Problem 1. Use Ito’s formula to show that if \( \sigma(t, \omega) \) is an adapted stochastic process which is bounded almost surely with \( |\sigma| \leq M \), then the stochastic integral \( I(t, \omega) = \int_0^t \sigma(s, \omega) dB(s, \omega) \) satisfies the following moment estimates
\[
E\{ |I(t)|^{2p} \} \leq 1 \cdot 3 \cdot 5 \cdot \cdots (2p-1)(M^2 t)^p
\]
for \( p = 1, 2, 3, \ldots \). How did you show that the mean of the martingale term in Ito’s formula is zero?

Problem 2. Let \( \sigma(t, \omega) \) be adapted and bounded by \( M \) as in problem 1. Let \( I(t, \omega) = \int_0^t \sigma(s, \omega) dB(s, \omega) \). Use the exponential martingale \( \exp\{\alpha I(t) - \frac{\alpha^2}{2} \int_0^t \sigma^2(s) ds \} \) and the Kolmogorov-Doob inequality to get the estimate
\[
P\{ \sup_{0 \leq t \leq T} |I(t)| \geq \lambda \} \leq 2 \exp \left\{ -\frac{\lambda^2}{2M^2 T} \right\}
\]
First express the event of interest in terms of the exponential martingale, then use the Kolmogorov-Doob inequality and after this choose the parameter \( \alpha \) to get the best bound.

Problem 3. (The Stratonovich integral: A first example) Let us denote the Stratonovich integral of a standard Brownian motion \( W(t) \) with respect to itself by
\[
\int_0^t W(s) \circ dW(t).
\]
we then define the integral buy
\[
\int_0^t W(s) \circ dW(t) = \lim_{n \to \infty} \sum_k \frac{1}{2}(W(t_{k+1}^n) + W(t_k^n))(W(t_{k+1}^n) - W(t_k^n))
\]
where \( t_k^n = tk/n \). Prove that with probability one
\[
X_t = \int_0^t W(s) \circ dW(s) = \frac{1}{2} W(t)^2.
\]
Observe that this is what one would have if one used standard (as opposed to Itô) calculus. Calculate \( E[X_t | \mathcal{F}_s] \) for \( s < t \) where \( \mathcal{F}_t \) is the \( \sigma \)-algebra generated by the Brownian motion. Is \( X_t \) a martingale with respect to \( \mathcal{F}_t \).
Problem 4. Let $X_t$ be an Itô processes with

$$dX_t = f_t dt + g_t dW_t$$

and $B_t$ be a second (possibly correlated with $W$ ) Brownian motion. We define the Stratanovich integral $\int X_t d dB_t$ by

$$\int_0^T X_t d dB_t = \int_0^T X_t dB_t + \frac{1}{2} \int_0^T d\langle X, B \rangle_t$$

Recall that if $B_t = W_t$ then $d\langle B, W \rangle_t = dt$ and it is zero if they are independent. Use this definition to calculate:

a) $\int_0^1 B_t d dB_t$ (Compare this to the answer you obtained is Problem 3).

b) The equation satisfied by $Y_t = f(X_t)$ written in terms of Stratanovich integrals. (Use Itô’s formula to find the equation for $dY_t$ in terms of Itô integrals and then use the above definition to rewrite the Itô integrals as Stratanovich integrals “$\circ dB_t$”.)

c) (Integration by parts) Let $Z_t$ be a second Itô process satisfying

$$dZ_t = b_t dt + \sigma_t dW_t$$

Calculate $d(X_t Z_t)$ using Itô’s formula and then write it in terms of Stratanovich integrals. Why is this part of the problem labeled integration by parts ? (Write the integral form of the expression you derived for $d(X_t Z_t)$ in the two cases. What are the differences ?)

Problem 5. (Definition of stochastic integrals by integration by parts) In 1959, Paley, Wiener, and Zygmund gave a definition of the stochastic integral based on integration by parts. The resulting integral will agree with the Itô integral when both are defined. However the Itô integral will have a much large domain of definition. We will now follow the develop the integral as outlined by Paley, Wiener, and Zygmund:

a) Let $f(t)$ be a deterministic function with $f'(t)$ continuous. Prove that

$$\int_0^1 f(t) dW(t) = f(1) W(1) - \int_0^1 f'(t) W(t) dt$$

where the first integral is the Itô integral and the last integral is defined path-wise as the standard Riemann integral since the integrands are a.s. continuous.

b) Now let $f$ we as above with in addition $f(1) = 0$ and “define” the stochastic integral $\int_0^1 f(t) \ast dW(t)$ by the relationship

$$\int_0^1 f(t) \ast dW(t) = - \int_0^1 f'(t) W(t) dt$$

Where the integral on the right hand side is the standard Riemann integral.
c) Show by direct calculation (not by the Itô isometry) that

$$E \left[ \left( \int_0^1 f(t) * dW(t) \right)^2 \right] = \int_0^1 f^2(t) dt ,$$

Paley, Wiener, and Zygmund then used this isometry to extend the integral to any deterministic function in $L^2[0, 1]$. This can be done since for any $f \in L^2[0, 1]$, one can find a sequence of deterministic functions in $\phi_n \in C^1[0, 1]$ with $\phi_n(1) = 0$ so that

$$\int_0^1 (f(s) - \phi_n(s))^2 ds \to 0 \text{ as } n \to 0.$$