Math 219, Stochastic Differential Equations
Problem set 2

Problem 1.
To check the process is stationary let $\xi = (-1)^i$, and $k, l \in \{0, 1\}$

\[ P\left(X(s) = (-1)^k, X(t) = (-1)^l\right) = P\left(N(s) \text{ mod } 2 = (k + i) \text{ mod } 2, N(t) \text{ mod } 2 = (l + i) \text{ mod } 2\right) \]

Since the Poisson process $N(t)$ is stationary, the transition probability for its parity depends only on $t - s$ as well. For the covariance of $X(t)$ let $s \leq t$

\[ E(X(s)X(t)) = E((-1)^{N(s)+N(t)} = E((-1)^{N(t) - N(s)}) = \sum_{k=0}^{\infty} (-1)^k(t - s)^k e^{-(t-s)/k!} = e^{-2(t-s)} \]

For $s > t$ just interchange the roles of $s$ and $t$ to get $E(X(s)X(t)) = e^{-2|t-s|}$.

Note that for any $s < t$ $|X(t) - X(s)| \in \{0, 2\}$, depending on whether $N(t)$ has an even or an odd number of jumps in $[s, t]$, so for any $\beta > 0$

\[ E|X(t) - X(s)|^\beta = 2^\beta \sum_{k=0}^{\infty} e^{-(t-s)}(t - s)^{2k+1} (2k+1)! = 2^\beta e^{-(t-s)} \sinh(t - s) = 2^{\beta-1}(1 - e^{-2(t-s)}) \]

and for any choice of $\beta, \alpha > 0, K < \infty$ we can find $s < t$ so that the right hand side above will be $> K|t - s|^{1+\alpha}$. Kolmogorov’s condition for path continuity is not satisfied by $X(t)$.

The OU process is stationary and Gaussian, so for any $s < t$ $X(t) - X(s)$ has a Normal distribution with mean 0 and variance

\[ E(X(t) - X(s))^2 = E(X(t)^2) + E(X(s)^2) - 2E(X(t)X(s)) = 2(1 - e^{-2|t-s|}) \leq 2|t - s| \quad (1) \]

Now since $X(t) - X(s)$ is Normal we have that

\[ E(X(t) - X(s))^4 = 3\left[E(X(t) - X(s))^2\right]^2 \leq 12|t - 2|^4 \]

which implies that Kolmogorov’s condition for path continuity is satisfied.
Problem 2.

Two possible choices are $Y_n = \sum_{k=0}^{n} \sigma_k^2$ and $\tilde{Y}_n = (\sum_{k=0}^{n} Z_k)^2$ since

$$E\left( (X_n - Y_n) - (X_{n-1} - Y_{n-1}) | F_{n-1} \right) = E(Z_n^2) - \sigma_n^2 = 0,$$

and

$$E\left( (X_n - \tilde{Y}_n) - (X_{n-1} - \tilde{Y}_{n-1}) | F_{n-1} \right) = E(Z_n^2) - E\left( Z_n^2 + 2Z_n \left( \sum_{k=0}^{n-1} Z_k \right)^2 | F_{n-1} \right) = 0$$

Problem 3.

a) Since $E(W_t^2 - W_s^2 | F_s) = E((W_t - W_s)^2 + 2(W_t - W_s)W_s | F_s) = t - s$ the right choice is $X_t = t$.

b) Since $E(e^{iW_t + \alpha t} | F_s) = e^{iW_s + \alpha t} E(e^{i(W_t - W_s)}) = e^{iW_s + \alpha t} e^{-\frac{1}{2}(t-s)}$ the right choice is $\alpha = 1/2$. 