that the expectation in the right-hand side of (13) is finite use Burkholder's inequality and the uniform estimates on $E_{Q_{T_m}} V_t^p$ given by Lemma 3.1:

$$\begin{aligned}
\mathsf{E}_{Q_{T_m}}\left[\sup_{t\in[n,n+1]}\int_n^t f(\pi_s X)d\widetilde{W}(s)\right]^{2p} &\le K_{2p}\mathsf{E}_{Q_{T_m}}\left[\int_n^{n+1} f^2(\pi_s X)ds\right]^p \quad (14)\\
&= K_{2p}\mathsf{E}_{Q_{T_m}}\int_n^{n+1}\dots\int_n^{n+1} f^2(\pi_{s_1}X)\dots f^2(\pi_{s_1}X)ds_1\dots ds_p\\
&\le K_{2p}\mathsf{E}_{Q_{T_m}}\int_n^{n+1}\dots\int_n^{n+1}\left[f^{2p}(\pi_{s_1}X)+\dots+f^{2p}(\pi_{s_1}X)\right]ds_1\dots ds_p\\
&\le K_{2p}pC < \infty\ .
\end{aligned}$$

To complete the proof apply (12)–(13) and the Borel–Cantelli lemma as in the first part. □

Completion of Proof of Theorem 1: All that remains in the proof of Theorem 1 is to show that Q_∞ is concentrated on solutions which solve the equation. From Lemma 3.4, we see that for any $\rho > 0$ and $r > \frac{1}{2}$, $Q_\infty\{X \in C_-^\rho \cap \mathcal{N}_r\} = 1$. Since the drift coefficient $a(\,\cdot\,)$ is continuous on the set of such paths, the reasoning from [IN64, p.21–25] shows that Q_∞ defines a stationary solution of the equation (1). Theorem 1 is proved. □

4 Properties of Stationary Solutions.

Before turning to the question of what additional requirements are sufficient to guarantee the uniqueness of the stationary measure, we extract a number of important properties which any stationary measure must possess given the assumptions already made. Specifically, we give bounds on the moments and growth to the Lyapunov function in time, prove that the averaging property is fulfilled a.s. and characterize the marginals of any stationary measure at a fixed given time.

4.1 Control of Moments and Asymptotic Path Behavior

Theorem 4 *Under the conditions of Theorem 1, for any $\kappa \in \mathbb{R}$ there exist a single, fixed constant M_k so that*

$$\mathsf{E}_Q V_t^\kappa \le M_\kappa < \infty$$

under any measure Q which defines a stationary $\mathcal{D}(V)$-valued solution X of equation (1)*.*

Proof: Let $g_{N,m}(x) = (x \wedge N)^\kappa$ for $x \in \mathbb{R}, m, N > 0$. Apply the Itô-Meyer formula (see [Pro90, Theorem 51]) to $g_{N,m}(V_t)$:

$$\begin{aligned} &g_{N,m}(V((T)) - g_{N,m}(V((T)) \\ &\quad\le \int_0^T \mathbf{1}\{V_t \le N\} m \Big[V_t^{m-1}(C_1 - C_2 V_t^\gamma) + \frac{m-1}{2} C_3^2 V_t^{m-2+2\delta} \Big] dt + \\ &\qquad\qquad m \int_0^T V_t^{m-1} f(\pi_t X) \mathbf{1}\{V_t \le N\} d\widetilde{W}(t) - \psi(T). \end{aligned}$$

Here ψ is a non-decreasing function such that $\psi(0) = 0$.

Take expectations of both sides of the last inequality with respect to the stationary measure Q:

$$\begin{aligned} C_2 m \mathsf{E}_Q \int_0^T V_t^{m-1+\gamma} \mathbf{1}\{V_t \le N\} dt \\ \le C_1 m \mathsf{E}_Q \int_0^T V_t^{m-1} \mathbf{1}\{V_t \le N\} dt \\ + \frac{m(m-1)}{2} C_3^2 \mathsf{E}_Q \int_0^T V_t^{m-2+2\delta} \mathbf{1}\{V_t \le N\} dt. \end{aligned}$$

Take the limit $N \to \infty$ and use stationarity of Q to get

$$\mathsf{E}_Q V_t^{m-1+\gamma} \le \frac{C_1}{C_2} \mathsf{E}_Q V_t^{m-1} + \frac{(m-1)C_3^2}{2C_2} \mathsf{E}_Q V_t^{m-2+2\delta}.$$

As in the previous section since $\varepsilon = 1 + \gamma - 2\delta > 0$ we can apply the last moment inequality iteratively to see that moments of V_t are bounded under the stationary measure Q. The proof is complete. □

Theorem 5 *Let the SDE* (1) *admit a Lyapunov function* V. *Suppose measure* Q *defines a stationary* $\mathcal{D}(V)$*-valued solution for* (1). *Then for any* $\rho > 0$

$$Q\left\{ \sup_t \frac{V(\pi_t X)}{1 + |t|^\rho} < \infty \right\} = 1.$$

PROOF: We proceed as in the proof of the second claim in Lemma 3.4. As in (12), For $n \in \mathbb{Z}$

$$\begin{aligned} Q\left\{ \sup_{t\in[n,n+1]} V_t > K(|n|^\rho + 1) \right\} \le Q\left\{ V(n) > \frac{K}{2}(|n|^\rho + 1) \right\} \\ + Q\left\{ \sup_{t\in[n,n+1]} V_t - V(n) > \frac{K}{2}(|n|^\rho + 1) \right\} = I_1 + I_2. \end{aligned} \quad (15)$$

Next we estimate both terms using Chebyshev's inequality.

$$I_1 \le \frac{\mathsf{E}_Q V(n)^\kappa}{\frac{K}{2}(|n|^\rho + 1)^\kappa} \le \frac{C}{n^2 + 1} \quad (16)$$

for some constant $C > 0$ if $\kappa > 2/\rho$. Calculations analogous to (13) and (14) give

$$I_2 \le \frac{K_{2p} p \mathsf{E}_Q f^{2p}(\pi_T X)}{n^2 + 1} \le \frac{K_{2p} p C_3^{2p} \mathsf{E}_Q V(T)^{2p\delta}}{n^2 + 1} \quad (17)$$

for some constant $C > 0$ and $p \in \mathbb{N}$, such that $2p\rho > 2$, and for any time $T \in \mathbb{R}$. (By stationarity the choice of T does not matter.) To complete the proof apply (15)–(17) and the Borel–Cantelli lemma. $\Box$.

Theorem 6 *Let the SDE* (1) *admit a Lyapunov function* V. *Suppose* Q *is a measure which defines a stationary* $\mathcal{D}(V)$*-valued solution to equation* (1). *Then for any* $\varkappa > \frac{1}{2}$

$$Q\left\{\sup_t \frac{|\mathcal{F}V(X,t)|}{1+|t|^{\varkappa}} < \infty\right\} = 1 \ .$$

PROOF: The needed calculations parallel those in the proof of the second part of Lemma 3.4; we give most of the details nonetheless. Using the definition of Lyapunov function we have

$$V_0 + \int_{-T}^{0} C_2 V_t^{\gamma} dt - C_1|T| \le V_{-T} + \int_{-T}^{0} f(\pi_t X) d\widetilde{W}(t)$$
$$V_T + \int_{0}^{T} C_2 V_t^{\gamma} dt - C_1|T| \le V_0 + \int_{0}^{T} f(\pi_t X) d\widetilde{W}(t)$$

Since by Theorem 5, V_{-T} grows slower than $T^{\frac{1}{2}}$ with probability 1 it suffices to show that the last term in each of the above lines grows no faster than $T^{\varkappa}$. Denote $I(t_1,t_2) = \int_{t_1}^{t_2} f(\pi_t X) d\widetilde{W}(t)$ for $t_1 < t_2$. We shall prove that

$$\sum_{n=1}^{\infty} Q\Big\{|I(-n,0)| > n^{\varkappa}\Big\} + Q\Big\{\sup_{s\in[-n,-n+1]} |I(-n,s)| > n^{\varkappa}\Big\} + \sum_{n=1}^{\infty} Q\Big\{\sup_{s\in[0,n+1]} |I(0,s)| > 2n^{\varkappa}\Big\} < \infty. \quad (18)$$

