MATHEMATICS OF COMPUTATION
Volume 70, Number 234, Pages 595–606
S 0025-5718(00)01271-0
Article electronically published on April 13, 2000

CONVERGENCE OF THE POINT VORTEX METHOD FOR 2-D VORTEX SHEET

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Abstract. We give an elementary proof of the convergence of the point vortex method (PVM) to a classical weak solution for the two-dimensional incompressible Euler equations with initial vorticity being a finite Radon measure of distinguished sign and the initial velocity of locally bounded energy. This includes the important example of vortex sheets, which exhibits the classical Kelvin-Helmholtz instability. A surprise fact is that although the velocity fields generated by the point vortex method do not have bounded local kinetic energy, the limiting velocity field is shown to have a bounded local kinetic energy.

1. Introduction

The dynamics of point vortices in an incompressible flow can be described by a Hamiltonian system, which was known to Helmholtz [14]. When the initial vorticity is approximated by a cluster of point vortices, this Hamiltonian system can be used as a computation method for inviscid incompressible flow, known as the point vortex method (PVM), and was first proposed by Rosenhead in 1932 [20]. This method is particularly effective for the vortex sheet problem since the vorticity in this problem is concentrated on a free curve, known as a vortex sheet, and the PVM has little numerical viscosity. The computation is reduced to a 1-D problem and the sheet is tracked by the Hamiltonian system. The vortex sheet problem is an ill-posed problem, known as Kelvin-Helmholtz instability, and a curvature singularity develops in finite time from an initial smooth vortex sheet [6, 19]. It is observed both experimentally and numerically that the vortex sheet rolls up immediately after the formation of curvature singularity. This peculiar roll-up phenomena was successfully computed by Rosenhead in 1932 [20] with the point vortex method (PVM). In the early 1960's, G. Birkhoff performed a systematic and careful computation with the PVM [3]. Nevertheless, the PVM is quite unstable in general. A vortex blob was introduced by Chorin in 1973 [7] to replace the point vortex, known as vortex blob method, and was successfully used in a computation of the flow past cylinder along with two other seminal ideas: creation of vorticity on the boundary to enforce the no-slip boundary condition and using random walk to approximate the viscosity. Krasny performed careful studies for the vortex sheet problem with the vortex blob methods [15].

Received by the editor May 24, 1999.
2000 *Mathematics Subject Classification.* Primary 65M06, 76M20.
Key words and phrases. Point vortex method, vortex sheet, incompressible Euler equations, classical weak solution.

The purpose of this paper is to give a simple proof of the convergence of the PVM for the vortex sheet problem. This is a very singular problem since the vorticity is only a bounded Random measure in general. The velocity field corresponding to the limiting vorticity generated by PVM is shown to be a classical weak solution to the Euler equations. This generalizes the results obtained by the authors for the vortex blob method in [16], where the main difficulty is the consistency analysis since the flow may be extremely singular in the roll-up process. The new difficulties in the analysis of the PVM mainly come from the fact that the velocity fields generated by the PVM do not have bounded local kinetic energy. The interesting fact is that the limiting velocity field does indeed have bounded local kinetic energy. It should be noted that a similar result was presented in [21] with a different approach. However, we are not able to follow some of the arguments in [21].

It should be noted that the convergence analysis of the PVM also proves the hydrodynamic limit to the 2-D incompressible fluids of a Hamiltonian system of interacting particles [11].

We recall that there are many works in the literature on the convergence theory of the vortex methods; however, most results concern the smooth flows. Some of the notable results for the vortex blob methods are due to Hald [13], Beale and Majda [2], and Cottet [8], and the results for the PVM are due to Goodman, Hou, and Lowengrub [12]. For the discontinuous flows, we refer to Beale [1], Brenier and Cottet [4], Caflisch and Lowengrub [5], Liu and Xin [16] and Schochet [22]. The convergence for a discontinuous Galerkin method with L^2 vorticity data was recently obtained by the authors in [17].

