1 HW 6

p 380 - 8 Let $X_i$ be 1 of the $i$-th person sits at a new table and zero if not. Notice that if $T$ is the number of tables used then

$$ T = \sum_{k=1}^{N} X_i $$

and so $ET = \sum_{k=1}^{N} EX_i$

If we assume that each person knows each pair of persons knows each other with probability $p$ independently of all of the other pairs then each time some one shows up for them to sit at a new table they need to not know any of the other people there. So

$$ X_i = \begin{cases} 
1 & (1-p)^{i-1} \\
0 & 1 - (1-p)^{i-1} 
\end{cases} $$

Hence $EX_i = (1-p)^{i-1}$ and

$$ ET = \sum_{k=1}^{N} (1-p)^{i-1} = \sum_{k=0}^{N-1} (1-p)^i = \frac{1 - (1-p)^N}{p} $$

p 380 - 24 We skipped this one. So it was not graded.

p 380 - 50 We will need the Marginal density in $y$.

$$ f_Y(y) = \int_{0}^{\infty} f(x, y) dx = e^{-y} $$

so the density of $x$ conditioned on $y$ is

$$ f(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{e^{-\frac{x}{y}}}{y} $$

so since $E(g(X)|Y) = \int g(x)f(x|Y)dx$ we have

$$ E(X^2|Y) = \int_{0}^{\infty} x^2 e^{-\frac{x}{Y}} dx = 2Y^2 $$

This at least looks right since the answer should be a function of $Y$ since that was not averaged out by the expectation.

p 380 - 59 The point is that by conditioning on the $T$ defined in the hint we can reduce the problem to simple calculations. Let $X$ be the number of flips to get $H, H, H$

$$ E(X|T = 0) = 3 \quad E(X|T = 1) = 1 + E(X) \quad E(X|T = 2) = 2 + E(X) \quad E(X|T = 3) = 3 + E(X) $$

also

$$ P(T = 0) = p^3 \quad P(T = 1) = 1 - p \quad P(T = 2) = p(1-p) \quad P(T = 3) = p^2(1-p) $$

so

$$ E(X) = E(X|T = 0)P(T = 0) + E(X|T = 1)P(T = 1) + E(X|T = 2)P(T = 2) + E(X|T = 3)P(T = 3) $$

$$ = 3p^3 + [1 + E(X)](1-p) + [2 + E(X)]p(1-p) + [3 + E(X)]p^2(1-p) $$

$$ = \frac{p^2 + p + 1}{p^3} $$
Looking at the moment generating function we see that $X$ is distributed Poisson with parameter $\lambda = 2$ and $Y$ is distributed binomial with $n = 10$ and $p = \frac{3}{4}$. Also we were told that the two were independent.

$$P(X + Y = 2) = P(X = 1, Y = 1) + P(X = 2, Y = 0) + P(X = 0, Y = 2)$$

$$= 10 \left( \frac{3}{4} \right)^9 \left( \frac{1}{4} \right)^1 + 2e^{-2} + \frac{3^2}{8} \left( \frac{1}{4} \right)^4 + \frac{2^2}{2} e^{-2} \left( \frac{1}{4} \right)^{10}$$

$$P(XY = 0) = P(X = 0, Y = 0) = \left( \frac{1}{4} \right)^{10} e^{-2}$$

Now since $X$ and $Y$ are independent $E(XY) = E(X)E(Y) = 2(\frac{15}{2}) = 15$

We have

$$E[(X - a)^2] = \int_{-\infty}^{\infty} (x - a)^2 f(x) dx$$

so

$$\frac{\partial}{\partial a} E[(X - a)^2] = \int_{-\infty}^{\infty} \frac{\partial}{\partial a} (x - a)^2 f(x) dx = -\int_{-\infty}^{\infty} 2(x - a)f(x) dx$$

$$= a \int_{-\infty}^{\infty} 2f(x) dx - \int_{-\infty}^{\infty} 2xf(x) dx = 2(a - E(X))$$

Setting the derivative equal to zero we see that $E(X) = a$. It is simple to see that it is a minimum. (Notice that when $a < E(X)$ the derivative is negative and when $a > E(X)$ the derivative is positive.)

Now

$$E[|X - a|] = \int_{-\infty}^{\infty} |x - a| f(x) dx = \int_{-\infty}^{a} (x - a)f(x) dx + \int_{a}^{\infty} (a - x)f(x) dx$$

so

$$\frac{\partial}{\partial a} E[|X - a|] = \frac{\partial}{\partial a} \int_{-\infty}^{a} (x - a)f(x) dx + \frac{\partial}{\partial a} \int_{a}^{\infty} (a - x)f(x) dx$$

and

$$\frac{\partial}{\partial a} \int_{-\infty}^{a} (x - a)f(x) dx = (a - a)f(a) - \int_{-\infty}^{a} f(x) dx = -\int_{-\infty}^{a} f(x)$$

$$\frac{\partial}{\partial a} \int_{a}^{\infty} (a - x)f(x) dx = -(a - a)f(a) + \int_{a}^{\infty} f(x) dx = \int_{a}^{\infty} f(x) dx$$

putting this all together and setting the derivative to zero gives

$$\int_{-\infty}^{a} f(x) dx = \int_{a}^{\infty} f(x) dx$$

so $a$ must be the point which splits the probability mass in two equal parts. This is the definition of the medium.

