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EXISTENCE OF GLOBAL WEAK SOLUTIONS OF p-NAVIER-STOKES EQUATIONS

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Abstract. This paper investigates the global existence of weak solutions for the incompressible p-Navier-Stokes equations in $\mathbb{R}^d$ $(2 \leq d \leq p)$. The p-Navier-Stokes equations are obtained by adding viscosity term to the p-Euler equations. The diffusion added is represented by the p-Laplacian of velocity and the p-Euler equations are derived as the Euler-Lagrange equations for the action represented by the Benamou-Brenier characterization of Wasserstein-p distances with constraint density to be characteristic functions.

1. **Introduction.**

In this paper, we show the global existence of weak solutions for the p-Navier-Stokes (p-NS) equations in $\mathbb{R}^d$:

$$\begin{cases} \partial_t u_p + u \cdot \nabla u_p + \nabla \pi = \nu \Delta_\gamma u, \\ u_p = |u|^{p-2} u, \\ \Delta_\gamma u = \nabla \cdot (|\nabla u|^{\gamma-2} \nabla u), \\ \nabla \cdot u = 0, u(x,0) = u_{in}. \end{cases} \tag{1}$$

Here $u(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ is the velocity field. u_p is the signed power of velocity u and is called the momentum. $\pi(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ denotes the unknown scalar pressure. Δ_γu is γ-Laplacian of velocity. Physically it is related to the shear thinning effect and is reminiscent of the shear thickening fluid [2]. $\nu > 0$ and $\gamma > 1$ are constants, which denote viscosity and strength of diffusion respectively. The parameter γ measures the level of diffusion. It is the fast diffusion when $\gamma \in (1,2)$, whereas $\gamma \in (2,\infty)$ corresponds to the slow diffusion. If $\gamma = 2$, it is the usual diffusion in the Newtonian fluid. In addition, we require the solution to decay fast enough at infinity.

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The p-Laplace equation ($\Delta_p u = 0$) is a far-reaching generalization of the ordinary Laplace equation, but it is non-linear and degenerate ($p > 2$) or singular ($p < 2$). More physical situations and applications about p-Laplacian can also be found in [1, 18]. The systems of p-Naiver-Stokes equations and p-Euler equations were proposed by Li and Liu in [16]. p-Euler equations are derived as the Euler-Lagrange equations for the action represented by the Benamou-Brenier characterization of Wasserstein-p distances with constraint density to be a characteristic function. The difference between p-Euler equations and usual Euler equations is the nonlinear p-momentum in the velocity. The p-Navier-Stokes equations are derived by adding a viscosity with γ-Laplacian ($\Delta_\gamma u$) to p-Euler equations. It should be mentioned that the name of 'p-Navier-Stokes equations' is reminiscent of the models for non-Newtonian fluids, which were studied by Breit in [1, 2] based on a power law model for the viscosity term. Their models are the usual Navier-Stokes equations with the viscous term replaced by $div(|\epsilon|^{p-2}\epsilon)$ where $\epsilon = \frac{1}{2}(\nabla u + \nabla u^T)$. With this term, it follows that

$$\langle div(|\epsilon|^{p-2}\epsilon), u\rangle = -\|\epsilon\|^p_{L^p}.$$

However, in Equations (1) we have

$$\langle \Delta_\gamma u, u\rangle = \langle div(|\nabla u|^{\gamma-2}\nabla u), u\rangle = -\|\nabla u\|^\gamma_{L^\gamma}.$$

Hence, if $\gamma = p$, we can give a *priori* estimates of (1)

$$u \in L^\infty(0,T;L^p) \cap L^p(0,T;W^{1,p}).$$

The p-Laplacian diffusion has been studied by many authors. Lindqvist studied stationary p-Laplace equation in [18]. Matas and Merker in [19] investigated existence of weak solutions to doubly degenerate diffusion equations via Faedo-Galerkin approximation. A degenerate p-Laplacian Keller-Segel model was studied by Cong and Liu in [9]. In addition, there have been extensive numerical works for various gradient flows, such as [8, 12, 22, 24]. The convexity of such a p-Laplacian energy has played a crucial role in the energy stability estimates in these numerical works.

About p-Navier-Stokes equations, it is worth noting that Li and Liu [16] have studied the existence of global weak solution by using the time-shift estimate and compactness criterion. In this paper we give a completely different proof about the global existence of weak solutions. More precisely, compared to the results in [16] we construct a sequence of approximate solutions by use of a semi-implicit scheme at first. In this construction, we establish a well-defined operator Φ and apply the Leray-Schauder fixed point theorem and the monotone operator theory to prove the existence of a solution to the discrete problem. The monotone operators theory was proposed by Minty [20, 21], which was used to obtain the existence results for quasi-linear elliptic and parabolic partial differential equations, see for instance [3, 4, 10, 14, 15]. Then, it is sufficient to get uniform estimates for the time shifts $u^\tau - u^\tau(\cdot - \tau)$ for $\tau > 0$ instead of all time shifts $u^\tau - u^\tau(\cdot - h)$ for $h > 0$. Next, we employ compactness and the monotone operator theory to show that the constructed solution converges to the weak solution. Some compactness results for piecewise constant functions in time can be found in [5, 11].

The rest of the paper is organized as follows. In Section 2, we give the definition of weak solution for (1) and the proof of some essential inequalities. In Section 3, we prove Theorem (4) by using a semi-implicit scheme to construct approximate solutions, which indeed converge to a weak solution. Finally, in the appendix, we establish existence and uniqueness of the weak solution for a class of stationary p-Laplacian equations.

2. **Preliminaries.** The global weak solution to the p-Navier-Stokes equations is defined as below:

Definition 2.1. We say u is a global weak solution to the p-Navier-Stokes equations (1) with an initial data $u_{in} \in L^p(\mathbb{R}^d)$ and $\nabla \cdot u_{in} = 0$ when $\gamma = p$, if for $\forall\ T > 0$,

$$u \in L^\infty(0,T; L^p(\mathbb{R}^d)) \cap L^p(0,T; W^{1,p}(\mathbb{R}^d));$$

and for $\forall\ \varphi \in C_c^\infty([0,T); \mathbb{R}^d)$ and $\nabla \cdot \varphi = 0$, it satisfies that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} u_p \cdot \partial_t \varphi dxdt + \int_0^T \int_{\mathbb{R}^d} \nabla\varphi : (u \otimes u_p) dxdt \\ = &\int_0^T \int_{\mathbb{R}^d} \nabla\varphi : (\nabla u |\nabla u|^{p-2}) dxdt - \int_{\mathbb{R}^d} (u_{in})_p \cdot \varphi(x,0) dx, \end{aligned} \tag{2}$$

where the term about pressure π vanishes: $\int_{\mathbb{R}^d} \nabla\pi\varphi dx = -\int_{\mathbb{R}^d} \pi\nabla \cdot \varphi dx = 0$, because φ is divergence free. Furthermore,

$$\nabla\varphi : (u \otimes u_p) = \sum_{ij} \partial_i \varphi_j v_i (v_p)_j, \nabla\varphi : \nabla u = \sum_{ij} \partial_i \varphi_j \partial_i u_j.$$

We prove the following essential lemma:

Lemma 2.2 ([18]). *Let $p \geq 2$, then there exists $C(p) > 0$ such that $\forall\ a, b \in \mathbb{R}^d$, then*

$$C(p)|a-b|^p \leq (|a|^{p-2}a - |b|^{p-2}b) \cdot (a-b), \tag{3}$$

Proof.

$$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a-b) = \frac{|a|^{p-2} + |b|^{p-2}}{2}|a-b|^2 + \frac{(|a|^{p-2} - |b|^{p-2})(|a|^2 - |b|^2)}{2}.$$

If $p \geq 2$,

$$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a-b) \geq \frac{|a|^{p-2} + |b|^{p-2}}{2}|a-b|^2 \geq 2^{2-p}|a-b|^p.$$

□

3. **Existence of the global weak solution.** In this section, we state our theorem and give a proof.

Theorem 3.1. *When $2 \leq d \leq p$, the p-Navier-Stokes equations (1) have a global weak solution with $\gamma = p$ and an initial data $u_{in} \in L^p(\mathbb{R}^d)$.*

Remark 3.1. Note that we allow $2 = d = p$. This is the classical incompressible Navier-Stokes equations in $\mathbb{R}^2$, which is well-known to have global weak solutions and strong solutions. Actually, we mainly focus on $2 < d \leq p$ or $2 = d < p$ in our proof. However, we include the result for the case that $2 = d = p$ in the statement of the theorem, for the sake of completeness.

We complete the proof through four subsections. Firstly, we use a semi-implicit scheme to construct a sequence of approximate solutions in the bounded domain by the Leray-Schauder fixed point theorem, and prove the existence of the approximate solution in the whole space by using maximal monotone operator theory. Then uniform estimates for approximate solutions are obtained. Next, we show that the sequence of constructed approximate solutions has a subsequence which converges to u by the compactness argument. We finally prove that u is indeed a global weak solution of the equations (1) when $\gamma = p$.