2. Euler equations

In the vorticity formulation, the 2-D incompressible Euler equations can be written as

$$\partial_t \omega + \nabla\cdot(\boldsymbol{u}\,\omega) = 0\,, \tag{1}$$

where the vorticity $\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1$ is the curl of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ and satisfies the Biot-Swart Law

$$\boldsymbol{u}(\boldsymbol{x}, t) = (K * \omega)(\boldsymbol{x}, t), \tag{2}$$

with the kernel K given by

$$K(\boldsymbol{x}) = \frac{1}{2\pi|\boldsymbol{x}|^2}(-x_2, x_1) \equiv \frac{\boldsymbol{x}^\perp}{2\pi|\boldsymbol{x}|^2}, \tag{3}$$

which is the curl of the fundamental solution

$$G(\boldsymbol{x}) = \frac{1}{2\pi}\ln|\boldsymbol{x}| \tag{4}$$

to the 2-D Laplace equation.

3. Initial data

We will consider the Cauchy problem for (1) and (2) with initial data

$$\omega(\boldsymbol{x}, 0) = \omega_0(\boldsymbol{x})\,. \tag{5}$$

It will be assumed thoughout this paper that the initial vorticity satisfies the following conditions:

(i) $\omega_0 \geq 0$.

(ii) ω_0 has compact support, or $\int |\boldsymbol{x}|^2 \omega_0(\boldsymbol{x})\, d\boldsymbol{x} \le C$.
(iii) $\omega_0 \in M(R^2) \cap H^{-1}_{\mathrm{loc}}(R^2)$, is a bounded Randon measure.
One checks easily that vortex sheet data satisfy these three conditions, see [10].

4. Grids

We cover the support of the initial vorticity by nonoverlapping squares, R_j, with side length h and centered at $\alpha_j = jh$ with $j \in Z \times Z$. Denote by ξ_j the total amount of the initial vorticity in R_j, i.e.,

$$\xi_j = \int_{R_j} \omega_0(\boldsymbol{x})\, d\boldsymbol{x}\,. \tag{6}$$

One can then verify the following lemma easily.

Lemma 1. *The following properties hold:*

$$\sum_j \xi_j |\alpha_j|^2 \le C; \tag{7}$$

$$\beta(\omega_0, h) \equiv \max_j \xi_j \le C(\ln \frac{1}{h})^{-\frac{1}{2}}; \tag{8}$$

$$I_0 = -\sum_{i \ne j} \xi_i \xi_j \ln(\alpha_i - \alpha_j) \le C\,. \tag{9}$$

Proof. (i). (7) follows from the assumption (ii) on the initial vorticity that ω_0 in Section 3.

(ii). To prove (8), one chooses a nonnegative function $\chi_j \in C_0^\infty(\mathbb{R}^2)$ which is equal to one on the cell R_j as follows:

$$\chi_j(\boldsymbol{x}) = \begin{cases} 1 & \text{if } |\boldsymbol{x} - \boldsymbol{x}_j| \le h\sqrt{2}, \\ \frac{2\log|\boldsymbol{x} - \boldsymbol{x}_j| - \log(h)}{\log(2h)} & \text{if } h\sqrt{2} < |\boldsymbol{x} - x_j| \le \sqrt{h}, \\ 0 & \text{if } |\boldsymbol{x} - \boldsymbol{x}_j| > \sqrt{h}, \end{cases}$$

after some slight modification to fit into $C_0^\infty(\mathbb{R}^2)$. Directly, computation gives

$$\int |\nabla \chi_j|^2\, d\boldsymbol{x} = 2\pi \int_{h\sqrt{2}}^{\sqrt{h}} \left| \frac{2}{r\log(2h)} \right|^2 r\, dr = \frac{-2}{\log(2h)}\,.$$

Thus we have

$$\max \|\chi_j\|_{H^1} \le C(\ln \frac{1}{h})^{-\frac{1}{2}}.$$

Hence

$$\max_j \xi_j \le \max_j \int \chi_j(\boldsymbol{x}) \omega_0(\boldsymbol{x})\, d\boldsymbol{x} \le \max_j \|\chi_j\|_{H^1} \|\omega_0\|_{H^{-1}_{\mathrm{loc}}} \le C(\ln \frac{1}{h})^{-\frac{1}{2}}$$

due to the assumption that $\omega_0 \in M(R^2) \cap H^{-1}_{\mathrm{loc}}(R^2)$. Thus, (8) follows.

