1 HW 5

p. 293, # 33 Let \( X = \text{Jill} \sim N(170, 20^2), Y = \text{Jack} \sim N(160, 15^2), \) and \( Z \sim N(0, 1) \) Then \( Y - X \sim N(-10, 20^2 + 15^2) \)

\[
\mathbb{P}(Y > X) = \mathbb{P}(Y - X > 0) = \Phi\left( \frac{Y - X + 10}{25} \right) = P(Z > .4) = 1 - \Phi(.4) = 1 - .655 = .345
\]

p. 294, # 39 \( p(a, b) = P(X = a, Y = b) = \frac{1}{5} \frac{1}{b} \) for \( a \in \{1, \ldots, 5\}, b \in \{1, \ldots, a\}. \) \( \mathbb{P}(Y = b) = \sum_{k=b}^{5} p(k, b) = \sum_{k=b}^{5} \frac{1}{k}. \) So \( p(a|b) = P(X = a|Y = b) = P(X = a, Y = b) / P(Y = b) = \frac{1}{5} (\sum_{k=b}^{5} \frac{1}{k})^{-1}. \) They are not independent. For example if you know that \( X = 1 \) then \( Y \) must equal one.

p. 294, # 40 Coming soon.

p. 294, # 41

\[
p_Y(1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \quad p_Y(2) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}
\]

\[
p(1|1) = \frac{p(1, 1)}{p_Y(1)} = \frac{1}{8} / 4 = \frac{1}{2} \quad p(2|1) = \frac{p(2, 1)}{p_Y(1)} = \frac{1}{8} / 4 = \frac{1}{2}
\]

\[
p(1|2) = \frac{p(1, 2)}{p_Y(2)} = \frac{1}{4} / 3 = \frac{1}{3} \quad p(2|2) = \frac{p(2, 2)}{p_Y(2)} = \frac{1}{4} / 3 = \frac{2}{3}
\]

Check: as must happen \( p(1|1) + p(2|1) = p(1|2) + p(2|2) = 1 \)

No, they are not independent. Notice that \( p_X(1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8} \) so \( p_X(1)P_Y(1) = \frac{3}{8} \frac{1}{8} = \frac{3}{8} = p(1, 1) \)

p. 294, # 42

\[
f_X(x) = \int_{0}^{\infty} f(x, y)dy = e^{-x} \quad f_Y(y) = \int_{0}^{\infty} f(x, y)dx = \frac{1}{(y+1)^2}
\]

\[
f_Y(x) = \frac{f(x, y)}{f_Y(y)} = e^{-xy} \quad f(x|y) = \frac{f(x, y)}{f_X(x)} = xe^{-x(y+1)}(y+1)^2
\]

Now if \( Z = XY \) then

\[
f_Z(z) = \int_{0}^{\infty} f(x, \frac{z}{x})dx = \int_{0}^{\infty} xe^{-xy-x}dx = e^{-z}
\]

Alternatively we can find the cumulative distribution function of \( Z \) first and then differentiate it. One gets the same thing.

p. 295, # 53 To find the joint density of \((X, Y)\) we need to find the Jacobian of the transformation which takes \((U, Z)\) to \((X, Y)\).

\[
J = \det \left( \begin{array}{cc} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial Z} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial Z} \end{array} \right) = -(\sin^2(U) + \cos^2(U))
\]

So \( f_{X,Y}(x, y) = \frac{1}{|J|} f_{U,Z}(U, Z) \). Since \( Z \) and \( U \) are independent \( f_{U,Z}(U, Z) = f_U(U) f_Z(Z) = \frac{1}{2\pi} e^{-z} \).

Now we need the inverse of the transform, but we only need \( Z \) as a function of \( X \) and \( Y \) since the
density does not depend on $U$. $X^2 + Y^2 = 2Z(\sin^2(U) + \cos^2(U)) = 2Z$ so $Z = \frac{1}{2}(X^2 + Y^2)$ and $f_{X,Y}(x,y) = 1\sqrt{2\pi}e^{-\frac{x^2}{2}}e^{-\frac{y^2}{2}}$. Since this is the product of $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ with $\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$ we conclude that $X$ and $Y$ are distributed as independent Normal random variables with mean 0 and variance 1.

p. 296, # 5 First let $Z = \frac{X}{Y}$, the cumulative distribution function is

$$F_Z(a) = \int_0^a \int_0^{\infty} f_X(x)f_Y(y)dxdy$$

so

$$f_Z(a) = \frac{\partial}{\partial a} F_Z(a) = \int_0^a \frac{\partial}{\partial a} \int_0^{\infty} f_X(x)f_Y(y)dxdy = \int_0^{\infty} y f_X(ya)f_Y(y)dy$$

where the last equality follows from the fundamental theorem of calculus. $(y = \frac{\partial}{\partial a} ay)$ Now evaluating this expression when $f_X(u) = f_Y(u) = \exp(-u)$ gives

$$f_Z(a) = \int_0^{\infty} ye^{-(y+ya)}dy = \frac{1}{(1+a)^2}$$

Now let $Z = XY$, the cumulative distribution function is

$$F_Z(a) = \int_0^{\infty} \int_0^{\frac{a}{x}} f_X(x)f_Y(y)dxdy$$

so

$$f_Z(a) = \frac{\partial}{\partial a} F_Z(a) = \int_0^{\infty} \frac{\partial}{\partial a} \int_0^{\frac{a}{x}} f_X(x)f_Y(y)dxdy = \int_0^{\infty} \frac{1}{x} f_X(x)f_Y\left(\frac{a}{x}\right)dx$$

where the last equality follows from the fundamental theorem of calculus. $(\frac{1}{x} = \frac{\partial}{\partial a} \frac{a}{x})$ Now evaluating this expression when $f_X(u) = f_Y(u) = \exp(-u)$ gives

$$f_Z(a) = \int_0^{\infty} \frac{1}{x} e^{-(x+\frac{a}{x})}dx$$

This integral can not be evaluated exactly in closed form. It can be express in terms of a modified Bessel function however which is just another special function like sin and cos.