3.1. **The approximate solutions.** The following notation will be used: $A \hookrightarrow D$ ($or\, A \hookrightarrow\hookrightarrow D$) denotes that A is continuously (or compactly) embedded in D. $f_\tau \to (\rightharpoonup \; or \; \overset{*}{\rightharpoonup}) f$ in A denotes that a sequence $\{f_\tau\}_{\tau>0} \subset A$ converges strongly (weakly or weakly star) to f in A as $\tau \to 0$. $C(a, b, ...)$ denotes a constant only dependent on $a, b, ...$

In this subsection, we will use a semi-implicit time scheme to obtain a sequence of approximate solutions in $B_n(x)$ by employing the Leray-Schauder fixed point theorem and then give the extension of solutions in B_n to $\mathbb{R}^d$. The extended solutions actually converge to the approximate solutions in whole space as $n \to \infty$.

Let $\forall\, T > 0$, $N \in \mathbb{N}$ and set $\tau = T/N$, $0 < \tau \ll 1$. We divide the time interval [0,T) into $\cup_{k=1}^N [(k-1)\tau, k\tau)$. For any $k = 1, ..., N$, given $\tilde{u}^{k-1}$, the approximate problem reads

$$\int_{\mathbb{R}^d} \frac{\tilde{u}_p^k - \tilde{u}_p^{k-1}}{\tau} \cdot \varphi dx + \int_{\mathbb{R}^d} \tilde{u}^{k-1} \cdot \nabla \tilde{u}_p^k \varphi dx + \int_{\mathbb{R}^d} |\nabla \tilde{u}^k|^{p-2} \nabla \tilde{u}^k \nabla \varphi dx = 0, \tag{4}$$

for any test function $\varphi \in W_0^{1,p}(\mathbb{R}^d)$ and $\nabla \cdot \varphi = 0$. In addition, the term about pressure π vanishes because φ is divergence free,

$$\int_{\mathbb{R}^d} \nabla \pi \varphi dx = -\int_{\mathbb{R}^d} \pi \nabla \cdot \varphi dx = 0.$$

We now prove the existence of solutions to approximate equation (4) by the following proposition:

Proposition 3.1. *Let $\tilde{u}^{k-1} \in W_0^{1,p}(\mathbb{R}^d)$ and $\nabla \cdot \tilde{u}^{k-1} = 0$. Then there exists a solution $\tilde{u}^k$, $\tilde{u}^k \in W_0^{1,p}(\mathbb{R}^d)$ and $\nabla \cdot \tilde{u}^k = 0$, which solves (4) when $2 < d \le p$ or $2 = d < p$.*

Proof of Proposition 3.1. We give the proof in two steps. In Step 1, we show the existence of $\tilde{u}^k$ to (4) in a bounded domain $B_n(x)$ for a fixed radius R_n by the Leray-Schauder fixed point theorem (see Theorem 11.3, [13]). In Step 2, we extend the function $\tilde{u}^k$ by zero outside B_n and denote such extended functions by $\tilde{u}^{k,n}$. Then we make use of the monotone operator theory to get the approximate solutions on the whole space.

Step 1. For fixed R_n, B_n denotes a ball centered at 0 of radius $R_n \in \mathbb{N}$, and

$$V := \{u \in L^p(B_n) : \nabla \cdot u = 0, u|_{\partial B_n} = 0\},\ H := \{u \in W^{1,p}(B_n) : \nabla \cdot u = 0\}.$$

Firstly, we construct a mapping. Let $u^{k-1} \in H$ and define the mapping Φ : $V \times [0, 1] \to V$ by $\Phi(\bar{u}^k, \sigma) = u^k$ via the following procedure

$$\int_{B_n} u_p^k \cdot \varphi dx + \tau \int_{B_n} |\nabla u^k|^{p-2} \nabla u^k \nabla \varphi dx = \sigma \int_{B_n} u_p^{k-1} \varphi dx - \sigma\tau \int_{B_n} u^{k-1} \cdot \nabla \bar{u}_p^k \varphi dx. \tag{5}$$

It can be deduced that the mapping Φ is well-defined through the existence and uniqueness of the weak solution to stationary p-Laplacian equation (49) in the Appendix.

Moreover, we find $u^k \in W^{1,p}(B_n)$ by *a priori* estimate. Indeed, it follows that by taking $\varphi = u^k$ in (5)

$$\begin{aligned}
\|u^k\|_{L^p}^p + \tau\|\nabla u^k\|_{L^p}^p &\leq \sigma \int u_p^{k-1} u^k dx - \sigma\tau \int u^{k-1} \cdot \nabla \bar{u}_p^k u^k dx \\
&\leq \sigma \|u^{k-1}\|_{L^p}^{p-1} \|u^k\|_{L^p} + \sigma\tau \|u^{k-1}\|_{L^\infty} \|\bar{u}^k\|_{L^p}^{p-1} \|\nabla u^k\|_{L^p} \\
&\leq C(p)\|u^{k-1}\|_{L^p}^p + C(p)\|\nabla u^{k-1}\|_{L^p}^{\frac{\frac{d}{p}}{p-1}} \|u^{k-1}\|_{L^p}^{\frac{\frac{p-d}{p}}{p-1}} \|\bar{u}^k\|_{L^p}^p \\
&\quad + \frac{\tau}{p}\|\nabla u^k\|_{L^p}^p + \frac{1}{p}\|u^k\|_{L^p}^p, \qquad (6)
\end{aligned}$$

where we have used the Hölder inequality, the Young inequality and the Gagliardo-Nirenberg inequality: $\|u^{k-1}\|_{L^\infty} \leq C\|\nabla u^{k-1}\|_{L^p}^{\frac{d}{p}} \|u^{k-1}\|_{L^p}^{1-\frac{d}{p}}$, $2 < d \leq p$ or $2 = d < p$. Then, we obtain

$$\|u^k\|_{L^p}^p + \tau\|\nabla u^k\|_{L^p}^p \leq C(p, \|u^{k-1}\|_H, \|\bar{u}^k\|_V).$$

Secondly, we shall prove that the mapping Φ_1 of V into itself given by $\Phi(\bar{u}^k, 1) = u^k$ has a fixed point $u^k \in V$ by the Leray-Schauder fixed point theorem. To achieve this goal, we need to show the following four claims (this format was used in [6, 7]):

Claim 1. *$\Phi(\bar{u}_i, 0) = 0$ for any $\bar{u}_i \in V$.*

Claim 2. *$\Phi : V \times [0, 1] \to V$ is continuous.*

Claim 3. *Φ is compact.*

Claim 4. *$\Gamma := \{u^k \in V : u^k = \Phi(u^k, \sigma)$ for all $\sigma \in [0, 1]\}$ is bounded in V.*

Proof of Claim 1. If $\Phi(\bar{u}_i, 0) = u_i$ for any $\bar{u}_i \in V$,

$$\int_{B_n} (u_i)_p \cdot \varphi dx + \tau \int_{B_n} |\nabla u_i|^{p-2} \nabla u_i \nabla \varphi dx = 0,$$

taking $\varphi = u_i$ yields

$$\|u_i\|_{L^p}^p + \tau\|\nabla u_i\|_{L^p}^p = 0,$$

whence, $u_i = 0$. Thus, $\Phi(\bar{u}_i, 0) = 0$ for any $\bar{u}_i \in V$. This ends the proof of Claim 1. ☐

Proof of Claim 2. Assume $\Phi(\bar{u}, \sigma) = u$ and $\Phi(\bar{u}_i, \sigma) = u_i$, $i \in \mathbb{N}$, $\sigma \neq 0$, then

$$\int_{B_n} (u_i)_p \cdot \varphi dx + \tau \int_{B_n} |\nabla u_i|^{p-2} \nabla u_i \nabla \varphi dx = \sigma \int_{B_n} u_p^{k-1} \varphi dx - \sigma\tau \int_{B_n} u^{k-1} \cdot \nabla(\bar{u}_i)_p \varphi dx. \qquad (7)$$

If

$$\bar{u}_i \to \bar{u} \text{ in } V \text{ as } i \to \infty,$$

we claim that

$$u_i \to u \text{ in } V \text{ as } i \to \infty.$$

Indeed, it follows from subtraction between equation (7) and (5) that

$$\begin{aligned}
&\int_{B_n} (|u_i|^{p-2} u_i - |u|^{p-2} u) \varphi dx + \tau \int_{B_n} (|\nabla u_i|^{p-2} \nabla u_i - |\nabla u|^{p-2} \nabla u) \nabla \varphi dx \\
= \;& \sigma\tau \int_{B_n} (u^{k-1} \cdot \nabla \bar{u}_p - u^{k-1} \cdot \nabla(\bar{u}_i)_p) \varphi dx.
\end{aligned}$$

Taking $\varphi = u_i - u$, using Lemma 2.2 and integrating by parts gives

$$\begin{aligned}
& \|u_i - u\|_{L^p}^p + \tau\|\nabla u_i - \nabla u\|_{L^p}^p \\
\leq\ & C(p)\tau \int u^{k-1}((\bar{u}_i)_p - \bar{u}_p)(\nabla u_i - \nabla u)dx \\
\leq\ & C(p)\tau \int u^{k-1}|\bar{u}_i - \bar{u}|(|\bar{u}_i|^{p-2} + |\bar{u}|^{p-2})(\nabla u_i - \nabla u)dx \\
\leq\ & C(p)\tau\|\nabla(u_i - u)\|_{L^p}\|\bar{u}_i + \bar{u}\|_{L^p}^{p-2}\|\bar{u}_i - \bar{u}\|_{L^p}\|u^{k-1}\|_{L^\infty} \\
\leq\ & \frac{\tau}{p}\|\nabla(u_i - u)\|_{L^p}^p + C(p)\|\bar{u}_i - \bar{u}\|_{L^p}^{\frac{p}{p-1}}\|\bar{u}_i + \bar{u}\|_{L^p}^{\frac{p(p-2)}{p-1}} \\
& \times\|\nabla u^{k-1}\|_{L^p}^{\frac{\frac{d}{p}}{p-1}}\|u^{k-1}\|_{L^p}^{\frac{p-d}{p-1}},
\end{aligned}$$

where we have used the Hölder inequality, the Young inequality and the Gagliardo-Nirenberg inequality $\|u^{k-1}\|_{L^\infty} \leq C\|\nabla u^{k-1}\|_{L^p}^{\frac{d}{p}}\|u^{k-1}\|_{L^p}^{1-\frac{d}{p}}$, $2 < d \leq p$ or $2 = d < p$.