Since the sum

$$\sum_{n=1}^{\infty} Q\left\{\sup_{s\in[-n,-n+1]} |I(s,0)| > 2n^{\varkappa}\right\} + \sum_{n=0}^{\infty} Q\left\{\sup_{s\in[n,n+1]} |I(0,s)| > 2n^{\varkappa}\right\}$$

is majorized by by the previous one, the applying the Borel–Cantelli lemma will conclude the proof. To derive (18) use Burkholder's inequality and the fact that moments of V are uniformly bounded:

$$Q\left\{|I(-n,0)| > n^{\varkappa}\right\} \le n^{-2p\varkappa} C \mathsf{E}_Q \left[\int_{-n}^{0} V_t^{2\delta} dt\right]^p \le C n^{-2p\varkappa} n^p \le C n^{p(1-2\varkappa)}$$

and

$$Q\left\{\sup_{s\in[-n,-n+1]}|I(-n,s)|>n^{\varkappa}\right\}\le n^{-2p\varkappa}C\mathsf{E}_Q\left[\int_{-n}^{-n+1}V_t^{2\delta}dt\right]^p\le Cn^{-2p\varkappa}$$

$$Q\left\{\sup_{s\in[n,n+1]}|I(0,s)|>n^{\varkappa}\right\}\le n^{-2p\varkappa}C\mathsf{E}_Q\left[\int_{0}^{n}V_t^{2\delta}dt\right]^p\le Cn^{p(1-2\varkappa)}$$

where by C we denote possibly different constants. Since $\varkappa > 1/2$, (18) is finite as claimed and the theorem is proved. □

4.2 Regularity of Time t Marginals

For any $x \in C^-$, let $P_t(\,\cdot\,|\,x)$ denote the measure induced on $\mathbb{R}^d$ at time t by the dynamics starting from the past x at time 0. Similarly for any stationary measure Q denote the marginal at time t on $\mathbb{R}^d$ by $Q_t(A) = Q\{X(t) \in A\}$. By stationarity the measure Q_t is independent of t. The following Theorem along with Lemma 5.2 in the next section are the key elements in the uniqueness proof.

Theorem 7 *Let Q be any measure defining a stationary solution of equation (1). For Q-almost every $x \in C^-$ and every $t > 0$, $P_t(\,\cdot\,|\,x)$ is equivalent to the Lebesgue measure on $\mathbb{R}^d$. In addition, Q_s is equivalent to the Lebesgue measure on $\mathbb{R}^d$ for any $s \in \mathbb{R}$.*

PROOF: Since, by stationarity, $Q_t(A) = \int P_t(A\,|\,\pi_0 x)Q(dx)$ the equivalence of Q_t to the Lebesgue measure follows from that of P_t. To prove that $P_t(\,\cdot\,|\,x)$ is equivalent to the Lebesgue measure, it is sufficient to show that the distribution $P^X_{[0,t]}(\,\cdot\,|\,x)$ of the process X on the interval $[0,t]$ is equivalent to the distribution $P^W_{[0,t]}(\,\cdot\,|\,x)$ of standard Wiener process W started at $x(0)$. To apply Lemma A.1 from the appendix we need a truncation to guarantee the Novikov like condition in (30). We define the adapted function

$$T_R(X) = \inf\left\{s > 0 : \int_0^s |a(\pi_s X)|^2 ds > R\right\}\,.$$

By stationarity, we know that for Q-almost every initial condition x,

$$P(\exists R \text{ so } T_R(X) > t|\,x) = 1.$$

On the other hand for Q-almost every initial condition $x \in C^-$ and any

$$P^X_{[0,t]}(\,\cdot\,; T_R(X) > t|\,x) \sim P^W_{[0,t]}(\,\cdot\,; T_R(W) > t|\,x) \tag{19}$$

where $\sim$ denotes equivalence of measures. Indeed, the definition of the stopping time $T_R(X)$ guarantees the Novikov condition (30) and the equivalence in (19) is implied by Lemma A.1 with $\mathcal{B} = \{T_R(X) > t\}$. If $R \to \infty$ then the sequences of measures $P^X_{[0,t]}(\,\cdot\,; T_{R_n}(X) > t|\, x)$ and $P^W_{[0,t]}(\,\cdot\,; T_{R_n}(W) > t|\, x)$ increase to $P^X_{[0,t]}(\,\cdot\,|\, x)$ and $P^W_{[0,t]}(\,\cdot\,|\, x)$ respectively and equivalence is preserved under this limit. $\square$

5 Uniqueness of Stationary Solutions

The proof of Theorem 2 rests on the following lemma which in turn relies on the two lemmas which follow it and contain the heart of the matter. Given any stationary measure Q, for $B \subset C^+$, we define $Q_+(B;\mathcal{B}) = Q\{\pi_+ X \in B \cap \mathcal{B})$.

Lemma 5.1 *In the setting of Theorem 2, $Q_{1+}(\,\cdot\,;\mathcal{B}_\infty) \sim Q_{2+}(\,\cdot\,;\mathcal{B}_\infty)$ for any two measures Q_1 and Q_2 defining stationary $\mathcal{D}$-valued solutions to* (1).

Using this lemma, one quickly obtains a proof of Theorem 2. Once two stationary measures are shown not to be singular on the future then they must be the same measure. Lemma 5.1 gives the necessary equivalence, one way to see that this ensures uniqueness is given next.
Proof of Theorem 2: Consider two ergodic measures Q_1 and Q_2 defining stationary $\mathcal{D}$-valued solutions. Let $\phi : C \to \mathbb{R}$ be an arbitrary bounded functional which depends on values of its argument within a finite interval. By the Birkhoff–Khinchin ergodic theorem (see for instance [Sin94]), there exists deterministic constants ϕ_i so that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(\pi_t X)dt = \int \phi(x)dQ_i(x) = \phi_i$$

Q_i-almost surely. Let $C^{(i)}$ be subsets of C^+ of full Q_i-measure so that the above limit on the left hand side converges to the given constant ϕ_i. Since $Q_1\{\pi_+ X \in C^{(1)}\} = 1$, we know that $Q_1\{\pi_+ X \in C^{(1)} \cap \mathcal{B}_\infty\} > 0$. Then Lemma 5.1 implies that $Q_2\{\pi_+ X \in C^{(1)} \cap \mathcal{B}_\infty\} > 0$. Since $Q_2\{\pi_+ X \in C^{(2)}\} = 1$, we have that $Q_2\{\pi_+ X \in C^{(1)} \cap C^{(2)} \cap \mathcal{B}_\infty\} > 0$. Hence $C^{(1)} \cap C^{(2)}$ is non-empty implying that $\phi_1 = \phi_2$. Since the ϕ was arbitrary, we conclude that the distribution of X under Q_1 and Q_2 is the same. Due to the general fact that any stationary measure can be represented as a convex combination of ergodic ones the distribution of X is the same for any measure Q defining a stationary $\mathcal{D}$-valued solution (the ergodic decomposition of a measure defining a $\mathcal{D}$-valued solution gives zero weight to ergodic measures defining solutions which are not $\mathcal{D}$-valued).

Since the trajectory of X on $(0,\infty)$ is fully determined by $\pi_0 X$ and the trajectory of B, the distribution of $\pi_0 X$ determines uniquely the joint distribution of X and W and the theorem is proved. $\square$

Lemma 5.1 which was pivotal in the preceding proof is itself a consequence of the following lemma which along with Theorem 7 contain the central steps in the uniqueness proof. We will give its proof directly after its statement, returning to the proof of Lemma 5.1 at the end of the section.

Lemma 5.2 *Suppose that assumptions of Theorem 2 are satisfied. For fixed* $n \in \mathbb{N}$, *if* $x_1, x_2 \in \mathcal{A}_n$ *and* $x_1(0) = x_2(0)$ *then* $P(\,\cdot\,;\mathcal{B}_n|x_1) \sim P(\,\cdot\,;\mathcal{B}_n|\,x_2)$ *where* $P(B;\mathcal{B}_n|\,x) = P\{X \in B \cap \mathcal{B}_n|\,x\}$ *for* $x \in \mathcal{A}_n$ *and* $B \subset C^+$.