(iii). We now turn to (9). First we show that

$$J_0 = -\int\int \omega_0(\boldsymbol{x}) \omega_0(\boldsymbol{y}) G(\boldsymbol{x} - \boldsymbol{y})\, d\boldsymbol{x}\, d\boldsymbol{y} \le C\,.$$

Since the initial vorticity is assumed to have compact support and $\omega_0 \in H^{-1}_{\rm loc}$, one has

$$J_0 = -\int (G*\omega_0)(\boldsymbol{x})\,\omega_0(\boldsymbol{x})\,d\boldsymbol{x} \le C\|\omega_0\|_{H^{-1}_{\rm loc}}\,\|G*\omega_0\|_{H^1_{\rm loc}}$$
$$\le C_1\|\nabla^\perp(G*\omega_0)\|_{L^2_{\rm loc}} + C_1\|G*\omega_0\|_{L^2_{\rm loc}} \le C_2\,.$$

In the last step above, we used the following simple estimate

$$\|G*\omega_0\|_{L^2_{\rm loc}} \le \|\omega_0\|_{L^1}\,\|G\|_{L^2_{\rm loc}} \le C\,,$$

where and from now on, we use C to denote any generic positive constant independent of h.

Next, it is not too hard to verify directly that

$$-\sum_{i\neq j,|\alpha_i-\alpha_j|\le 1/2} \xi_i\xi_j \ln(\alpha_i-\alpha_j)$$
$$= -\sum_{i\neq j,|\alpha_i-\alpha_j|\le 1/2} \ln(\alpha_i-\alpha_j)\int_{R_i}\int_{R_j}\omega_0(\boldsymbol{x})\omega_0(\boldsymbol{y})\,d\boldsymbol{x}\,d\boldsymbol{y}$$
$$\le -C\sum_{i\neq j,|\alpha_i-\alpha_j|\le 1/2}\int_{R_i}\int_{R_j}\omega_0(\boldsymbol{x})\omega_0(\boldsymbol{y})\ln(\boldsymbol{x}-\boldsymbol{y})\,d\boldsymbol{x}\,d\boldsymbol{y} \le C\,.$$

On the other hand, (7) yields that

$$-\sum_{|\alpha_i-\alpha_j|>1/2}\xi_i\xi_j G(\boldsymbol{x}_i-\boldsymbol{x}_j) \le \sum_{|\alpha_i-\alpha_j|>1/2}\xi_i\xi_j(1+|\alpha_i|^2+|\alpha_j|^2) \le C\,.$$

Collecting all the estimates gives (9). The proof of the lemma is complete. $\square$

Remark 1. In our previous analysis of the vortex blob method in [16], the condition on the computational grid size $h \le C\epsilon$, where ϵ is the blob size, it can be easily checked that this requirement can now be removed thanks to the estimate (9). However, when $\epsilon = 0$, the approximate solution generated by the PVM has unbounded local kinetic energy. Thus the analysis in [16] cannot be applied directly here. This is the main new difficulty. Nevertheless, we will show in Section 11 that the limiting velocity indeed has bounded local kinetic energy.

5. Point vortex method

The classical point vortex method, approximating (1) and (2), is to look for particle paths $\boldsymbol{x}_j(t)$ which solve the following problem:

$$\frac{d}{dt}\boldsymbol{x}_j(t) = \sum_{\ell\neq j}\xi_\ell\,K\big(\boldsymbol{x}_j(t)-\boldsymbol{x}_\ell(t)\big)\,,\quad \boldsymbol{x}_j(0)=\alpha_j\,. \tag{10}$$

It is easy to show that the finite system of ordinary differential equations (10) has a unique global solution. Furthermore, the system in (10) is a Hamiltonian system with the Hamiltonian given by

$$I_h(t) \equiv -\sum_{j\neq\ell}\xi_j\,\xi_\ell\,G(\boldsymbol{x}_j-\boldsymbol{x}_\ell)\,. \tag{11}$$

Indeed, one can show that

$$I_h(t) \equiv I_h(0) \equiv I_0 \leq C \tag{12}$$

for all t, where the first inequality can be verified directly by using the symmetry properties of ∇G and K, and the trivial identity that $\nabla a \cdot \nabla^{\perp} b = -\nabla^{\perp} a \cdot \nabla b$ for any smooth functions a and b, while the last inequality has been proved in the previous lemma.