Hence, $\|u_i - u\|_{L^p}^p \leq C\|\bar{u}_i - \bar{u}\|_{L^p}^{\frac{p}{p-1}}$. Thus, $u_i \to u$ in V when $\bar{u}_i \to \bar{u}$ in V as $i \to \infty$, i.e. Φ is continuous. This completes the proof of Claim 2. $\square$

Proof of Claim 3. It holds from (6) that

$$\exists C(p) > 0,\ \forall\ \bar{u}^k \in V,\ \|\Phi(\bar{u}^k, \sigma)\|_{W^{1,p}(B_n)} \leq C(p)(1 + \|\bar{u}^k\|_{L^p(B_n)}).$$

This inequality above together with $W^{1,p}(B_n) \hookrightarrow\hookrightarrow L^p(B_n)$ (for any bounded Ω, the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact by the Rellich-Kondrachov theorem) gives the proof of Claim 3. $\square$

Proof of Claim 4. Assuming that for any $\sigma \in [0, 1]$, it holds that

$$\int_{B_n} u_p^k \cdot \varphi dx + \tau \int_{B_n} |\nabla u^k|^{p-2}\nabla u^k \nabla\varphi dx = \sigma \int_{B_n} u_p^{k-1}\varphi dx - \sigma\tau \int_{B_n} u^{k-1} \cdot \nabla(u^k)_p \varphi dx.$$

Taking $\varphi = u^k$ yields

$$\|u^k\|_{L^p(B_n)}^p + \tau\|\nabla u^k\|_{L^p(B_n)}^p \leq C(p)\|u^{k-1}\|_{L^p},$$

where

$$\tau \int_{B_n} u^{k-1} \cdot \nabla(u^k)_p u^k dx = -\tau\frac{1}{p}\int_{B_n} u^{k-1} \cdot \nabla|u^k|^p dx = 0.$$

This establishes the Claim 4.

Thus, there exists a fixed point for Φ_1 given $\Phi_1 = \Phi(u, 1)$ of V into itself, i.e. $\Phi(u^k, 1) = u^k$, by Leray-Schauder fixed point theorem. Moreover, $\|u^k\|_{W^{1,p}(B_n)} \leq C$. Therefore, $u^k \in H$.

Step 2. Since B_n is a ball of radius R_n centred at zero, R_n is an increasing sequence with $R_n \to \infty$, and zero Dirichlet boundary condition in B_n: $u^k|_{\partial B_n} = 0$, we extend the u^k by zero outside B_n and denote such extended function by $\tilde{u}^{k,n}$, i.e.

$$\tilde{u}^{k,n}(x) := \begin{cases} u^k(x), & if\ x \in B_n, \\ 0, & if\ x \notin B_n. \end{cases}$$

Due to step 1, in fact we can show $\tilde{u}^{k,n}$ is a solution solving (4), i.e. $\forall\varphi \in W_0^{1,p}(\mathbb{R}^d)$ where the support of φ is larger than B_n,

$$\int_{\mathbb{R}^d} \tilde{u}_p^{k,n} \cdot \varphi dx + \tau \int_{\mathbb{R}^d} \tilde{u}^{k-1,n} \cdot \nabla\tilde{u}_p^{k,n}\varphi dx + \tau \int_{\mathbb{R}^d} |\nabla\tilde{u}^{k,n}|^{p-2}\nabla\tilde{u}^{k,n}\nabla\varphi dx = \int_{\mathbb{R}^d} \tilde{u}_p^{k-1}\varphi dx. \quad (8)$$

We can obtain *a priori* estimate from (6) and (8) that

$$\|\tilde{u}^{k,n}\|^p_{L^p(\mathbb{R}^d)} + \tau\|\nabla\tilde{u}^{k,n}\|^p_{L^p(\mathbb{R}^d)} \le C,$$

where C is independent of n.

Since $\{\tilde{u}^{k,n}\}$ is bounded in $W^{1,p}(\mathbb{R}^d)$, there exist a subsequence $\{\tilde{u}^{k,n}\}$, not relabeling and $\tilde{u}^k \in W^{1,p}(\mathbb{R}^d)$ such that

$$\tilde{u}^{k,n} \rightharpoonup \tilde{u}^k \text{ weakly in } W^{1,p}(\mathbb{R}^d) \text{ as } n \to \infty.$$

By the Rellich-Kondrashov theorem (see Theorem 8.9, [17]), there exist a subsequence of $\{\tilde{u}^{k,n}\}$, not relabeling, and $\tilde{u}^k \in W^{1,p}(\mathbb{R}^d)$ such that the following uniformly strong convergence holds true

$$\tilde{u}^{k,n}(x) \to \tilde{u}^k \text{ in } L^p_{loc}(\mathbb{R}^d) \text{ as } n \to \infty.$$

Through the argument about strong convergence, we easily obtain the convergence of linear terms in (8), i.e. as $n \to \infty$

$$\int_{\mathbb{R}^d} \tilde{u}^{k,n}_p \cdot \varphi dx \to \int_{\mathbb{R}^d} \tilde{u}^k_p \cdot \varphi dx;$$
$$\int_{\mathbb{R}^d} \tilde{u}^{k-1,n} \cdot \nabla\tilde{u}^{k,n}_p \varphi dx \to \int_{\mathbb{R}^d} \tilde{u}^{k-1} \cdot \nabla\tilde{u}^k_p \varphi dx.$$

Now we only need to prove $\int_{\mathbb{R}^d} |\nabla\tilde{u}^{k,n}|^{p-2}\nabla\tilde{u}^{k,n}\nabla\varphi dx \to \int_{\mathbb{R}^d} |\nabla\tilde{u}^k|^{p-2}\nabla\tilde{u}^k\nabla\varphi dx$ as $n \to \infty$.

Indeed, due to $\tilde{u}^{k,n} \rightharpoonup \tilde{u}^k$ weakly in $W^{1,p}_{loc}(\mathbb{R}^d)$ as $n \to \infty$, without loss of generality, we assume $|\nabla\tilde{u}^{k,n}|^{p-2}\nabla\tilde{u}^{k,n} \rightharpoonup \tilde{\chi}$ in $L^{\frac{p}{p-1}}(\mathbb{R}^d)$. Hence, let $n \to \infty$ in (8) and obtain

$$\int_{\mathbb{R}^d} \tilde{u}^k_p \cdot \varphi dx + \tau\int_{\mathbb{R}^d} \tilde{u}^{k-1} \cdot \nabla\tilde{u}^k_p \varphi dx + \tau\int_{\mathbb{R}^d} \tilde{\chi}\nabla\varphi dx = \int_{\mathbb{R}^d} \tilde{u}^{k-1}_p \varphi dx. \tag{9}$$

Choosing $\tilde{\phi}(x) \in C^\infty_c(\mathbb{R}^d)$ with $0 \le \phi \le 1$ and taking $\varphi = \tilde{u}^k\tilde{\phi}$ in (9), we obtain

$$\int_{\mathbb{R}^d} |\tilde{u}^k|^p \cdot \tilde{\phi} dx + \tau\int_{\mathbb{R}^d} \tilde{u}^{k-1} \cdot \nabla\tilde{u}^k_p \tilde{u}^k\tilde{\phi} dx + \tau\int_{\mathbb{R}^d} \tilde{\chi}\nabla(\tilde{u}^k\tilde{\phi}) dx = \int_{\mathbb{R}^d} \tilde{u}^{k-1}_p \tilde{u}^k\tilde{\phi} dx. \tag{10}$$

We take $\varphi = \tilde{u}^{k,n}\tilde{\phi}$ in (8) and have

$$\begin{aligned} \int_{\mathbb{R}^d} |\tilde{u}^{k,n}|^p\tilde{\phi} dx \quad &+ \quad \tau\int_{\mathbb{R}^d} \tilde{u}^{k-1,n} \cdot \nabla\tilde{u}^{k,n}_p \tilde{u}^{k,n}\tilde{\phi} dx + \tau\int_{\mathbb{R}^d} |\nabla\tilde{u}^{k,n}|^p\tilde{\phi} dx \\ &+\tau\int_{\mathbb{R}^d} |\nabla\tilde{u}^{k,n}|^{p-2}\nabla\tilde{u}^{k,n}\tilde{u}^{k,n}\nabla\tilde{\phi} dx = \int_{\mathbb{R}^d} \tilde{u}^{k-1,n}_p \tilde{u}^{k,n}\tilde{\phi} dx. \end{aligned} \tag{11}$$

