Math 219, Stochastic Differential Equations

Problem set 3

Problem 1.
Rewrite \( W(t_{j+1}) = W(t_j^a) + W(t_{j+1}) - W(t_j^a) \) and \( W(t_j^a) = W(t_j) + W(t_j^a) - W(t_j) \)

\[
\mathbf{E}I_N^\alpha(t) = \sum_{j=0}^{N-1} \left( \mathbf{E}(W(t_j^a)^2 + \mathbf{E}(W(t_j^a)(W(t_{j+1}) - W(t_j^a))) - \mathbf{E}(W(t_j)^2 - \mathbf{E}(W(t_j)(W(t_j^a) - W(t_j)))) \right)
\]

\[
= \sum_{j=0}^{N-1} (t_j^a + 0 - t_j - 0) = \sum_{j=0}^{N-1} (at_j + (1-\alpha)t_{j+1} + t_j) = \sum_{j=0}^{N-1} (1-\alpha)(t_{j+1} - t_j) = (1-\alpha)t
\]
so \( \lim_{N \to \infty} \mathbf{E}I_N^\alpha(t) = (1-\alpha)t \) as well.

For \( \alpha = 1 \) we get the standard Ito integral, and for \( \alpha = \frac{1}{2} \) the Stratonovich integral.

Problem 2.

To prove a process \( X(t) \) is a standard Brownian motion we need to check that:
(i) \( X(0) = 0 \) and continuity of \( t \mapsto -W(t) \) is obvious; (ii) for \( 0 < t_0 < t_1 < \ldots < t_n \) increments \( -(W(t_1) - W(t_0)), \ldots, -(W(t_n) - W(t_{n-1})) \) are just \(-\)increments of \( W(t) \) hence are independent; (iii) \( -(W(t') - W(t)) \sim N(0, (-1)^2(t' - t)) \)

a) (i) \(-W(0) = 0 \) and continuity of \( t \mapsto -W(t) \) is obvious; (ii) for \( 0 < t_0 < t_1 < \ldots < t_n \) increments \( c(W(t_1/c^2) - W(t_0/c^2)), \ldots, c(W(t_n/c^2) - W(t_{n-1}/c^2)) \) are increments of \( cW(t) \) at \( t_0/c^2, t_1/c^2, \ldots, t_n/c^2 \) hence are independent; (iii) \( Y(t') - Y(t) = c(W(t'/c^2) - W(t/c^2)) \sim N(0, c^2(t' - t)) \)

b) (i) \( Y(0) = cW(0) = 0 \) and continuity of \( t \mapsto cW(t/c^2) \) is obvious; (ii) for \( 0 < t_0 < t_1 < \ldots < t_n \) increments \( c(W(t_1/c^2) - W(t_0/c^2)), \ldots, c(W(t_n/c^2) - W(t_{n-1}/c^2)) \) are increments of \( cW(t) \) at \( t_0/c^2, t_1/c^2, \ldots, t_n/c^2 \) hence are independent; (iii) \( Y(t') - Y(t) = c(W(t'/c^2) - W(t/c^2)) \sim N(0, c^2(t' - t)) \)

c) (i) \( Z(0) = W(s) - W(s) = 0 \) and continuity of \( t \mapsto W(t + s) - W(s) \) is obvious; (ii) for \( t_0 < t_1 < \ldots < t_n \) increments \( W(t_0 + s) - W(s), W(t_1 + s) - W(t_0 - s), \ldots, W(t_n + s) - W(t_{n-1} + s) \)
are just increments of $W(t)$ at $t_0 + s < t_1 + s < \ldots < t_n + s$ hence are independent; (iii) 
$Z(t') - Z(t) = W(t' + s) - W(t + s) \sim N(0, t' + s - t - s)$

d) Here it is more convenient to use the equivalent definition of BM which replaces (ii) and (iii) by: (ii’) $X(t)$ is Gaussian process, and (iii’) $E[X(t)] = 0$ and $E[X(t')X(t)] = \min(t', t)$.

(i) $X(0) = 0$ by definition and continuity of $t \mapsto tW(1/t)$ is obvious except at $t = 0$ where we observe $\lim_{t \to 0} X(t) = \lim_{s \to \infty} W(s)/s = 0$ by the SLLN for Brownian motion; (ii’) for $t_0 < t_1 < \ldots < t_n$ the vector $((1/t_0)W(1/t_0), \ldots, (1/t_n)W(1/t_n))$ is a scalar combination of a multivariate Gaussian hence has multivariate Gaussian distribution; (iii) $E[X(t')X(t)] = (1/t')(1/t')E[W(1/t')W(1/t)] = (1/t')(1/t)\min(1/t', 1/t) = \min(t', t)$

Problem 3.
We have $\eta_k \sim N(0, 1)$ i.i.d. hence

$$E[X(t)] = \frac{t}{\sqrt{\pi}}E\eta_0 + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{\sin(kt)}{k} E\eta_k = 0$$

and due to independence of $\eta_k$

$$\text{Var}X(t) = \frac{t^2}{\pi} \text{Var}\eta_0 + 2 \sum_{k=1}^{\infty} \frac{\sin^2(kt)}{k^2} \text{Var}\eta_k = t^2 \left( \frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin^2(kt)}{(kt)^2} \right) = t^2 \frac{1}{t}$$

using the Fourier expansion of the function $1/t$ on $[0, \pi]$. Also

$$E[X(t')]X(t) = \frac{t't}{\pi} \text{Var}\eta_0 + 2 \sum_{k=1}^{\infty} \frac{\sin(kt')\sin(kt)}{k^2} \text{Var}\eta_k = \frac{t't}{\pi} + 2 \sum_{k=1}^{\infty} \frac{\sin(kt')\sin(kt)}{k^2} = \min(t', t)$$

Formally the derivative of $X(t)$ is

$$\frac{dX(t)}{dt} = \frac{1}{\sqrt{\pi}} \eta_0 + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \cos(kt) \eta_k$$

Notice that all of the frequencies have the same weight. that is why it is called “white”. White light is a mixture of light were all colors (frequencies) are equally included.