Proof of Lemma 5.2 : We are going to derive this lemma from Lemma A.1 in the appendix. Set $f_i(t, y[0,t]) = a(\pi_t(x_i{:}y)), i = 1, 2$. Notice that for a fixed t, $f_i(t,\,\cdot\,)$ can be thought of as functions from $C([0,t];\mathbb{R}^d)$ to $\mathbb{R}^d$ since the part of each $\pi_t(x_i{:}y)$ before 0 is fixed.

Since

$$\exp\left\{\frac{1}{2}\int_0^\infty \left|a\big(\pi_t(x_1{:}y)\big) - a\big(\pi_t(x_2{:}y)\big)\right|^2 dt\right\} \mathbf{1}_{\mathcal{B}_n}(x) \leq \exp\left\{\frac{1}{2}\int_0^\infty \mathcal{K}_n(t)dt\right\} < \infty \quad (20)$$

condition (30) is satisfied and Lemma A.1 now implies the desired equivalence of measures. $\square$

With Lemma 5.2 proved, we return to the proof of Lemma 5.1.
Proof of Lemma 5.1: It is sufficient to show that for all $n \in \mathbb{N}$ if B is a set in C^+ such that $Q_1\{\pi_+ X \in B \cap \mathcal{B}_n\} = 0$ then $Q_2\{\pi_+ X \in B \cap \mathcal{B}_n\} = 0$. From the relation

$$\begin{aligned} 0 =& Q_1\{\pi_+ X \in B \cap \mathcal{B}_n, \pi_0 X \in \mathcal{A}_\infty\} \\ =& \int_{\mathbb{R}^d} Q_1\{\pi_+ X \in B \cap \mathcal{B}_n, \pi_0 X \in \mathcal{A}_\infty | X(0) = x_0\} Q_1\{X(0) \in dx_0\} \end{aligned}$$

and Theorem 7 we see that for x_0 in some set $E \subset \mathbb{R}^d$ with complement in $\mathbb{R}^d$ of zero Lebesgue measure

$$Q_1\{\pi_+ X \in B \cap \mathcal{B}_n, \pi_0 X \in \mathcal{A}_\infty | X(0) = x_0\} = 0.$$

This means that for all $m \geq n$ and for $Q_1\{\pi_0 X \in \ \cdot \ \cap \mathcal{A}_m | X(0) = x_0\}$-almost all $x \in C^-$

$$Q_1\{\pi_+ X \in B \cap \mathcal{B}_n | \pi_0 X = x\} = 0.$$

Lemma 5.2 now implies that if $x_0 \in E$ then for *all* $x \in \mathcal{A}_m$ with $x(0) = x_0$

$$Q_2\{\pi_+ X \in B \cap \mathcal{B}_n | \pi_0 X = x\} = 0,$$

which in turn implies

$$Q_2\{\pi_+ X \in B \cap \mathcal{B}_n, \pi_0 X \in \mathcal{A}_m | X(0) = x_0\} = 0 \quad \text{for } x_0 \in E$$

and

$$Q_2\{\pi_+ X \in B \cap \mathcal{B}_n, \pi_0 X \in \mathcal{A}_\infty | X(0) = x_0\} = 0 \quad \text{for } x_0 \in E$$

To complete the proof integrate the last relation over $\mathbb{R}^d$ with respect to $Q_2\{X(0) \in dx_0\}$ and use Theorem 7. □

Proof of Theorem 3: Theorem 2 would imply Theorem 3, if $Q\{\pi_0 X \in \mathcal{A}_\infty(\rho, r)\} = Q\{\pi_+ X \in \mathcal{B}_\infty(\rho, r)\} = 1$ for every measure Q which defines a stationary $\mathcal{D}(V)$-valued solution. However this is precisely the content of Theorems 5 and 6. □

6 Basic Examples

In this section we consider two simple examples to illustrate our theory. The first example is quite uniform in its behavior. The second one which is similar to the first has a twist which makes the estimates less uniform. It can be seen as a warm up for applying our results to stochastically forced partial differential equations. Finally in the next section we illustrate the point by considering stochastically forced dissipative partial differential equations.

6.1 A Uniform Example

Consider the equation (1) with $X(t) \in \mathbb{R}$ and $a(x) = -x(0)(1 + \Psi(x))$ where

$$\Psi(x) = \int_{-\infty}^{0} e^{-s^2+s} x(s)^2 ds \ .$$

We now show that Theorem 1 and Theorem 3 apply, thus the system has a unique stationary solution.

First note that $V(x) = x(0)^2 + \Psi(x)^2$ is a Lyapunov function for the system with $\gamma = 1$, $\delta = \frac{1}{2}$, $C_1 = 1$ and $C_2 = 1$. To see this, apply Itô's formula to V giving

$$dV(\pi_t X) = h(\pi_t X)dt + 2X(t)dW(t)$$

where

$$h(x) = -2x(0)^2 + 1 + \int_{\mathbb{R}_-} (2s-1)e^{-s^2+s}ds \leq -V(x) + 1$$

and notice that $|x(0)| \leq V(x)^{\frac{1}{2}}$. Next we establish continuity in C_ρ, for any $\rho > 0$, of the functional a. For $x, \tilde{x} \in C_\rho$, a straightforward calculation gives

$$\begin{aligned} |\Psi(x) - \Psi(\tilde{x})| \leq & \int_{-\infty}^{0} e^{-s^2+s}\big(|x(s)| + |\tilde{x}(s)|\big)|x(s) - \tilde{x}(s)|ds \qquad (21) \\ \leq & \big(\|x\|_\rho + \|\tilde{x}\|_\rho\big)\|x - \tilde{x}\|_\rho \int_{-\infty}^{0} e^{-s^2+s}(1+|s|^\rho)^2 ds \end{aligned}$$

Since $|a(x)| \leq C(1 + V(x))$ for some positive C and all x the existence of a stationary solution is implied by Theorem 1 which applies with $\beta = 1$ and ν taken to be the delta measure concentrated at zero.

To show uniqueness of the stationary solution we consider $|a(\pi_t(x{:}y)) - a(\pi_t(\tilde{x}{:}y))|$ for $t > 0$ where $x, \tilde{x} \in \mathcal{A}_n(\frac{3}{4}, \frac{3}{4})$ and $y \in \mathcal{B}_n(\frac{3}{4}, \frac{3}{4})$ where $\mathcal{A}_n$ and $\mathcal{B}_n$ are as defined in Theorem 3. From (21) we have

$$\begin{aligned} \big|a\big(\pi_t(x{:}y)\big) - a\big(\pi_t(\tilde{x}{:}y)\big)\big| \leq & e^{-t} \int_{-\infty}^{0} e^{-|s|}\big(|x(s)| + |\tilde{x}(s)|\big)|x(s) - \tilde{x}(s)|ds \\ \leq & e^{-t} \int_{-\infty}^{0} e^{-|s|} 4n^2 (1 + |s|^{\frac{3}{4}})^2 ds < C(n)e^{-t} \ . \end{aligned}$$

Taking $\mathcal{K}_n(t) = C^2(n)e^{-2t}$ we can apply Theorem 3 and hence the stationary $\mathcal{D}(V)$-valued solution is unique. It is important to stress that Theorem 3 gives only uniqueness of $\mathcal{D}(V)$-valued solutions which is natural since the very dynamics is defined only on the set where $a(x) = -x(0)(1 + \Psi(x))$ is finite.