Since the inequality (3), for any $\tilde{\omega} \in W^{1,p}_{loc}(\mathbb{R}^d)$ we have

$$\begin{aligned} &\tau\int_{\mathbb{R}^d} (|\nabla\tilde{u}^{k,n}|^{p-2}\nabla\tilde{u}^{k,n} - |\nabla\tilde{\omega}|^{p-2}\tilde{\omega})(\nabla\tilde{u}^{k,n} - \nabla\tilde{\omega})\tilde{\phi} dx \\ \ge \quad &C(p,\tau)\int_{\mathbb{R}^d} |\nabla\tilde{u}^{k,n} - \nabla\tilde{\omega}|^p\tilde{\phi} dx \ge 0, \end{aligned}$$

i.e.

$$-\tau\int_{\mathbb{R}^d} |\nabla\tilde{u}^{k,n}|^p\tilde{\phi} - |\nabla\tilde{u}^{k,n}|^{p-2}\nabla\tilde{u}^{k,n}\nabla\tilde{\omega}\tilde{\phi} - |\nabla\tilde{\omega}|^{p-2}\nabla\tilde{\omega}\nabla\tilde{u}^{k,n}\tilde{\phi} + |\nabla\tilde{\omega}|^p\tilde{\phi} dx \le 0. \tag{12}$$

Combining (11) with (12) yields

$$\begin{aligned}\int_{\mathbb{R}^d}|\tilde{u}^{k,n}|^p\tilde{\phi}dx \quad &+ \quad \tau\int_{\mathbb{R}^d}\tilde{u}^{k-1,n}\cdot\nabla\tilde{u}_p^{k,n}\tilde{u}^{k,n}\tilde{\phi}+|\nabla\tilde{u}^{k,n}|^{p-2}\nabla\tilde{u}^{k,n}\tilde{u}^{k,n}\nabla\tilde{\phi}\\ &+ \quad |\nabla\tilde{u}^{k,n}|^{p-2}\nabla\tilde{u}^{k,n}\nabla\tilde{\omega}\tilde{\phi}+|\nabla\tilde{\omega}|^{p-2}\nabla\tilde{\omega}(\nabla\tilde{u}^{k,n}-\nabla\tilde{\omega})\tilde{\phi}dx\\ &- \quad \int_{\mathbb{R}^d}\tilde{u}_p^{k-1,n}\tilde{u}^{k,n}\tilde{\phi}dx\le 0.\end{aligned}\tag{13}$$

Letting $n\to\infty$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}|\tilde{u}^{k}|^p\tilde{\phi}dx \quad &+ \quad \tau\int_{\mathbb{R}^d}\tilde{u}^{k-1}\cdot\nabla\tilde{u}_p^{k}\tilde{u}^{k}\tilde{\phi}+\tilde{\chi}\tilde{u}^{k}\nabla\tilde{\phi}\\ &+ \quad \tilde{\chi}\nabla\tilde{\omega}\tilde{\phi}+|\nabla\tilde{\omega}|^{p-2}\nabla\tilde{\omega}(\nabla\tilde{u}^{k}-\nabla\tilde{\omega})\tilde{\phi}dx\\ &- \quad \int_{\mathbb{R}^d}\tilde{u}_p^{k-1}\tilde{u}^{k}\tilde{\phi}dx\le 0.\end{aligned}\tag{14}$$

Combining (10) with (14) yields

$$\int_{\mathbb{R}^d}(\nabla\tilde{\omega}-\nabla\tilde{u}^k)(\tilde{\chi}-|\nabla\tilde{\omega}|^{p-2}\nabla\tilde{\omega})\tilde{\phi}dx\le 0.$$

Taking $\tilde{\omega}=\tilde{u}^k-\lambda\varphi$ with $\lambda\ge 0$ yields that

$$\int_{\mathbb{R}^d}(\tilde{\chi}-|\nabla\tilde{u}^k-\lambda\nabla\varphi|^{p-2}\nabla(\tilde{u}^k-\lambda\nabla\varphi))\nabla\varphi\tilde{\phi}dx\ge 0.$$

Choosing $\tilde{\phi}$ such that supp $\varphi\subset$ supp $\tilde{\phi}$ and $\tilde{\phi}=1$ on supp $\tilde{\phi}$. When $\lambda\to 0$, it follows that

$$\int_{\mathbb{R}^d}(\tilde{\chi}-|\nabla\tilde{u}^k|^{p-2}\nabla\tilde{u}^k)\nabla\varphi dx\ge 0.$$

Employing the same method with $\tilde{\omega}=\tilde{u}^k-\lambda\varphi$ and $\lambda\le 0$, we obtain

$$\int_{\mathbb{R}^d}(\tilde{\chi}-|\nabla\tilde{u}^k|^{p-2}\nabla\tilde{u}^k)\nabla\varphi dx\le 0.$$

Thus, $\tilde{\chi}=|\nabla\tilde{u}^k|^{p-2}\nabla\tilde{u}^k$.

That proves $\int_{\mathbb{R}^d}|\nabla\tilde{u}^{k,n}|^{p-2}\nabla\tilde{u}^{k,n}\nabla\varphi dx\to\int_{\mathbb{R}^d}|\nabla\tilde{u}^{k}|^{p-2}\nabla\tilde{u}^{k}\nabla\varphi dx$.

Therefore, if $\tilde{u}^{k-1}\in W_0^{1,p}(\mathbb{R}^d)$, we will obtain $\tilde{u}^k\in W^{1,p}(\mathbb{R}^d)$ solving (4) through discussion of Step 1 and Step 2. That completes the proof of Proposition 3.1. □

3.2. **Uniform estimates.** In this subsection, we aim to obtain the uniform estimates of the approximate solutions $\{\tilde{u}^k\}$.

At first, we regularize u_{in} by $\tilde{u}^0$ which is the weak solution of $\tilde{u}_p^0-\tau\Delta_p\tilde{u}^0=(u_{in})_p$, with $u_{in}\in L^p(\mathbb{R}^d)$ (Meanwhile, the existence and uniqueness for weak solution to this stationary p-Laplacian equation is given in Appendix by calculus of variations). Therefore, multiplying above equation with $\tilde{u}^0$ and integrating in $\mathbb{R}^d$ gives

$$\|\tilde{u}^0\|^p_{L^p(\mathbb{R}^d)}+\tau\|\nabla\tilde{u}^0\|^p_{L^p(\mathbb{R}^d)}\le\|u_{in}\|^p_{L^p(\mathbb{R}^d)}.\tag{15}$$

By the weak compactness, there exists a subsequence (without relabeling) $\tilde{u}^0\rightharpoonup u_{in}$ in $L^p_{loc}(\mathbb{R}^d)$ as $\tau\to 0_+$. Moreover, we get $\tilde{u}^0\in W^{1,p}(\mathbb{R}^d)$.

It follows from Proposition 3.1 that a sequence of approximate solutions

$$\{\tilde{u}^0,\tilde{u}^1,\tilde{u}^2,\tilde{u}^3,...\}\in W^{1,p}(\mathbb{R}^d),$$

which satisfy the approximate problem of equations (4).

The first uniform estimate: Taking $\varphi = \tilde{u}^k$ in (4), we deduce

$$\|\tilde{u}^k\|_{L^p(\mathbb{R}^d)}^p + \tau\|\nabla\tilde{u}^k\|_{L^p(\mathbb{R}^d)}^p \quad \leq \quad \int_{\mathbb{R}^d} \tilde{u}_p^{k-1}\tilde{u}^k dx \leq \|\tilde{u}^{k-1}\|_{L^p(\mathbb{R}^d)}^p, \tag{16}$$

where the term of $\int u^{k-1}\cdot\nabla u_p^k u^k dx$ vanishes because of divergence free,

$$\int u^{k-1}\cdot\nabla u_p^k u^k dx = -\tfrac{1}{p}\int u^{k-1}\cdot\nabla|u^k|^p dx = 0.$$

Then, summing up (16) from $k=1$ to N yields

$$\|\tilde{u}^N\|_{L^p(\mathbb{R}^d)}^p + \tau\Sigma_{k=1}^N\|\nabla\tilde{u}^k\|_{L^p(\mathbb{R}^d)}^p \leq \|\tilde{u}^0\|_{L^p(\mathbb{R}^d)}^p \leq \|u_{in}\|_{L^p(\mathbb{R}^d)}^p. \tag{17}$$

The second uniform estimate: Taking $\varphi = \tilde{u}^k - \tilde{u}^{k-1}$ in (4) and using Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{1}{\tau}\|\tilde{u}^k - \tilde{u}^{k-1}\|_{L^p(\mathbb{R}^d)}^p + \|\nabla\tilde{u}^k\|_{L^p(\mathbb{R}^d)}^p \\
\leq\ & \int_{\mathbb{R}^d}|\nabla\tilde{u}^k|^{p-2}\nabla\tilde{u}^k\nabla\tilde{u}^{k-1}dx + \int_{\mathbb{R}^d}\tilde{u}^{k-1}\cdot\nabla\tilde{u}_p^k\tilde{u}^{k-1}dx \\
\leq\ & \|\nabla\tilde{u}^k\|_{L^p}^{p-1}\|\nabla\tilde{u}^{k-1}\|_{L^p} + \|\tilde{u}^{k-1}\|_{L^\infty}\|\tilde{u}^k\|_{L^p}^{p-1}\|\nabla\tilde{u}^{k-1}\|_{L^p}.
\end{aligned}$$