6. The Birkhoff-Rott equation

In both theoretical and numerical analysis and engineering applications [3, 6, 19, 20], the evolution of the vortex sheets is studied through the well-known Birkhoff-Rott equation which is an integro-differential equation derived from the Euler equations using the fact that the vorticity vanishes except on the vortex sheet [3]. Direct discretization of the Birkhoff-Rott equation gives exactly same approximation as the one in (10) by discretizing the Euler equations except that the index j in (6) and (10) are reduced to one-dimension and hence the computation is extremely efficient [15]. It is noted that the validity of the Birkhoff-Rott equation after the formation of a singularity is not well understood. The convergence result in [16] to a classical weak solution for Krasny's desingularization procedure of the Birkhoff-Rott equation gives an appropriate interpretation of this integro-differential equation even past the formation of a singularity.

We refer to Caflisch [6] and Moore [19] for the analysis of the Birkhoff-Rott equation in the class of analytic functions. Assuming the sheet is analytic and close to horizontal, Caflisch and Lowengrub [5] show that the vortex blob approximation to the sheet converges strongly for a time interval before the singularity formation.

7. Weak form

In order to study the convergence of the PVM, we need a weak PDE formulation of the Hamiltonian system (10). To this end, one defines the vorticity measure and the corresponding velocity field and stream function, respectively, as

$$\begin{aligned}
\omega_h(\boldsymbol{x}, t) &\equiv \sum_j \xi_j \, \delta\big(\boldsymbol{x} - \boldsymbol{x}_j(t)\big) \,, \\
\boldsymbol{u}_h(\boldsymbol{x}, t) &\equiv \sum_j \xi_j \, K\big(\boldsymbol{x} - \boldsymbol{x}_j(t)\big) \,, \\
\psi_h(\boldsymbol{x}, t) &\equiv \sum_j \xi_j \, G\big(\boldsymbol{x} - \boldsymbol{x}_j(t)\big) \,.
\end{aligned} \tag{13}$$

It follows from (10) that for any $\phi(\boldsymbol{x}, t) \in C_0^2(\mathbb{R}^2 \times \mathbb{R}_+)$, one has

$$\begin{aligned}
\partial_t \langle \phi, \, \omega_h \rangle &= \sum_j \xi_j \left(\partial_t \phi(\boldsymbol{x}_j(t), t) + \nabla \phi(x_j(t), t) \dot{\boldsymbol{x}}_j \right) \\
&= \sum_j \xi_j \partial_t \phi(\boldsymbol{x}_j) + \sum_{j \neq \ell} \nabla \phi(\boldsymbol{x}_j) \xi_j \xi_\ell K(\boldsymbol{x}_j - \boldsymbol{x}_\ell) \,.
\end{aligned}$$

Denote by I the second-order term on the right-hand side of the above equation. It is then easy to verify that

$$\begin{aligned} I &= \sum_{j\neq\ell} \xi_j\xi_\ell \int \nabla\phi(\boldsymbol{x})\cdot K(\boldsymbol{x}-\boldsymbol{x}_\ell)\delta(\boldsymbol{x}-\boldsymbol{x}_j)\,d\boldsymbol{x} \\ &= -\frac{1}{2}\sum_{j\neq\ell}\xi_j\xi_\ell \int \nabla^\perp\otimes\nabla\phi(\boldsymbol{x}) : K(\boldsymbol{x}-\boldsymbol{x}_j)\otimes K(\boldsymbol{x}-\boldsymbol{x}_\ell)\,d\boldsymbol{x}. \end{aligned} \tag{14}$$

Define a matrix-valued function $\mathcal{K}(\boldsymbol{y},\boldsymbol{z};\theta)$ by

$$\mathcal{K}(\boldsymbol{y},\boldsymbol{z};\theta) = \int \theta(\boldsymbol{x})\,K(\boldsymbol{x}-\boldsymbol{y})\otimes K(\boldsymbol{x}-\boldsymbol{z})\,d\boldsymbol{x}\,, \qquad \boldsymbol{y}\neq\boldsymbol{z}\,, \tag{15}$$

for any $\theta(\boldsymbol{x})\in C_0^\infty(\mathbb{R}^2)$. Then

$$\begin{aligned} I &= -\frac{1}{2}\sum_{j\neq\ell}\xi_j\xi_\ell\Big(\mathcal{K}_{12}(\boldsymbol{x}_j,\boldsymbol{x}_\ell,(\partial_x^2-\partial_y^2)\phi) + (\mathcal{K}_{22}-\mathcal{K}_{11})(\boldsymbol{x}_j,\boldsymbol{x}_\ell,\partial_{xy}\phi)\Big) \\ &= -\frac{1}{2}\int_{\boldsymbol{y}\neq\boldsymbol{z}} \omega_h(\boldsymbol{y})\omega_h(\boldsymbol{z})\Big(\mathcal{K}_{12}(\boldsymbol{y},\boldsymbol{z},(\partial_x^2-\partial_y^2)\phi) \\ &\qquad\qquad + (\mathcal{K}_{22}-\mathcal{K}_{11})(\boldsymbol{y},\boldsymbol{z},\partial_{xy}\phi)\Big)\,d\boldsymbol{y}\,d\boldsymbol{z}. \end{aligned}$$

