Problem 1.

Let \( f_p(x) = x^{2p} \) for \( p = 1, 2, \ldots \) then by Ito’s formula

\[
I(t)^{2p} = f_p(I(t)) = \int_0^t f'_p(I(s))dI(s) + \int_0^t \frac{1}{2} f''_p(I(s))d\langle I, I \rangle(s)
\]

\[
= \int_0^t 2pI(s)^{2p-1} \sigma(s)dB(s) + \int_0^t p(2p-1)I(s)^{2p-2}\sigma^2(s)ds
\]

Now note that \( M_p(t) = \int_0^t I(s)^{2p-1} \sigma(s)dB(s) \) is a martingale, and using the Optional stopping theorem we have \( EM_p(t) = EM_p(0) = 0 \).

We’ll prove the required inequality by induction on \( p \). For \( p = 1 \) we have

\[
E I(t)^2 = E \int_0^t 2I(s)\sigma(s)dB(s) + E \int_0^t \sigma^2(s)ds \leq 0 + M^2 t
\]

Assume \( E I(t)^{2p-2} \leq (2p - 3)!!(M^2t)^{p-1} \) then

\[
E I(t)^{2p} = 2pEM_p(t) + p(2p-1)E \int_0^t I(s)^{2p-2}\sigma^2(s)ds
\]

\[
\leq 0 + p(2p-1)M^2 \int_0^t (2p-3)!!(M^2s)^{p-1}ds \leq (2p - 1)!!M^{2p}t^p
\]

thus proving the claim.

Problem 2.

Note that \( |\sigma| \leq M \) implies \( \int_0^t \sigma^2(s)ds \leq M^2t \) for all \( t \geq 0 \), so

\[
P \left[ \sup_{t \in [0,T]} I(t) \geq \lambda \right] \leq P \left[ \sup_{t \in [0,T]} e^{\alpha I(t)} - \frac{1}{\alpha^2} \int_0^t \sigma^2(s)ds \geq e^{\alpha \lambda - \frac{1}{\alpha^2}M^2T} \right]
\]

Since \( M(t) = \exp\{\alpha I(t) - \frac{1}{2} \alpha^2 \int_0^t \sigma^2(s)ds\} \) is a positive continuous martingale by Kolmogorov-Doob’s inequality

\[
P \left[ \sup_{t \in [0,T]} M(t) \geq e^{\alpha \lambda - \frac{1}{\alpha^2}M^2T} \right] \leq \frac{1}{e^{\alpha \lambda - \frac{1}{\alpha^2}M^2T}} \sup_{t \in [0,T]} EM(t) = e^{-\alpha \lambda + \frac{1}{\alpha^2}M^2T}
\]
In order to get the best possible estimate we look for \( \lambda \) so that the RHS is smallest: 
\[
 f(\lambda) = e^{\alpha \lambda - \frac{1}{2} \alpha M^2 T} 
\]
is convex with 
\[
 f'(\lambda) = f(\lambda)(-\lambda + \alpha M^2 T) 
\]
so 
\[
 \lambda_{\text{min}} = \alpha M^2 T
\]
and 
\[
 f(\lambda_{\text{min}}) = e^{-\lambda_{\text{min}}^2 / 2}
\]
so
\[
P\left( \sup_{t \in [0,T]} I(t) \geq \lambda \right) \leq e^{-\frac{\lambda^2}{2\alpha M^2 T}}
\]
Now note that 
\[
P\left( \inf_{t \in [0,T]} I(t) \leq -\lambda \right) = P\left( \sup_{t \in [0,T]} -I(t) \geq \lambda \right),
\]
but since 
\[
 -I(t) = \int_0^t -\sigma(s) dB_s
\]
has the same property \(|-\sigma| \leq M\), the quantity \(\sup -I(t)\) can be shown to satisfy the same inequality as \(\sup I(t)\), so
\[
P\left( \sup_{t \in [0,T]} |I(t)| \geq \lambda \right) \leq 2e^{-\frac{\lambda^2}{\alpha M^2 T}}
\]
Problem 3.
For any \( n \geq 1 \) we have a telescoping sum
\[
\sum_k \frac{1}{2}(W_{t_{k+1}}^n + W_{t_k}^n)(W_{t_{k+1}}^n - W_{t_k}^n) = \sum_k \frac{1}{2}(W_{t_{k+1}}^2 - W_{t_k}^2) = \frac{1}{2}W_t^2 - W_0^2 = \frac{1}{2}W_t^2
\]
so 
\[
X_t = \frac{1}{2}W_t^2.
\]
By writing 
\[
W_t = W_s + (W_t - W_s)
\]
\[
E[X_t|\mathcal{F}_s] = E\left[ \frac{1}{2}W_s^2 + W_s(W_t - W_s) + \frac{1}{2}(W_t - W_s)^2 \right| \mathcal{F}_s] = \frac{1}{2}W_s^2 + \frac{1}{2}(t - s)
\]
we see \( X_t \) is not a martingale, but \( X_t - \frac{1}{2}t \) is a martingale.