Exam 2: Math 135/Stats 104, Fall 2003

You may use your notes and the textbook. You may use a computer math program or calculator (though I really don’t think you need one.) No other books. No web resources. No other living person other than the me (Jonathan Mattingly). If you have any questions do not hesitate to ask. The exam is due Monday, December 1st.

Version 3: Problem 10 has been corrected. Problem 7 simplified to avoid a problem some people have (and make it shorter)

Please ! State your assumptions ! Do not expect full credit if you don’t. Answers without supporting work will receive no credit. Similarly, explain what you are doing ! Do not expect any partial credit if you don’t. I want to understand how you are doing the problem.

Your test will not be graded unless you affirm the Honor code on your exam !.

Abbreviations:

i.i.d. := independent, identically distributed
p.m.f. := probability mass function
m.g.f. := moment generating function

Question 1. (10 points) Suppose that \( X \sim Bern(1/2) \) and \( Y \sim U[0, 1] \) are independent.

a) Compute \( P[X = 1, Y \in (0, 1/3)] \).

b) Find the distribution of the random variable \( 3(X + Y) \). (Identify it as a well known distribution with a certain parameter.)

Question 2. (15 points) Suppose \( X \) has density \( f_X(x) = \frac{\lambda}{2}e^{-\lambda|x|} \), where \( \lambda > 0 \).

a) Find the m.g.f. of \( X \). Find \( EX^2 \) and \( EX^3 \).

b) Say \( Y \sim Exponential(\lambda_2) \), and is independent of \( X \). Find the m.g.f. of \( X + Y \). Find the Cov\((X, Y)\).

c) Suppose \( Z = 2X \). Find the m.g.f. of \( X + Z \). Find the Cov\((X, Z)\).

Question 3. (15 points) Mark the fisherman catches fish in Wallowa Lake at random times at a average rate of 4 / hour. Each fish caught has a \( 1/3 \) chance of being a trout.

a) If Mark fishes for five hours, what is the expectation and variance of the number of trout caught.

b) What is the expected amount of time needed to catch one trout?

Question 4. (15 points) Let \( X_1, X_2, \ldots \) be a collection of independent, identically distributed positive random variables. Explain why the limit

\[
\lim_{n \to \infty} \left[ X_1 X_2 \cdots X_n \right]^\frac{1}{n}
\]

exists. Show that the limit equals \( \exp(\mathbb{E} \log(X_1)) \). Calculate the limit when \( X_i = \exp(\exp(2Y_i)) \) if \( Y_i \) are independent normal with mean \( \mu \) and variance \( \sigma^2 \).
**Question 5.** (15 points) Let $X$ be a random variable taking its values in the positive integers.

a) Show that

$$E[X] = \sum_{k=1}^{\infty} P(X \geq k)$$

b) Let $X_1, X_2, \ldots$ be i.i.d contiguous random variables. We call $n$ the first time of increase if

$$X_1 > X_2 > X_3 > \cdots > X_{n-1} < X_n$$

Let $N$ be the time until the first increase. Use the first part of this problem to show that $E[N] = e$.

**Question 6.** (15 points) Jill is at a casino and has the chance to play the following game. (The pot starts at zero.)

a) $N$ cards are placed face down on the table and the player picks one.

b) All cards are then turned over.

c) If the player picked the biggest one the game is over and she gets the money in the pot.

d) If she picked the smallest one, the game is over and she wins nothing.

e) If she picked the $n$-th biggest, $n$ dollars are placed in the pot by the house and the game returns to a) and starts again.

A) What is the probability of winning the game?

B) What is the expected amount won by playing the game?

C) How much should the casino charge to play this game if they want to on average earn 5 dollars for each time the game is played.

**Question 7.** (15 points) If $X_1$ and $X_2$ are independent exponential random variables, each with parameter $\lambda > 1$, find the joint density of $Y_1 = X_1 + X_2$ and $Y_2 = \exp(X_1)$. What is $\text{Cov}(X_1, X_2)$? (2 points extra: What is $\text{Cov}(Y_1, Y_2)$?)

**Question 8.** (10 points) Suppose that your new Microsoft MP3 player is great except that it has to use a special batteries which you have to buy from Microsoft. Assume the mean life time of each battery is 4 weeks with a standard deviation of 1 week. If you replace a dead battery with a new one as soon as the old one dies, approximately what is the probability that more than 26 replacements will be needed over a two year period. (The first battery does not count as a replacement.)

**Question 9.** (15 points) Let $X_i$ be the last digit of $D^2_i$, where $D_i$ is a random digit between 0 and 9, inclusive. For instance if $D_i = 7$ then $D^2_i = 49$ so $X_i = 9$. Define

$$\bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$$

$$\bar{D}_n = \frac{D_1 + \cdots + D_n}{n}$$

where $D_i$ are independent choices of digits.

a) Predict the value of $\bar{X}_n$ for large $n$,

b) Find a number $\epsilon$ such that for $n = 10,000$ the chance that your prediction is off by more than $\epsilon$ is about 1 in 200.

c) Find approximately the least value of $n$ such that your prediction of $X_n$ is correct to within 0.01 with probability at least 0.99.

d) Which can be predicted with better accuracy for large $n$: the value of $\bar{X}_n$ or the value of $\bar{D}_n$?

**Question 10.** (10 points) Let $X_i$ be a sequence of mutually independent random variables with mean zero and $\text{Var}(X_i) < \frac{1}{i^2}$. Define $Y_n = X_1 + \cdots + X_n$. Show that for any $n$ and positive $\beta$, $P(Y_n > \beta) < 2\beta^{-2}$. 