6.2 A Less Uniform Example

We now modify the previous pedagogical example making it less uniform. We set $a(x) = -x(0)[1+\Psi(x)]+\Psi(x)^2$ where now Ψ equals $\hat{\Psi}(x)$ if $\hat{\Psi}(x) < \infty$

and zero otherwise. Here $\hat{\Psi}(x)$ is given by

$$\hat{\Psi}(x) = \int_{-\infty}^{0} \exp\Big(- 2|s| - \int_{-|s|}^{0} x(r)dr\Big) x(s)^2 ds \ .$$

We again take $V(x) = x(0)^2 + \hat{\Psi}(x)^2$ as our Lyapunov function which produces

$$dV(\pi_t X) = [-2|X(t)|^2 - 4\Psi(\pi_t X)^2 + 1]dt + 2X(t)dW(t)$$

for $t > s$ when $\hat{\Psi}(\pi_s X) < \infty$. Since $-2|x|^2 - 4\Psi(x)^2 + 1 \le 1 - 2V(x)$, we take $C_1 = 1$, $C_2 = 2$, $\beta = 1$, $\gamma = 1$, $\delta = \frac{1}{2}$ and ν a delta measure concentrated at zero. Next observe that $\frac{1}{s}\int_{-|s|}^{0} X(r)dr \le (\frac{1}{s}\int_{-|s|}^{0} X(r)^2 dr)^{\frac{1}{2}} \le (\frac{1}{s}\int_{-|s|}^{0} V(X(r))dr)^{\frac{1}{2}}$. Since the time average of the Lyapunov function is less than $\frac{C_1}{C_2} = \frac{1}{2}$ (see Theorem 6 and Lemma 3.4) we conclude that if $\hat{\Psi}(X(t_1)) < \infty$ then $\hat{\Psi}(X(t)) < \infty$ for all t. Showing that $a(\ \cdot\ )$ is locally Lipschitz on $C_\rho^- \cap \mathcal{N}_r$ reduces to showing that $\hat{\Phi}$ is locally Lipschitz on $C_\rho^- \cap \mathcal{N}_r$, for concreteness we choose $\rho = r = \frac{3}{4}$. Define $\Gamma(s,t) = \exp\Big(-2(t-s) + \int_s^t (x(s_1) + \tilde{x}(s_1))ds_1\Big)$ and take $x, \tilde{x} \in \mathcal{A}_n$ where $\mathcal{A}_n$ is as in Theorem 3. Then direct calculation gives

$$\begin{aligned} |\hat{\Psi}(x) - \hat{\Psi}(\tilde{x})| &\le \int_{-\infty}^{0} \Gamma(s,0)|x(s) - \tilde{x}(s)|[C + V(x(s)) + V(\tilde{x}(s))]ds \\ &\le C\|x-\tilde{x}\|_{\frac{3}{4}} \int_{-\infty}^{0} \exp\left\{-2|s|\left(1-\sqrt{\frac{1}{2} + n\frac{1+|s|^{\frac{3}{4}}}{|s|}}\right)\right\} n(1+|s|^{\frac{3}{4}})^2 ds \ . \end{aligned}$$

Hence Theorem 1 implies the existence of a stationary solution. An analogous calculation give for $t > 0$, $x, \tilde{x} \in \mathcal{A}_n$ and $y \in \mathcal{B}_n$

$$\begin{aligned} |\hat{\Psi}(\pi_t(x{:}y)) - \hat{\Psi}(\pi_t(\tilde{x}{:}y))| &\le |\hat{\Psi}(x) - \hat{\Psi}(\tilde{x})|\Gamma(0,t) \\ &\le K_n(t) = 2n \exp\left\{-2|t|\left(1 - \Big(\frac{1}{2} + n\frac{1+|t|^{\frac{3}{4}}}{|t|}\Big)^{1/2}\right)\right\} \end{aligned}$$

which through Theorem 3 give uniqueness of the stationary $\mathcal{D}(V)$-valued solution.

7 Application to Stochastically Forced Dissipative Partial Differential Equations

Consider the stochastic differential equation

$$du(t) = F(u(t))dt + GdB(t) \tag{22}$$
$$u(t_0) = u_0$$

on a Hilbert space $\mathbb{X}$ with a norm $\| \cdot \|$ and inner product $\langle \cdot , \cdot \rangle$. Here B is a $\mathbb{X}$-valued cylindrical Brownian motion on a probability space Θ and G is a Hilbert-Schmidt operator mapping the domain of B into $\mathbb{X}$. We assume that $B(t)$ exists for all positive and negative times. We also assume without further comment that (22) has strong, global, pathwise unique solutions. The goal of this section is to show that under certain assumptions this Markovian system on a possibly infinite dimensional phase space can be reduced to finite dimensional system with memory which has the same asymptotic behavior as the original system. In the stochastic setting, the reduction was proved in [Mat98]. The pathwise contraction of the small scales embodied in (23) were used in [Mat99]. These and related ideas were used to prove ergodic results for the stochasticly forced Navier-Stokes euqations in [EMS01, BKL01, KS00]. In [Mat01, BKL02], exponential mixing is proved. The first of these uses a coupling construction and gives explicit estimates on the dependence of initial conditions.

We make three basic assumptions:

1. The dynamics admits a Markovian Lyapunov function. Namely, in complete analogy to our previous definition, there exist a function $U : \mathbb{X} \to \mathbb{R} \cup \infty$ so that
   (a) $U(u) \geq C_0 \|u\|^l$ for some $C_0, l > 0$ and all $u \in \mathbb{X}$.
   (b) For a solution u of equation (1)
   $$dU(u(t)) = h(u(t))dt + f(u(t))d\widetilde{B}(t).$$
   if $t \geq s$ and $U(u(s)) < \infty$. Here $h : \mathbb{X} \to \mathbb{R}$ is a function satisfying
   $$h(u) < C_1 - C_2 U(u)^\gamma$$
   for some constants $C_1, C_2, \gamma > 0$ and $f : \mathbb{X} \to \mathbb{R}$ is a function satisfying
   $$|f(u)| \leq C_3 U(u)^\delta,$$
   for some $\delta \in [0, (1+\gamma)/2)$ and $C_3 > 0$. Finally $\widetilde{B}$ is a standard Wiener process adapted to the flow generated by (dB).

2. There exist a decomposition of $\mathbb{X} = \mathbb{L} \oplus \mathbb{H}$ where $\mathbb{L}$ is d-dimensional with $d < \infty$ with the property if Π_ℓ and Π_h are the respective projections on $\mathbb{L}$ and $\mathbb{H}$ then

$$\langle F(u) - F(\tilde{u}), \Pi_h(u - \tilde{u})\rangle \leq |||\Pi_h(u - \tilde{u})|||^2(-c_1 + c_2 U(u)^\gamma) \\ + c_3 |||\Pi_\ell(u - \tilde{u})|||^{p_1}(1 + U(u)^{p_2} + U(\tilde{u})^{p_2}) \quad (23)$$

for all $u, \tilde{u} \in \mathbb{X}$ and some $c_i, p_i \geq 0$ with $\frac{c_1}{c_2} > \frac{C_1}{C_2}$.

3. $\Pi_h GB(t) = 0$. This amounts to only saying that the subspace $\mathbb{L}$ is forced.

4. The Markovian dynamics (22) possesses an invariant probability measure μ such that $\mu\{u \in \mathcal{D}(U)\} = 1$.

Remark: The last condition requiring the existence of an invariant measure can almost certainly be derived from the first two. This would be interesting in that it would replace the normal compactness arguments with a dissipative argument much as the uniqueness theory removes most topological considerations by using the dissipative structure.

Remark: In fact many of the results follow without modification even if $\Pi_h GB \neq 0$. It would require the consideration of drifts of the form $a(\pi_t X, t)$ in (1) and further assumptions to control the behavior in t. Though it is instructive to explore this option, for the PDE applications it is more natural to stay in a Markovian framework, shifting to the memory point of view only as needed. See for example [Mat02, KPS02]

Remark: The asymmetry in the estimate on the differences in the F is an artifact of our memory setting. If we stayed in a more Markov setting it would not be needed. See [Mat02] for more details.