The idea is that we want to estimate the mean $\mu$. We construct $Y = \lambda X_1 + (1 - \lambda) X_2$. Notice that $EY = \mu$ so it on average will have the value $\mu$. We want the value it talks to be as close to $\mu$ as we can. Hence we want the definition of $Y$ which will be as highly concentrated around $\mu$ as possible. This mean we want the variance as small as possible. Now $\text{Var}(Y) = \lambda^2 \text{Var}(X_1) + (1 - \lambda)^2 \text{Var}(X_2) = \lambda^2 \sigma_1^2 + (1 - \lambda)^2 \sigma_2^2$. Differentiating with respect to $\lambda$, we find that the minimum is at $\lambda = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$. 

$$\text{Cov}(X, Y|Z) = E[(X - E(X|Z))(Y - E(Y|Z))|Z]$$

$$= E[XY - YE(X|Z) - XE(Y|Z) + E(X|Z)E(Y|Z)|Z]$$


Now we need that $E[E(X|Z)E(Y|Z)|Z] = E[X|Z]E[Y|Z]$ so the random variables are independent as the joint moment generating function is just the product of the independent ones.

First let us assume that $\text{Cov}(X, Y, Z) = 0$ if $i \neq j$ and show the random variables are independent. From above we have

$$M(t_1, \ldots, t_m) = \exp \left( \sum_{i=1}^{m} t_i \mu_i + \frac{1}{2} \sum_{i=1}^{m} t_i \sum_{j=1}^{m} t_j \text{Cov}(X_i, X_j) \right)$$

$$= \exp \left( \sum_{i=1}^{m} t_i \mu_i + \frac{1}{2} \sum_{i=1}^{m} t_i^2 \text{Var}(X_i) + \frac{1}{2} \sum_{i=1}^{m} \sum_{j \neq i}^{m} t_i t_j \text{Cov}(X_i, X_j) \right)$$

Since each of $X_i$ is just a Normal random variable (being the sum of independent Normal random variables),

$$M_i(t_i) = Ee^{t_i X_i} = \exp \left( t_i \mu_i + \frac{t_i^2}{2} \sum_{i=1}^{m} \text{Var}(X_i) \right)$$

First let us assume that $\text{Cov}(X_i, X_j) = 0$ if $i \neq j$ and show the random variables are independent. From above we have

$$M(t_1, \ldots, t_m) = \exp \left( \sum_{i=1}^{m} t_i \mu_i + \frac{1}{2} \sum_{i=1}^{m} t_i^2 \text{Var}(X_i) \right)$$

$$= \exp(t_1 \mu_1 + \frac{t_1^2}{2} \text{Var}(X_1)) \cdots \exp(t_1 \mu_m + \frac{t_m^2}{2} \text{Var}(X_m)) = M_1(t_1) \cdots M_m(t_m)$$

so the random variables are independent as the joint moment generating function is just the product of the individual ones.

Now let us assume that they are independent and show that $\text{Cov}(X_i, X_j) = 0$ if $i \neq j$. Since the $X_i$ are independent we know that $M(t_1, \ldots, t_m) = M_1(t_1) \cdots M_m(t_m)$ so

$$M(t_1, \ldots, t_m) = \exp \left( \sum_{i=1}^{m} t_i \mu_i + \frac{1}{2} \sum_{i=1}^{m} t_i^2 \text{Var}(X_i) + \right)$$

However the expression for the joint moment generating function given at the start still holds. Comparing the two implies that $\text{Cov}(X_i, X_j) = 0$ if $i \neq j$. 
Since $X$ is log normal with parameters $(\mu, \sigma^2)$, $Y = \log(X)$ is normal with mean $\mu$ and variance $\sigma^2$.

Now

$$\exp(\mu t + \frac{\sigma^2 t^2}{2} ) = \mathbb{E}(e^{tY}) = \mathbb{E}(e^{t \log(X)}) = \mathbb{E}((e^{t \log(X)})^t) = \mathbb{E}(X^t)$$

So to obtain the mean we evaluate the right hand side at $t = 1$ and to get the second moment we use $t = 2$. So

$$\mathbb{E}(X) = \exp(\mu + \frac{\sigma^2}{2}) \quad \mathbb{E}(X^2) = \exp(\mu^2 + \sigma^2 2)$$

$$\text{Var}(X) = \exp(\mu^2 + \sigma^2 2) - \exp(\mu^2 + \sigma^2)$$