We infer from (17) that

$$\|\tilde{u}^k - \tilde{u}^{k-1}\|_{L^p(\mathbb{R}^d)}^p + \frac{\tau}{p}\|\nabla\tilde{u}^k\|_{L^p(\mathbb{R}^d)}^p \leq \frac{\tau}{p}\|\nabla\tilde{u}^{k-1}\|_{L^p(\mathbb{R}^d)}^p + C\tau\|\nabla\tilde{u}^{k-1}\|_{L^p(\mathbb{R}^d)}^p + C\tau.$$

Then, summing up above inequality from $k=1$ to N yields

$$\sum_{k=1}^N\|\tilde{u}^k - \tilde{u}^{k-1}\|_{L^p(\mathbb{R}^d)}^p + \frac{\tau}{p}\|\nabla\tilde{u}^N\|_{L^p(\mathbb{R}^d)}^p \leq C + CN\tau. \tag{18}$$

3.3. **Convergence.**

Definition 3.2. Define the following piecewise function in t by

$$u^\tau(t,\cdot) := \tilde{u}^k(\cdot),\ \pi_\tau u^\tau(t,\cdot) := \tilde{u}^{k-1}(\cdot),$$
$$\partial_t u^\tau(t,\cdot) := \tfrac{\tilde{u}^k(\cdot)-\tilde{u}^{k-1}(\cdot)}{\tau},\ t\in[(k-1)\tau, k\tau),\ k=1,2,...,N.$$

In this subsection, we aim to obtain the compactness of $\{u^\tau\}$ in $L^p(0,T;L^p(\mathbb{R}^n))$. We will employ an important lemma about compactness (see Theorem 3, [23]):

Lemma 3.3 ([23])**.** *Let X and Y be Banach spaces such that the embedding $X \hookrightarrow\hookrightarrow Y$ is compact. If $1\leq p\leq\infty$, let $\{u^\tau\}$ be a sequence of functions in $L^p(0,T;Y)$, satisfying*
(1) $\{u^\tau\}$ is bounded in $L^1_{loc}(0,T;X)$; (2) $\|u^\tau(\cdot+\tau)-u^\tau\|_{L^p(0,T-\tau;Y)}\to 0$ as $\tau\to 0$, uniformly for u^τ.
Then $\{u^\tau\}$ is relatively compact in $L^p(0,T;Y)$(and in $C(0,T;Y)$ if $p=\infty$).

We have our results about convergence.

Proposition 3.2. *As $\tau\to 0_+$, there exist a subsequence of $\{u^\tau\}$, not relabeled, and u satisfying $u\in L^\infty(0,T;L^p(\mathbb{R}^d))\cap L^p(0,T;W^{1,p}(\mathbb{R}^d))$ for any $T>0$ with $d\leq p<\infty$ such that*

$$u^\tau \to u \text{ in } L^p(0,T;L^p_{loc}(\mathbb{R}^d)), \tag{19}$$
$$\pi_\tau u^\tau \to u \text{ in } L^p(0,T;L^p_{loc}(\mathbb{R}^d)), \tag{20}$$
$$\nabla u^\tau \rightharpoonup \nabla u \text{ in } L^p(0,T;\mathbb{R}^d), \tag{21}$$
$$u^\tau \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;L^p(\mathbb{R}^d)). \tag{22}$$

Proof of Proposition 3.2. It follows from the estimate of (17) that

$$\|u^\tau\|^p_{L^\infty(0,T;L^p(\mathbb{R}^d))} + \int_0^T \|\nabla u^\tau\|^p_{L^p(\mathbb{R}^d)} dt \le \|u_{in}\|^p_{L^p},$$

hence

$$\int_0^T \|u^\tau\|_{W^{1,p}(\mathbb{R}^d)} dt \le C(T) \int_0^T \|u^\tau\|^p_{W^{1,p}(\mathbb{R}^d)} dt \le C(T)\|u_{in}\|^p_{L^p}. \tag{23}$$

In addition, (18) yields

$$\int_0^{T-\tau} \|u^\tau(t+\tau) - u^\tau\|^p_{L^p} dt \le \tau \sum_{k=1}^{N-1} \|\tilde{u}^{k+1} - \tilde{u}^k\|^p_{L^p} \le C\tau,$$

whence

$$\|u^\tau(t+\tau) - u^\tau\|_{L^p(0,T-\tau;L^p(\mathbb{R}^d))} \to 0 \ as \ \tau \to 0_+. \tag{24}$$

For any compact set $K \subset \mathbb{R}^d$, $W^{1,p}(K) \hookrightarrow\hookrightarrow L^p(K)$, so it follows from Lemma 3.3 and (23)(24) that $\{u^\tau\}$ is relatively compact in $L^p(0,T : L^p_{loc}(\mathbb{R}^d))$. Then we obtain (19).

$$\int_\tau^T \|\pi_\tau u^\tau - u^\tau\|^p_{L^p} dt \le \tau \sum_{k=1}^{N} \|\tilde{u}^k - \tilde{u}^{k-1}\|^p_{L^p} \le C\tau.$$

This and (19) yield (20).

Since $\{\nabla u^\tau\}$ is bounded in $L^p(0,T;L^p(\mathbb{R}^d))$ from (23), the set is weakly precompact in $L^p(t_1,t_2;L^p(\mathbb{R}^d))$ for all $t_1,t_2 \in (0,T)$ and $t_1 < t_2$. By diagonal argument, we obtain (21). Indeed, we can pick out a subsequence (without relabeling) such that $\nabla u^\tau \rightharpoonup \zeta \in L^p(t_1,t_2;L^p(\mathbb{R}^d))$ as $\tau \to 0_+$. For any $\psi \in C_c^\infty(0,T;\mathbb{R}^d)$

$$\begin{aligned}
\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \zeta\cdot\psi dxdt &= \lim_{\tau\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \nabla u^\tau \psi dxdt \\
&= -\lim_{\tau\to 0}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^\tau \nabla\cdot\psi dxdt \\
&= -\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u\nabla\cdot\psi dxdt,
\end{aligned}$$

where the last equality employs the strong convergence $u^\tau \to u$ in $L^p([0,T);L^p_{loc}(\mathbb{R}^d))$. Thus, $\nabla u = \zeta$ for $\forall t \in [0,T)\setminus\Theta$ where Θ is a set of measure zero.

By above similar diagonal argument and employing the Banach-Alaoglu Theorem (any bounded set in the dual of a separable space in weak star precompact), we get (22). ☐

3.4. **Proof of Theorem 3.1.** Now, we are in the position to prove that u is a weak solution of the equations (1). The weak approximate form of (4) is rewritten that, for any $T > 0$, the test function $\varphi \in C_c^\infty([0,T);\mathbb{R}^d)$ and $\nabla\cdot\varphi = 0$,

$$\int_0^T\int_{\mathbb{R}^d} \partial_t u_p^\tau\cdot\varphi dxdt + \int_0^T\int_{\mathbb{R}^d} \pi_\tau u^\tau\cdot\nabla u_p^\tau \varphi dxdt + \int_0^T\int_{\mathbb{R}^d} |\nabla u^\tau|^{p-2}\nabla u^\tau \nabla\varphi dxdt = 0. \tag{25}$$

Next, we separate the proof of this step into three parts.

(i) We first claim that,

$$\int_0^T\int_{\mathbb{R}^d} \partial_t u_p^\tau \varphi dxdt \to -\int_0^T\int_{\mathbb{R}^d} u_p \partial_t\varphi dxdt - \int_{\mathbb{R}^d} (u_{in})_p \varphi(0)dx, \ as \ \tau \to 0_+. \tag{26}$$

Indeed,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\partial_t u_p^\tau\varphi dxdt\\
=\ &\int_\tau^T\int_{\mathbb{R}^d}\frac{u_p^\tau(t)-u_p^\tau(t-\tau)}{\tau}\varphi dxdt+\int_0^\tau\int_{\mathbb{R}^d}\frac{u_p^\tau-u_p^0}{\tau}\varphi dxdt\\
=\ &\int_0^T\int_{\mathbb{R}^d}\frac{u_p^\tau}{\tau}\varphi dxdt-\int_0^{T-\tau}\int_{\mathbb{R}^d}\frac{u_p^\tau}{\tau}\varphi(t+\tau)dxdt-\int_0^\tau\int_{\mathbb{R}^d}\frac{u_p^0}{\tau}\varphi dxdt\\
=\ &\int_{T-\tau}^T\int_{\mathbb{R}^d}u_p^\tau\frac{\varphi}{\tau}dxdt+\int_0^{T-\tau}\int_{\mathbb{R}^d}u_p^\tau\frac{\varphi}{\tau}dxdt-\int_0^{T-\tau}\int_{\mathbb{R}^d}u_p^\tau\frac{\varphi(t+\tau)}{\tau}dxdt\\
&-\int_0^\tau\int_{\mathbb{R}^d}u_p^0\frac{\varphi}{\tau}dxdt\\
=\ &\int_{T-\tau}^T\int_{\mathbb{R}^d}u_p^\tau\frac{\varphi}{\tau}dxdt-\int_0^{T-\tau}\int_{\mathbb{R}^d}u_p^\tau\frac{\varphi(t+\tau)-\varphi(t)}{\tau}dxdt-\int_0^\tau\int_{\mathbb{R}^d}u_p^0\frac{\varphi}{\tau}dxdt\\
=\ &\int_{T-\tau}^T\int_{\mathbb{R}^d}u_p^\tau\frac{\varphi}{\tau}dxdt-\int_0^{T-\tau}\int_{\mathbb{R}^d}u_p^\tau\partial_t\varphi dxdt-\int_0^\tau\int_{\mathbb{R}^d}u_p^0\frac{\varphi}{\tau}dxdt.
\end{aligned}$$