We can extend the invariant measure μ to a stationary measure $\mathcal{M}$ on $C((-\infty, \infty), \mathbb{X}) \times \Theta$. See [EMS01] equation (6) for more details. Then using precisely the same calculations as in Theorem 6 and 5 with $\mathcal{F}U(u,t) = |\int_0^t U(\pi_s x)^\gamma ds| - \frac{C_1}{C_2}|t|$ we obtain

Theorem 8 *Under the assumptions 1–4 for any* $\rho > 0$ *and* $\varkappa > \frac{1}{2}$

$$\mathcal{M}\left\{\sup_t \frac{|\mathcal{F}U(u,t)|}{1 + |t|^\varkappa} < \infty\right\} = \mathcal{M}\left\{\sup_t \frac{U(u(t))}{1 + |t|^\rho} < \infty\right\} = 1 \ .$$

We now prove the critical lemma which allows us to remap this problem to the setting of the first half of the paper. Let $\mathcal{M}\Pi_\ell^{-1}$ be the projection onto paths in $C((-\infty,\infty),\mathbb{L}) \times \Theta$. Similarly we split equation (22) into equations for $(h(t),\ell(t)) \in \mathbb{L} \oplus \mathbb{H}$, obtaining

$$\frac{dh(t)}{dt} = \Pi_h F(h(t)+\ell(t)) \tag{24}$$

$$d\ell(t) = \Pi_\ell F(h(t)+\ell(t))dt + GdB(t) \tag{25}$$

We now show that there exists a function $\Psi : C^- \to \mathbb{H}$ so that the equation

$$d\ell(t) = F(\ell(t)+\Psi(\pi_t \ell))dt + \Pi_\ell GdB(t) \tag{26}$$

has the same asymptotic behavior as (22). This equation is similar to what is called an inertial form in the theory of inertial manifolds; there however, the function Ψ depends only on the present and not on the past. In some settings but not in all cases, one can construct a stochastic inertial manifold (see [DPD96, CG94, BF95]). However the constructions of this section work in most dissipative settings. The reduced memory formulation (26) is akin to the reduction of a dynamical system done in the context of symbolic dynamics. By only having some coarse description of the dynamics, but for a time interval of infinite length, one can reconstruct the exact position. Usually the symbolic dynamics encodes the forward trajectory. Here we are encoding the state of some subset of the variables in the infinite past.

Theorem 9 *Under the same assumptions there exist an $\mathbb{H}$-valued function Ψ defined on $C((-\infty,0],\mathbb{L})$ so that the following holds:*

1. *For $\mathcal{M}\Pi_\ell^{-1}$-almost every $(\ell_{(-\infty,0]},B) \in C((-\infty,0],\mathbb{L}) \times \Theta$ if $\Psi(\pi_t \ell_{(-\infty,0]}) = h(t)$ with $t \le 0$ then $u(t) = (\ell(t),h(t))$ is a solution to (22) with noise realization $B(t)$.*

2. *If $\Psi_{s,t}(\ell_{(-\infty,0]},h_0)$ is the solution to (24) at time t with initial data $h_0 \in \mathbb{H}$ at time s and exogenous forcing ℓ, then for $\mathcal{M}\Pi_\ell^{-1}$-almost every $\ell_{(-\infty,0]}$ one has $\lim_{s\phi-\infty} \Psi_{s,t}(\ell_{(-\infty,0]},h_0) = \Psi(\pi_t \ell_{(-\infty,0]})$ for any h_0.*

3. *Fix $r > 0$ and $\kappa \in (\frac{1}{2},1)$. There exists a constant $C(r,\kappa,n)$ so that for $\mathcal{M}\Pi_\ell^{-1}$ almost every $\ell,\tilde{\ell} \in A_n(\kappa)$, $\|\Psi(\pi_0\ell)-\Psi(\pi_0\tilde{\ell})\|^2 \le C(r,\kappa,n)\|\ell-\tilde{\ell}\|_r^{p_1}$ where*

$$A_n(\kappa) = \left\{ \ell \in C\big((-\infty,0],\mathbb{L}\big) : \sup_{s\le 0} \frac{U(u(s)) + \mathcal{F}U(u,s)}{1+|s|^\kappa} < n \right\}$$

and $u(s) = \ell(s) + \Psi(\pi_s \ell)$. Furthermore there exist $\gamma_n > 0$ and $K_n > 0$so that if $t > 0$ and $\ell(s) = \tilde{\ell}(s)$ for $s \in [0,t]$ then $\|\Psi(\pi_t\ell)-\Psi(\pi_t\tilde{\ell})\|^2 \le K_n \exp(-\gamma_n t)$

Proof: For $\mathcal{M}\Pi_\ell^{-1}$-almost every $(\ell(t), B(t))$ there is a corresponding $h(t)$ so that $u(t) = (\ell(t), h(t))$ is a solution to (22) with forcing $B(t)$. For $s \le 0$, $\tilde{h}(s) = \Psi_{-t,s}(\ell_{(-\infty,0]}, \tilde{h}_0)$. Defining $\rho(s) = h(s) - \tilde{h}(s)$ we have

$$\begin{aligned}\frac{d|\!|\!|\rho(s)|\!|\!|^2}{dt} &= \langle F(\ell(s)+h(s)) - F(\tilde{\ell}(s)+h(s)), \rho(s)\rangle \\ &\le |\!|\!|\rho(s)|\!|\!|^2(-c_1 + c_2 U(u(s))^\gamma) \ .\end{aligned}$$

Hence using Theorem 8 to continue

$$\begin{aligned}|\!|\!|\rho(s)|\!|\!|^2 &\le (|\!|\!|\tilde{h}_0|\!|\!| + |\!|\!|h(-t)|\!|\!|)^2 \exp\left(-c_1(|t|-|s|) + c_2\int_{-t}^{s} U(u(\tau))^l d\tau\right) \\ &\le (|\!|\!|\tilde{h}_0|\!|\!| + \|h\|_\kappa(1+|t|^\kappa)^2 \exp\left(-(c_1 - c_2\frac{C_1}{C_2})(|t|-|s|) + C(\kappa)(1+|t|^\kappa)\right)\end{aligned}$$

for some $\kappa \in (\frac{1}{2}, 1)$. Since by assumption $c_1 - c_2\frac{C_1}{C_2} > 0$, $|\!|\!|\rho(s)|\!|\!| \to 0$ as $t \to -\infty$. This proves the first and second claim of the theorem. To see the third consider two pairs of solution (ℓ, B) and $(\tilde{\ell}, \tilde{B})$ in A_n. For $\mathcal{M}\Pi_\ell^{-1}$-almost every such pair, in the same way as the previous estimate one obtains

$$\begin{aligned}|\!|\!|\rho(0)|\!|\!|^2 \le c_3\|\ell - \tilde{\ell}\|_r^{p_1} \int_{-\infty}^{0} \exp\left(-(c_1 - c_2\frac{C_1}{C_2})|t| + nc_2(1+|t|^\kappa)\right) \times \\ (1+|t|^r)^{p_1}(1 + 2n^{p_2}(1+|t|^\kappa)^{p_2})dt.\end{aligned}$$

□

With Theorem 9 in hand, we can define dynamics with memory on finite dimensional space $\mathbb{L}$ which is isomorphic to $\mathbb{R}^d$. We set $Q = \mathcal{M}\Pi_\ell^{-1}$ and consider an equation of the form (1) with $a(x) = \Pi_\ell F(x(0) + \Phi(x))$ and with a Lyapunov function $V(x) = U(x(0) + \Psi(x))$. We then arrive at an equation of the form (1). Hence by invoking theorem 3, we obtain the following result.

Theorem 10 *Assuming all of the assumptions of this section and additionally that $\Pi_\ell G$ is of full rank and that*

$$|\Pi_\ell F(\ell+h) - \Pi_\ell F(\ell+\tilde{h})|^2 \le c_4(1 + U(\ell+h)^{p_3} + U(\ell+\tilde{h})^{p_3})|\!|\!|h - \tilde{h}|\!|\!|^{p_4} \ . \quad (27)$$

for all $\ell \in \mathbb{L}$ and $h, \tilde{h} \in \mathbb{H}$ and some $c_4 \ge 0$, and $p_i \ge 0$. Then the invariant solution $\mathcal{M}$ is the unique $\mathcal{D}(U)$-valued one.

Notice that ergodicity only requires that all of the modes up to a given scale are forced. In other words, it is sufficient for the system to be elliptic only up to a certain scale to ensure ergodicity. The arguments used are "soft" in that they do not require explicit geometric information as hypoelliptic arguments do. For this reason, it is reasonable to call the system "effectively elliptic" because the reduced memory equation is truly elliptic and hence the arguments are relatively "soft."