Integrating in time leads to the following weak form of the point vortex method in terms of only the vorticity field:

$$\begin{aligned} \langle\partial_t\phi,\omega_h\rangle + \frac{1}{2}\int_{\boldsymbol{y}\neq\boldsymbol{z}} \omega_h(\boldsymbol{y})\omega_h(\boldsymbol{z})\Big(\mathcal{K}_{12}(\boldsymbol{y},\boldsymbol{z},(\partial_x^2-\partial_y^2)\phi) \\ + (\mathcal{K}_{22}-\mathcal{K}_{11})(\boldsymbol{y},\boldsymbol{z},\partial_{xy}\phi)\Big)\,d\boldsymbol{y}\,d\boldsymbol{z} = 0\,. \end{aligned} \tag{16}$$

It should be clear now from the weak formulation (16) that in order to study the convergence of the PVM, it is necessary to estimate the singularities of the kernel $\mathcal{K}$ and the concentration property of the vorticity measures involved in the second term of (16). The weak form (16) was first formulated by Goodman, Hou, and Lowengrub in [12], and is analogous to the continuous counterpart in Diperna and Majda [10] and Delort [9] (see also [21]) where they established the theory for the weak solution to the 2-D Euler equations.

8. Kernel estimate

The necessary estimates on the singular kernel $\mathcal{K}$ defined in (15) are provided in the following lemma, and its proof can be found in [16], see also [9].

Lemma 2. *For $|\boldsymbol{y}-\boldsymbol{z}|\le 1/2$, it holds that*

$$\big|\mathcal{K}_{11}(\boldsymbol{y},\boldsymbol{z};\theta)\big| + \big|\mathcal{K}_{22}(\boldsymbol{y},\boldsymbol{z};\theta)\big| \le C\ln\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|}\,, \tag{17}$$

$$\big|\mathcal{K}_{12}(\boldsymbol{y},\boldsymbol{z};\theta)\big| + \big|\mathcal{K}_{21}(\boldsymbol{y},\boldsymbol{z};\theta)\big| \le C\,, \tag{18}$$

$$\big|\mathcal{K}_{11}(\boldsymbol{y},\boldsymbol{z};\theta) - \mathcal{K}_{22}(\boldsymbol{y},\boldsymbol{z};\theta)\big| \le C\,, \tag{19}$$

where C depends only on θ.

9. Estimate on maximal vorticity function

As in [16], the local concentration behavior of the sequence of vorticity measures ω_h is analyzed by studying the maximal vorticity function $M_r(\omega)$ of DiPerna and Majda [10], defined for any Radon measure ω as

$$M_r(\omega) = \sup_{\boldsymbol{x}\in\mathbb{R}^2, 0<t\le T} \int_{|\boldsymbol{y}-\boldsymbol{x}|\le r} |\omega(\boldsymbol{y},t)|\, d\boldsymbol{y} \qquad \text{for } 0 < r \le \frac{1}{2}\,. \tag{20}$$

Hence,

$$M_r(\omega_h) = \sup_{\boldsymbol{x}\in\mathbb{R}^2, 0<t\le T} \sum_{|x_j - x|\le r} \xi_j\,, \quad \text{for } 0 < r \le \frac{1}{2}\,. \tag{21}$$

Now we estimate the maximal vorticity function by using the special Hamiltonian structure of the particle system (10), see (11) and (12). Indeed, it follows from (11) and (12) that

$$\begin{aligned} -\sum_{j\ne\ell, |\boldsymbol{x}_j-\boldsymbol{x}_\ell|\le 1/2} \xi_j\,\xi_\ell\,\ln(\boldsymbol{x}_j - \boldsymbol{x}_\ell) \quad &= I_h(t) + \sum_{|\boldsymbol{x}_j-\boldsymbol{x}_\ell|>1/2} \xi_j\,\xi_\ell\,\ln(\boldsymbol{x}_j - \boldsymbol{x}_\ell) \\ &\le C + \sum_{j,\ell} \xi_j\,\xi_\ell \left(1 + |\boldsymbol{x}_j|^2 + |\boldsymbol{x}_\ell|^2\right) \le C_6\,. \end{aligned}$$