Then,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\partial_t u_p^\tau\varphi dxdt+\int_0^T\int_{\mathbb{R}^d}u_p\partial_t\varphi dxdt+\int_{\mathbb{R}^d}(u_{in})_p\varphi(0)dx\\
=\ &\int_{T-\tau}^T\int_{\mathbb{R}^d}u_p^\tau\frac{\varphi}{\tau}dxdt-\int_0^{T-\tau}\int_{\mathbb{R}^d}u_p^\tau\partial_t\varphi dxdt-\int_0^\tau\int_{\mathbb{R}^d}u_p^0\frac{\varphi}{\tau}dxdt\\
&+\int_0^T\int_{\mathbb{R}^d}u_p\partial_t\varphi dxdt+\int_{\mathbb{R}^d}(u_{in})_p\varphi(0)dx\\
=\ &\Big(\int_{T-\tau}^T\int_{\mathbb{R}^d}u_p^\tau\frac{\varphi}{\tau}+u_p\partial_t\varphi dxdt\Big)+\Big(\int_0^{T-\tau}\int_{\mathbb{R}^d}u_p\partial_t\varphi-u_p^\tau\partial_t\varphi dxdt\Big)\\
&+\Big(\int_{\mathbb{R}^d}(u_{in})_p\varphi(0)dx-\int_0^\tau\int_{\mathbb{R}^d}u_p^0\frac{\varphi}{\tau}dxdt\Big)\\
=:\ &J_1+J_2+J_3.
\end{aligned}$$

We estimate J_1, J_2 and J_3, respectively.

$$\begin{aligned}
|J_1|\ &\le\ \int_{T-\tau}^T\int_{\mathbb{R}^d}u_p^\tau\Big|\frac{\varphi(t)-\varphi(T)}{t-T}\Big|+u_p\partial_t\varphi dxdt\\
&\le\ \tau\|\partial_t\varphi\|_{L^\infty(0,T;L^p(\mathbb{R}^d))}(\|u^\tau\|^{p-1}_{L^\infty(T-\tau,T;L^p(\mathbb{R}^d))}+\|u\|^{p-1}_{L^\infty(T-\tau,T;L^p(\mathbb{R}^d))})\\
&\le\ C\tau\to 0,\ as\ \tau\to 0_+.
\end{aligned}$$

And

$$\begin{aligned}
|J_2|\ &\le\ \int_0^{T-\tau}\int_{\mathbb{R}^d}u_p\Big(\partial_t\varphi-\frac{\varphi(t+\tau)-\varphi(t)}{\tau}\Big)dxdt\\
&\quad+\int_0^{T-\tau}\int_{\mathbb{R}^d}|u_p-u_p^\tau|\Big|\frac{\varphi(t+\tau)-\varphi(t)}{\tau}\Big|dxdt\\
&\le\ \tau\|\partial_{tt}\varphi\|_{L^1(0,T;L^p(\mathbb{R}^d))}\|u\|^{p-1}_{L^\infty(0,T-\tau;L^p(\mathbb{R}^n))}\\
&\quad+\|u-u^\tau\|_{L^p_{loc}(0,T;\mathbb{R}^d)}\|u+u^\tau\|^{p-2}_{L^p(0,T-\tau;\mathbb{R}^d)}\|\partial_t\varphi\|_{L^p(0,T-\tau;\mathbb{R}^d)},
\end{aligned}$$

where we have used Lemma 2.2. Moreover, Proposition 3.2 yields

$$|J_2| \to 0, \ as \ \tau \to 0_+.$$

It follows from $u^0 \rightharpoonup u_{in}$ in V as $\tau \to 0_+$ and the mean value theorem that

$$\begin{aligned}
|J_3| &\leq \int_{\mathbb{R}^d} (u_{in})_p \varphi(0) dx - \int_0^\tau \int_{\mathbb{R}^d} u_p^0 \frac{\varphi}{\tau} dx dt \\
&\leq \int_{\mathbb{R}^d} ((u_{in})_p - u_p^0)\varphi(0) dx + \int_{\mathbb{R}^n} u_p^0 (\varphi(0) - \frac{1}{\tau}\int_0^\tau \varphi dt) dx \\
&\leq \int_{\mathbb{R}^d} |u_{in} - u^0||u_{in} + u^0|^{p+2} \varphi(0) dx + \int_{\mathbb{R}^d} u_p^0 (\varphi(0) - \frac{1}{\tau}\int_0^\tau \varphi dt) dx \\
&\to 0, \ as \ \tau \to 0.
\end{aligned}$$

Thus, we establish (26).

(ii) Then, we show for any $\varphi \in C_c^\infty(0,T;\mathbb{R}^d)$ and $\nabla \cdot \varphi = 0$, it holds that

$$\int_0^T \int_{\mathbb{R}^d} \pi_\tau u^\tau \cdot \nabla(u_p^\tau) \varphi dx dt \to \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla u_p \varphi dx dt, \ as \ \tau \to 0_+. \tag{27}$$

In fact,

$$\begin{aligned}
& |\int_0^T \int_{\mathbb{R}^d} \pi_\tau u^\tau \cdot \nabla u_p^\tau \varphi dx dt - \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla u_p \varphi dx dt| \\
= & |\int_0^T \int_{\mathbb{R}^d} (u \cdot u_p - \pi_\tau u^\tau \cdot u_p^\tau) \nabla \varphi dx dt| \\
\leq & |\int_0^T \int_{\mathbb{R}^d} (u^\tau - \pi_\tau u^\tau) u_p^\tau \nabla \varphi dx dt| + |\int_0^T \int_{\mathbb{R}^d} (u \cdot u_p - u^\tau \cdot u_p^\tau) \nabla \varphi dx dt| \\
= & K_1 + K_2,
\end{aligned}$$

where

$$\begin{aligned}
K_1 &\leq \int_0^T \int_{\mathbb{R}^d} |(u^\tau - \pi_\tau u^\tau) u_p^\tau \nabla \varphi| dx dt \\
&\leq \|u^\tau - \pi_\tau u^\tau\|_{L^p_{loc}(0,T;\mathbb{R}^d)} \|u^\tau\|_{L^p(0,T;\mathbb{R}^d)} \|\nabla \varphi\|_{L^\infty(0,T;\mathbb{R}^d)} \\
&\to 0, \ as \ \tau \to 0_+,
\end{aligned}$$

and

$$\begin{aligned}
K_2 &\leq \int_0^T \int_{\mathbb{R}^d} |(u|u|^{p-2} u \nabla \varphi - u^\tau |u^\tau|^{p-2} u^\tau \nabla \varphi)| dx dt \\
&\leq \int_0^T \int_{\mathbb{R}^d} (u - u^\tau)|u|^{p-2} u \nabla \varphi + u^\tau (|u|^{p-2} u - |u^\tau|^{p-2} u^\tau) \nabla \varphi dx dt \\
&\leq \|u - u^\tau\|_{L^p_{loc}(0,T;\mathbb{R}^d)} \|u\|^{p-1}_{L^p(0,T;\mathbb{R}^d)} \|\nabla \varphi\|_{L^\infty(0,T;\mathbb{R}^d)} \\
&\quad + \|u - u^\tau\|_{L^p_{loc}(0,T;\mathbb{R}^d)} \|u + u^\tau\|^{p-2}_{L^p(0,T;\mathbb{R}^d)} \|u^\tau\|_{L^p(0,T;\mathbb{R}^d)} \|\partial_t \varphi\|_{L^\infty(0,T;\mathbb{R}^d)} \\
&\to 0, \ as \ \tau \to 0_+.
\end{aligned}$$

(iii) Next, we establish that

$$\int_0^T \int_{\mathbb{R}^d} \Delta_p u^\tau \varphi dx dt \to \int_0^T \int_{\mathbb{R}^d} \Delta_p u \varphi dx dt, \ as \ \tau \to 0_+; \tag{28}$$

i.e.

$$\int_0^T\int_{\mathbb{R}^d}|\nabla u^\tau|^{p-2}\nabla u^\tau\cdot\nabla\varphi dxdt\to\int_0^T\int_{\mathbb{R}^d}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dxdt,\ as\ \tau\to 0_+.$$

Since $\|\nabla u^\tau\|_{L^p(0,T;L^p(\mathbb{R}^d))}\le C$ in (23), we obtain

$$\int_0^T\int_{\mathbb{R}^d}||\nabla u^\tau|^{p-2}\nabla u^\tau|^{\frac{p}{p-1}}dxdt=\int_0^T\int_{\mathbb{R}^d}|\nabla u^\tau|^p dxdt\le C.$$

Then there exists a χ such that

$$|\nabla u^\tau|^{p-2}\nabla u^\tau\rightharpoonup\chi,\ in\ L^{\frac{p}{p-1}}(0,T;L^{\frac{p}{p-1}}(\mathbb{R}^d)).\tag{29}$$

Letting $\tau\to 0$ in (25) and considering (26)(27)(29) yields

$$\begin{aligned}-\int_0^T\int_{\mathbb{R}^d}u_p\partial_t\varphi dxdt-\int_{\mathbb{R}^d}(u_{in})_p\varphi(0)dx &+\int_0^T\int_{\mathbb{R}^d}u\cdot\nabla u_p\varphi dxdt\\ &+\int_0^T\int_{\mathbb{R}^d}\chi\cdot\nabla\varphi dxdt=0.\end{aligned}\tag{30}$$

Then we will prove

$$\int_0^T\int_{\mathbb{R}^d}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dxdt=\int_0^T\int_{\mathbb{R}^d}\chi\cdot\nabla\varphi dxdt\tag{31}$$

to finish the proof of (28).