PROOF: Notice that the sets in 3 are just the projects of the $t \leq 0$ and $t \geq 0$ parts of A_n defined above. Let x_1, x_2, y be chosen as in theorem 2. Since $a(x) = \Pi_\ell F(x(0) + \Psi(x))$,

$$\begin{aligned}
&|a(\pi_t(x_1{:}y)) - a(\pi_t(x_2{:}y))|^2 \\
&\quad = |\Pi_\ell F(y(t) + \Psi(\pi_t(x_1{:}y))) - \Pi_\ell F(y(t) + \Psi(\pi_t(x_2{:}y)))|^2 \\
&\quad \leq c_4(1 + V(\pi_t(x_1{:}y))^{p_3} + V(\pi_t(x_2{:}y))^{p_3})|\!|\!|\Psi(\pi_t(x_1{:}y)) - \Psi(\pi_t(x_2{:}y))|\!|\!|^{p_4} \\
&\quad \leq c_4(1 + 2n^{p_3}(1 + |t|^\kappa)^{p_3})K_n \exp(-p_4\gamma_n t)
\end{aligned}$$

As this bound is integrable, Theorem 3 completes the proof. □

One interesting consequence of the memory point of view is following factorization of the invariant measure into a measure living on the path space $C((-\infty, 0], \mathbb{L})$ and an atomic measure living in $\mathbb{H}$ which depends only on the choice in $C((-\infty, 0], \mathbb{L})$. This factorization show that the random attractor projected into $\mathbb{H}$ space is a single point attractor fibered over the choice of trajectory in $C((-\infty, 0], \mathbb{L})$.

Theorem 11 *Assuming all of the conditions of Theorem 10 hold. Then the following factorization of the invariant measure μ holds: for any $A \subset \mathbb{X}$*

$$\mu(A) = \int \mathbf{1}_A(\ell(0) + \Psi(\ell))\mathcal{M}\Pi_\ell^{-1}(d\ell) \ .$$

7.1 The 2D Stochastic Navier Stokes Equation

Consider the incompressible Navier Stokes equation with mean zero flow on the two dimensional unit torus,$\mathbb{T}^2$, agitated by a stochastic forcing with no mean flow. By projecting out the pressure, we obtain the following Itô equation for $u(x,t) = (u^{(1)}(x,t), u^{(2)}(x,t)) \in \mathbb{X} = L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)$

$$du(x,t) = [\nu\Delta u + B(u,u)]\, dt + GdW(t)$$

where $B(u,v) = P_{div}(u \cdot \nabla)u$, P_{div} is the projection onto divergence-free vector fields, G a Hilbert-Schmidt operator mapping the cylindrical Weiner process W into $\mathbb{X}$. We assume that there exists $\mathcal{K}_{\cos}$, $\mathcal{K}_{\sin} \subset \mathbb{Z}^2$ so that $Im(G) = \mathbb{L} = \operatorname{span}(\sin(2\pi k \cdot x), \cos(2\pi m \cdot x) : k \in \mathcal{K}_{\sin}, m \in \mathcal{K}_{\cos})$. We define N_0 to be the largest integer multiple of 2π so that if $2\pi|k| < N_0$ then $k \in \mathcal{K}_{\cos} \cap \mathcal{K}_{\sin}$. Similarly we define N_1 to be the smallest integer so that if $2\pi|k| > N_1$ then $k \notin \mathcal{K}_{\cos} \cup \mathcal{K}_{\sin}$. We want to show that if N_0 is sufficiently large then Theorem 10 and 11 hold. We take $U(u) = \|\nabla u\|^2$ as the Lyapunov function. Standard calculations show that U satisfies our conditions for a Lyapunov function with $C_1 = \mathrm{Tr}GG^*$, $C_2 = 2\nu$, and $C_3 = 2$. (See the enstrophy calculations in [EMS01]). Lastly, we need to verify (23) and (27). We begin with the first, setting $F(u) = \nu\Delta u + B(u,u)$, $\mathbb{H} = \mathbb{X}/\mathbb{L}$ and $\rho = u - \tilde{u}$ one has

$$\begin{aligned}\langle F(u) - F(\tilde{u}), \Pi_h(u-\tilde{u})\rangle &\le -\nu\|\Pi_h\nabla(u-\tilde{u})\|^2 + \langle B(\rho,u), \Pi_h(u-\tilde{u})\rangle \\ &\quad + \langle B(u,\rho), \Pi_h(u-\tilde{u})\rangle \\ &\le \left(-\frac{\nu N_0^2}{2} + \frac{C}{\nu}\|\nabla u\|^2\right)\|\Pi_h(u-\tilde{u})\|^2 \\ &\quad + \frac{CN_0^2}{\nu}\left(\|\nabla u\|^2 + \|\nabla\tilde{u}\|^2\right)\|\Pi_\ell(u-\tilde{u})\|^2\end{aligned}$$

where the constant C is independent of N_0. Hence if $\frac{\nu N_0^2}{2} > \frac{C}{2\nu^2}\mathrm{Tr}GG^*$, the assumption in (23) holds.

Lastly we check the condition used to control the paths in $\mathbb{L}$. For any $h, \tilde{h} \in \mathbb{H}$, $\ell \in \mathbb{L}$, and $v \in \mathbb{L}$ with $\|v\| = 1$ one has

$$\begin{aligned}\langle F(\ell+h) - F(\ell+\tilde{h}), v\rangle &\le \langle B(h-\tilde{h}, \ell+h), v\rangle + \langle B(\ell+\tilde{h}, h-\tilde{h}), v\rangle \\ &\le C\|h-\tilde{h}\|\|\nabla^2 v\|\left(\|\nabla(\ell+h)\| + \|\nabla(\ell+\tilde{h})\|\right) \\ &\le CN_1^2\|h-\tilde{h}\|\left(\|\nabla(\ell+h)\| + \|\nabla(\ell+\tilde{h})\|\right)\end{aligned}$$

Hence (27), holds and we obtain the following result.

Theorem 12 *Let C be the constant from the above calculations. If $N_0^2 > C\frac{TrGG^*}{\nu^3}$ then Theorems 10 and 11 apply to the Stochastic Navier-Stokes equation.*

7.2 The 1D Stochastic Ginsburg-Landau Equation

As a second PDE example, we consider the stochastically forced Cahn-Allen/Ginsburg-Landau equation in a one dimensional periodic domain. Ergodic results for this equation have been proved in [EL02, Hai02]. The general framework of [Mat01] applies equally well to this setting.

Consider

$$du(x,t) = \left[\nu\Delta u + u - u^3\right] dt + dW(x,t) \tag{28}$$

where $W(x,t) = \sum_{\mathcal{K}} e_k(x)\sigma_k\beta(t)$, β_k are independent standard Brownian motions, σ_k are positive constants and e_k are the elements of the real Fourier basis $\{1, \sin(2\pi x), \cos(2\pi x), \sin(4\pi x), \cos(4\pi x), \cdots\}$. We denote by λ_k the eigenvalue of $-\Delta$ associated with e_k. As before we consider the case when $\sigma_k > 0$ for only a finite number of k and define N_0 by the smallest integer so that if $\lambda_k < 4\pi^2 N_0^2$ then $\sigma_k > 0$.

As in the previous example, we let $\mathbb{L}$ be the span of the e_k with $\sigma_k > 0$ and $\mathbb{H}$ the span of the remaining e_k. Then $\mathbb{X} = L^2([0,1]) = \mathbb{L} \oplus \mathbb{H}$ and by $\|\cdot\|$ we mean the L^2-norm on $\mathbb{X}$.

We use the Lyapunov function $U(u) = \|u\|^2 + \|\nabla u\|^2$. Direct calculation gives (see for instance [EL02, Smo94])

$$\begin{aligned} dU(t) = 2\left[\|u\|^2 - \|\Delta u\|^2 + (1-\nu)\|\nabla u\|^2 - 3\nu\|u\nabla u\|^2 - \int u^4 dx + \frac{1}{2}\mathcal{E}_0 + \frac{1}{2}\mathcal{E}_1\right] dt \\ + 2\langle u, dW\rangle - 2\langle \Delta u, dW\rangle. \end{aligned}$$

where $\mathcal{E}_m = \sum \lambda_k^m \sigma_k^2$. Using Jensen's inequality on the L^4 norm, the fact that $\frac{1}{2}K_0 - x^2 > x^2 - x^4$ for some positive K_0, $\|\Delta u\|^2 > 4\pi^2\|\nabla u\|^2$ and the non-positivity of the third term, we see that U is a Lyapunov function with $C_1 = K_0 + \mathcal{E}_0 + \mathcal{E}_1$, $C_2 = 2$, $\gamma = 1$, $\delta = \frac{1}{2}$ and $C_3 = 2\sigma_{max}^2(0) + 2\sigma_{max}^2(1)$. Here $\sigma_{max}^2(m) = \max \sigma_k^2 \lambda_k^m$.