Here we have also used Lemma 1 and the conservation of the second moment for the discrete vorticity field, i.e. $\sum_j \xi_j\,|\boldsymbol{x}_j(t)|^2 = \sum_j \xi_j\,|\alpha_j|^2$, which can be verified directly. This gives

$$\sum_{j\ne\ell, |\boldsymbol{x}_j-\boldsymbol{x}_\ell|\le 1/2} \xi_j\,\xi_\ell\,\ln\frac{1}{|\boldsymbol{x}_j - \boldsymbol{x}_\ell|} \le C\,.$$

This, together with Lemma 1, shows that

$$\begin{aligned} \sum_{|\boldsymbol{x}_j-\boldsymbol{x}_\ell|\le r} \xi_j\,\xi_\ell\,\ln\frac{1}{r} \quad &\le \quad \sum_{j\ne\ell, |\boldsymbol{x}_j-\boldsymbol{x}_\ell|\le r} \xi_j\,\xi_\ell\,\ln\frac{1}{|\boldsymbol{x}_j - \boldsymbol{x}_\ell|} + \sum_j \xi_j^2\,\ln\frac{1}{r} \\ &\le C + C(\ln\frac{1}{h})^{-\frac{1}{2}}\,\ln\frac{1}{r}. \end{aligned} \tag{22}$$

As a direct consequence, we know that any two particles will never run into each other. More importantly, (22) implies the following optimal decay estimate on the maximal vorticity function:

$$M_r(\omega_h) \le C\sqrt{(\ln\frac{1}{h})^{-\frac{1}{2}} - \frac{1}{\ln r}} \to 0 \quad \text{as } r, h \to 0. \tag{23}$$

10. Convergence analysis

We are now in the position to give the convergence analysis for the PVM. Since ω_h admits the uniform bound

$$\max_{0\le t\le T}\int_{\mathbb{R}^2}\omega_h(\boldsymbol{x},t)\,d\boldsymbol{x}\le C\,, \tag{24}$$

there exists an $\omega\in L^\infty\big([0,T],M(\mathbb{R}^2)\cap H^{-1}_{\rm loc}(\mathbb{R}^2)\big)$ such that passing to a subsequence, which we still denote as $\{\omega_h\}$, one has

$$\omega_h\to\omega\quad\text{in } M(\mathbb{R}^2\times\mathbb{R}_+)\,. \tag{25}$$

Furthermore, since $\omega_h\ge 0$, so $\omega\ge 0$. Choose a nonnegative function $\chi\in C_0^\infty(\mathbb{R}^2)$ which is equal to one on the ball $\{\boldsymbol{x}:|\boldsymbol{x}|\le r\}$, and equal to zero when $|\boldsymbol{x}|\ge 2r$. Then, one obtains that

$$M_r(\omega)\le\int\chi(\boldsymbol{x})\omega(\boldsymbol{x})\,d\boldsymbol{x}=\lim_{h\to0}\int\chi(\boldsymbol{x})\omega_h(\boldsymbol{x})\,d\boldsymbol{x}\le\lim_{h\to0}M_{2r}(\omega_h)\le\frac{C}{\ln 1/r}\,. \tag{26}$$

For any given $0<r\le 1/2$, $\mathcal{K}_{12}(\boldsymbol{y},\boldsymbol{z},(\partial_x^2-\partial_y^2)\phi)$ and $(\mathcal{K}_{11}-\mathcal{K}_{22})(\boldsymbol{y},\boldsymbol{z},\partial_{xy}\phi)$ are continuous bounded functions on the region $|\boldsymbol{y}-\boldsymbol{z}|\ge r$, as can be checked easily, and they are bounded for $|\boldsymbol{y}-\boldsymbol{z}|\le r$, which follows Lemma 2. Therefore, taking the limit in (16) and using the estimates (23) and (26), one can show that for any $\phi(\boldsymbol{x},t)\in C_0^2(\mathbb{R}^2\times\mathbb{R}_+)$, it holds that

$$\begin{aligned}\langle\partial_t\phi,\omega\rangle+\frac12\int\!\!\int\omega(\boldsymbol{y})\omega(\boldsymbol{z})\Big(&\mathcal{K}_{12}(\boldsymbol{y},\boldsymbol{z},(\partial_x^2-\partial_y^2)\phi)\\&+(\mathcal{K}_{22}-\mathcal{K}_{11})(\boldsymbol{y},\boldsymbol{z},\partial_{xy}\phi)\Big)\,d\boldsymbol{y}\,d\boldsymbol{z}=0\,.\end{aligned} \tag{27}$$