Choosing $\phi(x,t)\in C_c^\infty(0,T;\mathbb{R}^d)$ with $0\le\phi\le 1$, multiplying the equation (25) by $u^\tau\phi$ and integrating in $\mathbb{R}^n$ and $(0,T)$, we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\partial_t u_p^\tau u^\tau\phi dxdt &+\int_0^T\int_{\mathbb{R}^d}\pi_\tau u^\tau\cdot\nabla u_p^\tau u^\tau\phi dxdt\\ &+\int_0^T\int_{\mathbb{R}^d}|\nabla u^\tau|^p\phi+u^\tau|\nabla u^\tau|^{p-2}\nabla u^\tau\nabla\phi dxdt=0.\end{aligned}\tag{32}$$

For any $\omega\in L^p(0,T;W^{1,p}(\mathbb{R}^d))$ to be determined later, we obtain the following inequality by using Lemma 2.2

$$\int_0^T\int_{\mathbb{R}^d}(|\nabla u^\tau|^{p-2}\nabla u^\tau-|\nabla\omega|^{p-2}\nabla\omega)\nabla(u^\tau-\omega)\phi dxdt\ge 0,\tag{33}$$

i.e.

$$\begin{aligned}&-\int_0^T\int_{\mathbb{R}^d}|\nabla u^\tau|^p\phi dxdt+\int_0^T\int_{\mathbb{R}^d}|\nabla\omega|^{p-2}\nabla\omega\nabla u^\tau\phi dxdt\\ &+\int_0^T\int_{\mathbb{R}^d}|\nabla u^\tau|^{p-2}\nabla u^\tau\nabla\omega\phi dxdt-\int_0^T\int_{\mathbb{R}^d}|\nabla\omega|^p\phi dxdt\le 0.\end{aligned}\tag{34}$$

Combining (32) and (34), we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}\partial_t u_p^\tau u^\tau\phi dxdt+\int_0^T\int_{\mathbb{R}^d}\pi_\tau u^\tau\cdot\nabla u_p^\tau u^\tau\phi dxdt\\ &+\int_0^T\int_{\mathbb{R}^d}u^\tau|\nabla u^\tau|^{p-2}\nabla u^\tau\nabla\phi dxdt+\int_0^T\int_{\mathbb{R}^d}|\nabla\omega|^{p-2}\nabla\omega(\nabla u^\tau-\nabla\omega)\phi dxdt\\ &+\int_0^T\int_{\mathbb{R}^d}|\nabla u^\tau|^{p-2}\nabla u^\tau\nabla\omega\phi dxdt\le 0.\end{aligned}\tag{35}$$

Then we estimate every terms in (35) one by one.

From (25), (26) and (29), it easily follows that as $\tau \to 0_+$

$$\int_0^T \int_{\mathbb{R}^d} \partial_t u_p^\tau u^\tau \phi dxdt \to \int_0^T \int_{\mathbb{R}^d} \partial_t u_p u \phi dxdt, \tag{36}$$

$$\int_0^T \int_{\mathbb{R}^d} \pi_\tau u^\tau \cdot \nabla u_p^\tau u^\tau \phi dxdt \to \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla u_p u \phi dxdt \tag{37}$$

and

$$\int_0^T \int_{\mathbb{R}^d} |\nabla u^\tau|^{p-2} \nabla u^\tau \nabla \omega \phi dxdt \to \int_0^T \int_{\mathbb{R}^d} \chi \nabla \omega \phi dxdt. \tag{38}$$

It holds from (29) and Proposition 3.2 that

$$\begin{aligned} & |\int_0^T \int_{\mathbb{R}^d} u^\tau |\nabla u^\tau|^{p-2} \nabla u^\tau \nabla \phi dxdt - \int_0^T \int_{\mathbb{R}^d} u \chi \nabla \phi dxdt| \\ \leq \ & |\int_0^T \int_{\mathbb{R}^d} u^\tau (|\nabla u^\tau|^{p-2} \nabla u^\tau - \chi) \nabla \phi dxdt| + |\int_0^T \int_{\mathbb{R}^d} (u^\tau - u) \chi \nabla \phi dxdt| \\ \to \ & 0, \ as \ \tau \to 0_+. \end{aligned} \tag{39}$$

And (21) yields

$$\begin{aligned} & |\int_0^T \int_{\mathbb{R}^d} |\nabla \omega|^{p-2} \nabla \omega (\nabla (u^\tau - \omega) - \nabla (u - \omega)) \phi dxdt| \\ = \ & |\int_0^T \int_{\mathbb{R}^d} |\nabla \omega|^{p-2} \nabla \omega \nabla (u^\tau - u) \phi dxdt| \\ \to \ & 0, \ as \ \tau \to 0_+. \end{aligned} \tag{40}$$

Then, combining (36)-(40) and letting $\epsilon \to 0$, (35) becomes that

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^d} \partial_t u_p u \phi dxdt + \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla u_p u \phi dxdt + \int_0^T \int_{\mathbb{R}^d} u \chi \nabla \phi dxdt \\ & + \int_0^T \int_{\mathbb{R}^d} |\nabla \omega|^{p-2} \nabla \omega (\nabla u - \nabla \omega) \phi dxdt + \int_0^T \int_{\mathbb{R}^d} \chi \nabla \omega \phi dxdt \\ & \leq 0. \end{aligned} \tag{41}$$

Taking $\varphi = u\phi$ in (30), we have

$$\int_0^T \int_{\mathbb{R}^d} \partial_t u_p u \phi dxdt + \int_0^T \int_{\mathbb{R}^d} u \cdot \nabla u_p u \phi dxdt + \int_0^T \int_{\mathbb{R}^d} \chi \nabla u \phi + \chi u \nabla \phi dxdt = 0. \tag{42}$$

Combing (41) and (42) yields that

$$\int_0^T \int_{\mathbb{R}^d} |\nabla \omega|^{p-2} \nabla \omega (\nabla u - \nabla \omega) \phi dxdt + \int_0^T \int_{\mathbb{R}^d} \chi \nabla (\omega - u) \phi dxdt \leq 0,$$

i.e.

$$\int_0^T \int_{\mathbb{R}^d} (|\nabla \omega|^{p-2} \nabla \omega - \chi)(\nabla u - \nabla \omega) \phi dxdt \leq 0. \tag{43}$$

Taking $\omega = u - \lambda \varphi$ with $\lambda \geq 0$ yields that

$$\int_0^T \int_{\mathbb{R}^d} (|\nabla (u - \lambda \varphi)|^{p-2} \nabla (u - \lambda \varphi) - \chi) \nabla \varphi \phi dxdt \leq 0. \tag{44}$$

Choosing ϕ such that supp $\varphi \subset$ supp ϕ and $\phi = 1$ on supp ϕ. When $\lambda \to 0$, it follows that

$$\int_0^T \int_{\mathbb{R}^d} (|\nabla u|^{p-2} \nabla u - \chi) \nabla \varphi dxdt \leq 0. \tag{45}$$

Employing the same method with $\omega = u - \lambda\varphi$ and $\lambda \le 0$, we obtain

$$\int_0^T \int_{\mathbb{R}^d} (|\nabla u|^{p-2}\nabla u - \chi)\nabla\varphi dxdt \ge 0. \tag{46}$$

Thus, for any $\varphi \in C_c^\infty(0,T;\mathbb{R}^d)$,

$$\int_0^T \int_{\mathbb{R}^d} |\nabla u|^{p-2}\nabla u\nabla\varphi dxdt = \int_0^T \int_{\mathbb{R}^d} \chi\nabla\varphi dxdt. \tag{47}$$

Therefore, combining (i)-(iii), we have for $\forall T > 0$

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}^d} u_p\partial_t\varphi dxdt + \int_0^T \int_{\mathbb{R}^d} u\cdot\nabla u_p\varphi dxdt &+ \int_0^T \int_{\mathbb{R}^d} |\nabla u|^{p-2}\nabla u\nabla\varphi dxdt \\ &= \int_{\mathbb{R}^d} (u_{in})_p\varphi(0)dx. \end{aligned} \tag{48}$$

Therefore, we complete the proof of Theorem 3.1. □

4. **Appendix.** In this Appendix, we give a proof of existence for weak solutions to equation (49) by calculus of variations.

$$\begin{cases} u_p - \tau\Delta_p u = |u_{in}|^{p-2}u_{in}, x \in U \\ \nabla\cdot u = 0, x \in U \\ u(x) = 0, x \in \partial U \\ u_{in} \in L^p(U), \\ u_p = |u|^{p-2}u, \Delta_p u = \nabla\cdot(|\nabla u|^{p-2}\nabla u) \end{cases} \tag{49}$$

where the given positive constant $\tau > 0$.