All that remains is to prove the estimates (23) and (27) hold. To see the first recall that the Sobolev embedding theorem implies that $|f|_{L^\infty}^2 < CU(f)$ and hence if $F(u) = \Delta u + R(u)$ and $R(u) = u - u^3$ then

$$\begin{aligned} \langle F(u) - F(\tilde{u}), \Pi_h(u-\tilde{u})\rangle &= -\|\nabla\Pi_h(u-\tilde{u})\|^2 + \langle R(u) - R(\tilde{u}), u - \tilde{u}\rangle \\ &\quad + \langle R(u) - R(\tilde{u}), \Pi_\ell(u-\tilde{u})\rangle \\ &\le (1 - 4\nu\pi^2 N_0^2)\|\Pi_h(u-\tilde{u})\|^2 + \|\Pi_\ell(u-\tilde{u})\|^2 \\ &\quad + |\langle R(u) - R(\tilde{u}), \Pi_\ell(u-\tilde{u})\rangle| \ . \end{aligned}$$

To finish this estimate, observe that

$$\begin{aligned} |\langle R(u) - R(\tilde{u}), \Pi_\ell(u-\tilde{u})\rangle| &\le (1 + 2|u|_{L^\infty}^2 + 2|\tilde{u}|_{L^\infty}^2)\|\Pi_\ell(u-\tilde{u})\|^2 \\ &\le C(1 + U(u) + U(\tilde{u}))\|\Pi_\ell(u-\tilde{u})\|^2 \ . \end{aligned}$$

To see (27), set $u = \ell + h$ and $\tilde{u} = \ell + \tilde{h}$ and observe that

$$\begin{aligned}
\|\Pi_\ell(F(\ell+h) - F(\ell+\tilde{h}))\| =& \|(\ell+h)^3 - (\ell+\tilde{h})^3\| + \|h - \tilde{h}\| \\
=& \|\int_u^{\tilde{u}} 3v^2 dv\| + \|h - \tilde{h}\| \\
\le& \|(3u^2 + 3\tilde{u}^2)|h - \tilde{h}|\| + \|h - \tilde{h}\| \\
\le& C(|u|_{L^\infty}^2 + |\tilde{u}|_{L^\infty}^2 + 1)\|h - \tilde{h}\| \\
\le& C\left(U(u) + U(\tilde{u}) + 1\right)\|h - \tilde{h}\|.
\end{aligned}$$

In the last line we have used the Sobolev embedding theorem.

In light of the above, calculations we have

Theorem 13 *If $N_0^2 > \frac{1}{4\pi^2\nu}$ then Theorems 10 and 11 apply to the stochastically forced Ginsburg-Landau Equation* (28)*.*

8 Acknowledgments

We would like to thank Yakov Sinai, Toufic Suidan and Alexander Veretennikov for their input and comments. We would also like to thank the hospitality and support of the Institute for Advanced Study in Princeton where this work was undertaken. The second author was partial supported by the NSF under grant DMS-9971087. And both authors received support from the NSF under grant DMS-9729992.

A Comparison of Measures on Path Space

Suppose that we have two measures $P^{(1)}$ and $P^{(2)}$ on the space $C^+ \times C^+$ which define solutions for equations

$$\begin{aligned}
&dX_i(t) = f_i(t, X_i[0,t])dt + dW(t),\ t \ge 0, \quad i = 1,2 \text{ respectively,} \\
&X_i(0) = x_0.
\end{aligned} \tag{29}$$

Here for fixed t functions f_1 and f_2 map the space $C_{[0,t]} = C([0,t], \mathbb{R}^d)$ to $\mathbb{R}^d$. By $X[0,t]$ we mean the segment of the trajectory on $[0,t]$. Let $T \in (0,\infty]$ and $\mathcal{B} \subset C_{[0,T]}$. Define measures $P^{(i)}_{[0,T]}(\,\cdot\,;\mathcal{B})$ on the path space as:

$$P^{(i)}_{[0,T]}(A;\mathcal{B}) = P\{X_i[0,T] \in A \cap \mathcal{B}\},\ \text{for } A \subset C_{[0,T]}.$$

Also define $D(t,\,\cdot\,) = f_1(t,\,\cdot\,) - f_2(t,\,\cdot\,)$.

In this setting, we have the following result which is a variation on Lemma B.1 from [Mat02].

Lemma A.1 *Assume there exists a constant* $D_* \in (0, \infty)$ *such that*

$$\exp\left\{\frac{1}{2}\int_0^T \left|D\big(t, X[0,t]\big)\right|^2 dt\right\} \mathbf{1}_{\mathcal{B}}(X[0,t]) < D_* \tag{30}$$

almost surely with respect to both measures $P^{(1)}$ *and* $P^{(2)}$. *Then the measures* $P^{(1)}_{[0,T]}(\,\cdot\,;\mathcal{B})$ *and* $P^{(2)}_{[0,T]}(\,\cdot\,;\mathcal{B})$ *are equivalent.*

PROOF: Define the auxiliary SDEs

$$dY_i(t) = f_i\big(t, Y_i[0,t]\big)\mathbf{1}_{\mathcal{B}(t)}(Y_i[0,t])dt + dW(t)$$

where $\mathcal{B}(t) = \{x \in C_{[0,t]} : \exists \bar{x} \in \mathcal{B} \text{ such that } x(s) = \bar{x}(s) \text{ for } s \in [0,t]\}$. Solutions $Y_i(t)$ to these equation can be constructed as

$$Y_i(t) = X_i(t)\mathbf{1}_{\{t \le \tau\}} + [W(t) - W(\tau) + X_i(\tau)]\mathbf{1}_{\{t > \tau\}}.$$

Here $(X_i(t), W)$ is the solution to equation (29) and $\tau = \inf\{s > 0 : X_i[0,s] \notin \mathcal{B}(s)\}$.

Denote $D_{\mathcal{B}}(t,x) = [f_1(t,x) - f_2(t,x)]\mathbf{1}_{\mathcal{B}(t)}(x)$. The assumption on D in (30) and the definition of $\mathcal{B}(t)$ imply that

$$\exp\left\{\frac{1}{2}\int_0^T \left|D_{\mathcal{B}}\big(t, X[0,t]\big)\right|^2 dt\right\} < D_* \quad \text{a.s.}$$

under both measures $P^{(i)}_{Y[0,t]}$ defining solutions to auxiliary equation with $i = 1$ and $i = 2$. Hence Novikov's condition is satisfied for the difference of the drifts of the auxiliary equations and the Girsanov theorem implies that $\frac{dP^{(1)}_{Y[0,t]}}{dP^{(2)}_{Y[0,t]}}(x) = \mathcal{E}(x)$ where the Radon–Nikodym derivative evaluated at a trajectory x is defined by the stochastic exponent:

$$\mathcal{E}(x) = \exp\left\{\int_0^T \langle D_{\mathcal{B}}(s, x[0,s]), dW(s)\rangle - \frac{1}{2}\int_0^T |D_{\mathcal{B}}(s, x[0,s])|^2 ds\right\}.$$

Note that restrictions of the measures $P^{(i)}_{Y[0,t]}$ on the set $\mathcal{B}$ coinside with $P^{(i)}_{[0,t]}(\,\cdot\,,\mathcal{B})$. This proves that $P^{(1)}_{[0,t]}(\,\cdot\,,\mathcal{B})$ is absolutely continuous with respect to $P^{(2)}_{[0,t]}(\,\cdot\,,\mathcal{B})$. The reverse relation follows by symmetry and the proof is complete. $\square$

B Cauchy problem

In this appendix we study existence and uniqueness of solution to Cauchy problem for (1).

Theorem 14 *Suppose the functional $a(\ \cdot\ )$ is locally Lipschitz on C_ρ with respect to the norm $\|\ \cdot\ \|_\rho$. If $x \in C_\rho$ then for any realization of standard Wiener process W on any probability space there exist a positive stopping time T and a continuous process $X(t)$ on the same probability space with the following properties:*

1. *$X(t) = x(t)$ for $t \le 0$ almost surely.*
2. *The couple (X, W) solves equation (1) on $[0, T]$.*
3. *The process X is adapted to the flow generated by W.*

Any other process with this properties coincides with X almost surely.