Thus, in terms of weak formulation, we have shown that the limiting vorticity field yields a weak solution to the 2-D incompressible Euler equations. To recover the classical weak solution in terms of the velocity variables, we need an estimate on local kinetic energy.

11. Estimate on local kinetic energy

The limiting velocity can be defined as

$$\boldsymbol{u}=K*\omega\,. \tag{28}$$

We will show that $\boldsymbol{u}$ is a classical weak solution in the sense which will be clear at the end of this subsection, see also [10, 16]. We first need show that $\boldsymbol{u}$ is bounded in $L^2_{\rm loc}$. To this end, it suffices to derive the bound

$$\int_{|\boldsymbol{y}-\boldsymbol{z}|\le r}\omega(\boldsymbol{y})\omega(\boldsymbol{z})\log\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|}\,d\boldsymbol{y}\,d\boldsymbol{z}\le C\,. \tag{29}$$

Indeed, choosing a nonnegative function $\chi \in C_0^\infty(\mathbb{R}^2)$, which is equal to one on the ball $\{\boldsymbol{x} : |\boldsymbol{x}| \le R\}$, one may bound the local kinetic energy as

$$\begin{aligned}
\int_{|\boldsymbol{x}|\le R} |\boldsymbol{u}(\boldsymbol{x},t)|^2\,d\boldsymbol{x} &\le \int \chi(\boldsymbol{x})\,|\boldsymbol{u}(\boldsymbol{x},t)|^2\,d\boldsymbol{x}\\
&= \int\!\!\int d\boldsymbol{y}\,d\boldsymbol{z}\omega(\boldsymbol{y})\omega(\boldsymbol{z}) \int \chi(\boldsymbol{x})\,K(\boldsymbol{x}-\boldsymbol{y})\cdot K(\boldsymbol{x}-\boldsymbol{z})\,d\boldsymbol{x}\,.\\
&= \int\!\!\int d\boldsymbol{y}\,d\boldsymbol{z}\omega(\boldsymbol{y})\omega(\boldsymbol{z})\Big(\mathcal{K}_{11}+\mathcal{K}_{22}\Big)(\boldsymbol{y},\boldsymbol{z},\chi)\\
&= \left(\int_{|\boldsymbol{y}-\boldsymbol{z}|<r} + \int_{|\boldsymbol{y}-\boldsymbol{z}|\ge r}\right) d\boldsymbol{y}\,d\boldsymbol{z}\omega(\boldsymbol{y})\omega(\boldsymbol{z})\Big(\mathcal{K}_{11}+\mathcal{K}_{22}\Big)(\boldsymbol{y},\boldsymbol{z},\chi)\\
&= I_1 + I_2\,.
\end{aligned}$$

For I_2, since there is no singularity in the kernel for $|\boldsymbol{y}-\boldsymbol{z}| \ge r$, it is evident that $I_2 \le C$. As for I_1, one may use Lemma 2 to obtain

$$I_1 \le C \int_{|\boldsymbol{y}-\boldsymbol{z}|\le r} \omega(\boldsymbol{y})\omega(\boldsymbol{z}) \ln\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|}\,d\boldsymbol{y}\,d\boldsymbol{z} \le C\,,$$

where (17) and (29) have been used.