At first, we show the definition of weak solution for (49):

Definition 4.1. We suppose that a domain $U \subset \mathbb{R}^d$ is bounded, connected and has a smooth boundary. We say u is a weak solution of (49) if $u \in W^{1,p}(U)$ which satisfies

$$\int_U u_p\psi dx + \tau\int_U |\nabla u|^{p-2}\nabla u\nabla\psi dxdx = \int_U |u_{in}|^{p-2}u_{in}\psi dx \tag{50}$$

for any $\psi \in W_0^{1,p}(U)$ and given $u_{in} \in L^p(U)$.

Lemma 4.2. *For $u_{in} \in L^p(U)$. The problem (49) admits a unique weak solution.*

Proof. We first give the proof of existence in three steps.

Step 1. Define a functional $I[\cdot]$. We set

$$\mathcal{A} := \{w \in W^{1,p}(U) \mid \nabla\cdot w = 0, w(x)|_{\partial U} = 0\}.$$

For any function $w \in \mathcal{A}$, we define a functional

$$I[w] := \int_U L(Dw,w,x)dx = \int_U \frac{\tau}{p}|Dw|^p + \frac{1}{p}|w|^p - |u_{in}|^{p-2}u_{in}wdx.$$

Since

$$\int_U |u_{in}|^{p-2}u_{in}wdx \le \|u_{in}\|_p^p + \frac{1}{p}\|w\|_p^p,$$

we have

$$I[w] \ge \frac{\tau}{p}\|Dw\|_p^p - \|u_{in}\|_p^p. \tag{51}$$

Step 2. Existence of a minimizer for $I[\cdot]$. Due to (51), we set

$$m := \inf_{w \in \mathcal{A}} I[w]$$

and choose a sequence of functions $u_k \in \mathcal{A}$ such that

$$m = \lim_{k \to \infty} I[u_k].$$

Note that

$$\sup_k \|u_k\|_{W^{1,p}(U)} < \infty.$$

By weak compactness there exists a subsequence of $\{u_k\}_{k=1}^{\infty}$, without relabeling, and a function $u \in W^{1,p}(U)$ so that

$$u_k \rightharpoonup u \text{ weakly in } W^{1,p}(U). \tag{52}$$

By the Rellich-Kondrachov theorem we get that $u_k \to u$ strongly in $L^p(U)$; and choose a subsequence u_k(without relabeling) such that $u_k \to u$ a.e. in U. In fact, $u \in \mathcal{A}$. Furthermore, we obtain that

$$\|u\|_{W^{1,p}(U)} \leq \liminf_{k \to \infty} \|u_k\|_{W^{1,p}(U)},$$

therefore

$$I[u] \leq \liminf_{k \to \infty} I[u_k] = m,$$

i.e. $I[\cdot]$ is weakly lower semicontinuous on $W^{1,p}(U)$. Since $u \in \mathcal{A}$, it follows that

$$I[u] = m = \min_{w \in \mathcal{A}} I[w]. \tag{53}$$

Step 3. Existence of a weak solution for (1). Fix any $\psi \in W_0^{1,p}(U)$ and set $i(\epsilon) := I[u + \epsilon\psi](\epsilon \in \mathbb{R})$:

$$\begin{aligned} i(\epsilon) : &= \int_U L(\nabla(u + \epsilon\psi), u + \epsilon\psi, x)dx \\ &= \int_U \frac{\tau}{p}|\nabla u + \epsilon\nabla\psi|^p + \frac{1}{p}|u + \epsilon\psi|^p - |u_{in}|^{p-2}u_{in}(u + \epsilon\psi)dx. \end{aligned}$$

$$i'(\epsilon) = \int_U \tau|\nabla u + \epsilon\nabla\psi|^{p-2}(\nabla u + \epsilon\nabla\psi)\nabla\psi + |u + \epsilon\psi|^{p-2}(u + \epsilon\psi)\psi - |u_{in}|^{p-2}u_{in}\psi dx.$$

Since $i(\cdot)$ has a minimum for $\epsilon = 0$, we know $i'(0) = 0$:

$$\int_U \{-\tau\nabla \cdot (|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u - |u_{in}|^{p-2}u_{in}\}\psi dx = 0.$$

Hence, u is a weak solution to (49).

Now we prove the uniqueness. Assume that $u, \tilde{u} \in \mathcal{A}$ are two weak solutions satisfying (50), i.e. for any $\psi \in W_0^{1,p}(U)$,

$$\int_U u_p\psi dx + \tau\int_U |\nabla u|^{p-2}\nabla u\nabla\psi dx = \int_U |u_{in}|^{p-2}u_{in}\psi dx, \tag{54}$$

$$\int_U \tilde{u}_p\psi dx + \tau\int_U |\nabla\tilde{u}|^{p-2}\nabla\tilde{u}\nabla\psi dx = \int_U |u_{in}|^{p-2}u_{in}\psi dx. \tag{55}$$

By subtraction,

$$\int_U (u_p - \tilde{u}_p)\psi dx + \tau\int_U (|\nabla u|^{p-2}\nabla u - |\nabla\tilde{u}|^{p-2}\nabla\tilde{u})\nabla\psi dx = 0.$$

Taking $\psi = u - \tilde{u}$ yields

$$\begin{aligned} 0 &= \int_U (u_p - \tilde{u}_p)(u-\tilde{u})dx + \tau \int_U (|\nabla u|^{p-2}\nabla u - |\nabla \tilde{u}|^{p-2}\nabla \tilde{u})\nabla(u-\tilde{u})dx \\ &\geq C(p)\|u-\tilde{u}\|_p^p + C(p)\|\nabla(u-\tilde{u})\|_p^p \geq C(p)\|u-\tilde{u}\|_p^p. \end{aligned}$$

Hence, $u = \tilde{u}$ a.e. in U. □

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REFERENCES

[1] D. Breit, *Existence Theory for Generalized Newtonian Fluids*, Academic Press, 2017.
[2] D. Breit, Existence theory for stochastic power law fluids, *J. Math. Fluid. Mech.*, **17** (2015), 295–326.
[3] F. E. Browder, Nonlinear elliptic boundary value problems, *Bull. Amer. Math. Soc.*, **69** (1963), 862–874.
[4] F. E. Browder, Non-linear equations of evolution, *Ann. of Math.*, **80** (1964), 485–523.
[5] X. Chen, A. Jüngel and J.-G Liu, A note on Aubin-Lions-Dubinskii lemmas, *Acta Appl. Math.*, **133** (2014), 33–43.
[6] X. Chen and J.-G Liu, Global weak entropy solution to Doi-Saintillan-Shelley model for active and passive rod-like and ellipsoidal particle suspensions, *J. Differential Equations.*, **254** (2013), 2764–2802.
[7] X. Chen and J.-G Liu, Analysis of polymeric flow models and related compactness theorems in weighted spaces, *SIAM J. Math. Anal.*, **45** (2013), 1179–1215.
[8] K. Cheng, C. Wang and S. M. Wise, An energy stable BDF2 Fourier pseudo-spectral numerical scheme for the square phase field crystal equation, *Commun. Comput. Phys.*, **26** (2019), 1335–1364.
[9] W. Cong and J.-G. Liu, A degenerate p-Laplacian Keller-Segel model, *Kinet. Relat. Models.*, **9** (2016), 687–714.
[10] E. DiBenedetto and M. A. Herrero, Non-negative solutions of the evolution p-Laplacian equation. Initial traces and Cauchy problem when $1 < p < 2$, *Arch. Rational Mech. Anal.*, **111** (1990), 225–290.
[11] M. Dreher and A. Jüngel, Compact families of piecewise constant functions in $L^p(0,T;B)$, *Nonlinear Anal.*, **75** (2012), 3072–3077.
[12] W. Feng, A. J. Salgado, C. Wang and S. M. Wise, Preconditioned steepest descent methods for some nonlinear elliptic equations involving p-Laplacian terms, *J. Comput. Phys.*, **334** (2017), 45–67.
[13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.
[14] P. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta Math.*, **115** (1966), 271–310.
[15] J. Leray and J.-L. Lions, Quelques résultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder, *Bull. Soc. Math. France.*, **93** (1965), 97–107.
[16] L. Li and J.-G. Liu, p-Euler equations and p-Navier-Stokes equations, *J. Differential Equations.*, **264** (2018), 4707–4748.
[17] E. H. Lieb and M. Loss, *Analysis*, American Mathematical Society, 2001.
[18] P. Lindqvist, *Notes on the Stationary p-Laplace Equation*, Springer Briefs in Mathematics, Springer, 2019.
[19] A. Matas and J. Merker, Existence of weak solutions to doubly degenerate diffusion equations, *Appl. Math.*, **57** (2012), 43–69.
[20] G. J. Minty, On a monotonicity method for the solution of nonlinear equations in Banach spaces, *Proc. Nat. Acad. Sci. U.S.A.*, **50** (1963), 1038–1041.
[21] G. J. Minty, Monotone (nonlinear) operators in Hilbert space, *Duke Math. J.*, **29** (1962), 341–346.

[22] J. Shen, C. Wang, X. Wang and S. M. Wise, Second-order convex splitting schemes for gradient flows with Ehrlich-Schwoebel type energy: Application to thin film epitaxy, *SIAM J. Numer. Anal.*, **50** (2012), 105–125.

[23] J. Simon, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.*, **146** (1987), 65–96.

[24] C. Wang, X. Wang and S. M. Wise, Unconditionally stable schemes for equations of thin film epitaxy, *Discrete Contin. Dyn. Syst.*, **28** (2010), 405–423.

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