Proof: Fix a trajectory of W on $[0, \infty)$ and for any positive T define operator $\Phi : C_{[0,T]} \to C_{[0,T]}$ by

$$\Phi(y)(t) = x(0) + \int_0^t a(\pi_s(x{:}y))ds + W(t).$$

Then

$$|\Phi(y_1)(t) - \Phi(y_2)(t)| \le \int_0^t |a(\pi_s(x{:}y_1)) - a(\pi_s(x{:}y_2))|ds.$$

If $\sup_{s\in[0,T]} |y_i(s)| \le M, i = 1, 2$ for some $M > 0$, then

$$\|\pi_s(x{:}y_i)\|_\rho \le M + \|x\|_\rho, \quad i = 1, 2.$$

Hence

$$|a(\pi_r(x{:}y_1)) - a(\pi_r(x{:}y_2))| < K\|\pi_r(x{:}y_1) - \pi_r(x{:}y_2)\|_\rho$$

where $K = K(M + \|x\|_\rho)$ is the local Lipschitz constant. If $|y|$ is bounded by M then

$$\|\pi_r(x{:}y_1) - \pi_r(x{:}y_2)\|_\rho \le \sup_{s\in[0,t]} |y_1(s) - y_2(s)|(1 + t^\rho)$$

implies that

$$|\Phi(y_1)(t) - \Phi(y_2)(t)| \le t \cdot K(1 + t^\rho) \sup_{s\in[0,t]} |y_1(s) - y_2(s)| \quad \text{for } t \in [0, T].$$

Choose $M > |x_0| + 2$ and

$$T = \sup\left\{t : tK(1+t^\rho) < \frac{1}{2}, |W(t)| < 1 \text{ and } Ct < 1\right\} \wedge t_0$$

where $C = C(M) > 0$ and $t_0 > 0$ are such that for $t < t_0$ the drift $|a(\pi_t(x{:}y))|$ is bounded by C if $|y|$ is bounded by M.

Then Φ is a contracting in L^∞ map from compact set

$$D = \Big\{ y \in C_{[0,T]} : y(0) = 0, \sup_{s\in[0,T]} |y(s)| \leq M,$$
$$w_\delta(y) \leq C\delta + \omega_\delta(W) \text{ for all } \delta > 0 \Big\}$$

to itself where $w_\delta(y)$ is the δ-modulus of continuity of y.

There exists unique fixed point in D of this map which gives the desired solution. Since the choice of M is arbitrary, we conclude that this solution is unique. □

Theorem 14 implies that in its setting the dynamics corresponding to the equation (1) is defined at least up to some random moment T.

Let T_∞ be the largest time such that a solution exists on $[0, T_\infty)$. The question of global existence will be answered if one shows that if T_∞ is finite then the solution can be in fact extended beyond it.

Theorem 15 *If there exists a Lyapunov function (see section 2) for the dynamics corresponding to* (1) *built in Theorem 14 then the time T in this theorem can be chosen to be equal to ∞, i.e. pathwise uniquness and strong existence hold globally.*

PROOF: Essentially we need to show that $X(t)$ does not escape to infinity in finite time a.s. Introduce the stopping time $\tau_R = \inf\{t > 0 : V_t \geq R\}$. Then by the definition of Lyapunov function we have

$$V(\tau_R \wedge t) \leq Ct + \int_0^{\tau_R \wedge t} h(\pi_s(X))dW(s).$$

Hence

$$\mathsf{E}V(\tau_R \wedge t) \leq Ct.$$

This inequality with $\mathsf{E}V(\tau_R \wedge t) > R \cdot \mathbb{P}\{\tau_R \leq t\}$ implies that for any $t > 0$

$$\mathbb{P}\{\text{for every } R > 0 \text{ there exists } \ s \leq t \text{ such that } |V_s| > R\} = 0.$$

So V_s is finite for all $s \leq t$ which implies that $X(s)$ is finite for all $s \leq t$. □

References

[Bak02] Yu. Yu. Bakhtin. Existence and uniqueness of stationary solution of stochastic differential equation with memory. *Theory Probab. Appl*, 47(4):764–769, 2002.

[BF95] Alain Bensoussan and Franco Flandoli. Stochastic inertial manifold. *Stochastics Stochastics Rep.*, 53(1-2):13–39, 1995.

[Bil68] Patrick Billingsley. *Convergence of probability measures.* John Wiley & Sons, Inc., New York-London-Sydney, 1968.

[BKL01] J. Bricmont, A. Kupiainen, and R. Lefevere. Ergodicity of the 2D Navier-Stokes equations with random forcing. *Comm. Math. Phys.*, 224(1):65–81, 2001. Dedicated to Joel L. Lebowitz.

[BKL02] J. Bricmont, A. Kupiainen, and R. Lefevere. Exponential mixing of the 2D stochastic Navier-Stokes dynamics. *Comm. Math. Phys.*, 230(1):87–132, 2002.

[CG94] I. D. Chueshov and T. V. Girya. Inertial manifolds for stochastic dissipative dynamical systems. *Dopov./Dokl. Akad. Nauk Ukraïni*, 7:42–45, 1994.

[DPD96] Giuseppe Da Prato and Arnaud Debussche. Construction of stochastic inertial manifolds using backward integration. *Stochastics Stochastics Rep.*, 59(3-4):305–324, 1996.

[EL02] Weinan E and Di Lui. Gibbsian dynamics and invariant measures for stochastic dissipative PDEs. *Journal of Statistical Physics*, 108(5/6):1125–1156, 2002.

[EMS01] Weinan E, J. C. Mattingly, and Ya G. Sinai. Gibbsian dynamics and ergodicity for the stochastic forced navier-stokes equation. *Comm. Math. Phys.*, 224(1), 2001.

[FP67] C. Foiaş and G. Prodi. Sur le comportement global des solutions non-stationnaires des équations de Navier-Stokes en dimension 2. *Rend. Sem. Mat. Univ. Padova*, 39:1–34, 1967.

[Hai02] Martin Hairer. Exponential mixing properties of stochastic pdes through asymptotic coupling. *Probab Theory Relat Fields*, 124(3):345–380, 2002.

[IN64] Kiyoshi Itô and Makiko Nisio. On stationary solutions of a stochastic differential equation. *J. Math. Kyoto Univ.*, 4:1–75, 1964.

[KPS02] Sergei Kuksin, Andrey Piatnitski, and Armen Shirikyan. A coupling approach to randomly forced nonlinear PDEs. II. *Comm. Math. Phys.*, 230(1):81–85, 2002.

[KS00] Sergei Kuksin and Armen Shirikyan. Stochastic dissipative PDEs and Gibbs measures. *Comm. Math. Phys.*, 213(2):291–330, 2000.

[Mat98] Jonathan C. Mattingly. *The Stochastically forced Navier-Stokes equations: energy estimates and phase space contraction.* PhD thesis, Princeton University, 1998.

[Mat99] Jonathan C. Mattingly. Ergodicity of 2D Navier-Stokes equations with random forcing and large viscosity. *Comm. Math. Phys.*, 206(2):273–288, 1999.

[Mat01] Jonathan C. Mattingly. Ergodicity of stochastic reaction diffusion equations: Effectively elliptic systems. In preperation, 2001.

[Mat02] Jonathan C. Mattingly. Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics. *Comm. Math. Phys.*, 230(3):421–462, 2002.

[MY02] Nader Masmoudi and Lai-Sang Young. Ergodic theory of infinite dimensional systems with applications to dissipative parabolic PDEs. *Comm. Math. Phys.*, 227(3):461–481, 2002.

[Pro90] Philip Protter. *Stochastic Integration and Differential Equations: a new approach.* Springer-Verlag, 1990.

[Sin94] Ya. G. Sinaĭ. *Topics in ergodic theory*, volume 44 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 1994.

[Smo94] Joel Smoller. *Shock Waves and Reaction-Diffusion Equations.* Springer-Verlag, 2nd edition, 1994.

[Tem88] Roger Temam. *Infinite Dimensional Dynamical Systems in Mechanics and Physics.* Springer-Verlag, New York, 1988.

[Ver97] A. Yu. Veretennikov. On polynomial mixing bounds for stochastic differential equations. *Stochastic Process. Appl.*, 70(1):115–127, 1997.