It remains to prove (29). Denote the measure $\mu = d\omega(\boldsymbol{y})d\omega(\boldsymbol{z})$. For any $\delta > 0$ fixed, one has that

$$\begin{aligned}
\int_{|\boldsymbol{y}-\boldsymbol{z}|\le r} \log\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|+\delta}\,d\mu &= \int_{|\boldsymbol{y}-\boldsymbol{z}|\le r} \omega(\boldsymbol{y})\omega(\boldsymbol{z})\log\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|+\delta}\,d\boldsymbol{y}\,d\boldsymbol{z}\\
&= \lim_{h\to 0}\int_{|\boldsymbol{y}-\boldsymbol{z}|\le r} \omega_h(\boldsymbol{y})\omega_h(\boldsymbol{z})\log\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|+\delta}\,d\boldsymbol{y}\,d\boldsymbol{z} \le C\,.
\end{aligned} \tag{30}$$

It follows from

$$\lim_{r\to 0} M_r(\omega) \to 0$$

that the subset $\{(\boldsymbol{y},\boldsymbol{z})\,|\,\boldsymbol{y}=\boldsymbol{z}\}$ has zero measure in μ. Hence

$$\lim_{\delta\to 0}\log\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|+\delta} \to \log\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|} \qquad a.e.\text{ in } \mu.$$

Note also that $\log\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|+\delta}$ is monotonically increasing as $\delta \to 0$. So Levi's lemma implies that

$$\lim_{\delta\to 0}\int \log\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|+\delta}\mu \to \int \log\frac{1}{|\boldsymbol{y}-\boldsymbol{z}|}\mu\,.$$

Consequently, (29) follows by taking limit of $\delta \to 0$ in (30).

Thus, we have shown that the limiting velocity field $\boldsymbol{u}$ has local finite energy, i.e.,

$$\boldsymbol{u}(\boldsymbol{x},t) \in L^2_{\mathrm{loc}}\,. \tag{31}$$

It should be noted that (31) is a remarkable fact since the approximating velocity fields $\boldsymbol{u}_h(\boldsymbol{x},t)$ themselves are unbounded in L^2_{loc}. It is now easy to verify that

$\boldsymbol{u}(\boldsymbol{x},t) \in L^2_{\rm loc}$ is a classical weak solution to the 2-D incompressible Euler equation in the sense that

(i) for all test functions $\theta(\boldsymbol{x},t) \in C_0^\infty\big(\mathbb{R}^2 \times (0,T)\big)$,

$$\iint \Big(\nabla^\perp \theta_t \cdot \boldsymbol{u} + (\nabla^\perp \otimes \nabla\theta) : (\boldsymbol{u} \otimes \boldsymbol{u}) \Big)\, d\boldsymbol{x}\, dt = 0,$$

(ii) div $\boldsymbol{u} = 0$ in the sense of distribution,

(iii) $\boldsymbol{u}(\boldsymbol{x},t) \in \mathrm{Lip}\big([0,T), H^{-m}_{\rm loc}(\mathbb{R}^2)\big)$ for some $m > 0$ and $\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x})$,

provided that the condition in (iii) is satisfied, which will be verified in the next section.

12. Estimate in time

It remains to prove that $\boldsymbol{u}(\boldsymbol{x},t) \in \mathrm{Lip}\big([0,T), H^{-3}_{\rm loc}(\mathbb{R}^2)\big)$. It suffices to show that $\boldsymbol{u}_h(\boldsymbol{x},t) \in \mathrm{Lip}\big([0,T), H^{-3}_{\rm loc}(\mathbb{R}^2)\big)$ uniformly (see [16]). Let $\theta = \theta(\boldsymbol{x}) \in C_0^\infty(\mathbb{R}^2)$. One then has

$$\begin{aligned}
&\int \nabla^\perp\theta\, \partial_t \boldsymbol{u}_h\, d\boldsymbol{x} = -\partial_t \langle \theta\, \omega_h \rangle \\
&= \frac{1}{2}\int_{\boldsymbol{y}\neq\boldsymbol{z}} \omega_h(\boldsymbol{y})\omega_h(\boldsymbol{z}) \Big(\mathcal{K}_{12}(\boldsymbol{y},\boldsymbol{z},(\partial_x^2 - \partial_y^2)\theta) + (\mathcal{K}_{22} - \mathcal{K}_{11})(\boldsymbol{y},\boldsymbol{z},\partial_{xy}\theta) \Big)\, d\boldsymbol{y}\, d\boldsymbol{z} \\
&\to \frac{1}{2}\int_{\boldsymbol{y}\neq\boldsymbol{z}} \omega(\boldsymbol{y})\omega(\boldsymbol{z}) \Big(\mathcal{K}_{12}(\boldsymbol{y},\boldsymbol{z},(\partial_x^2 - \partial_y^2)\theta) + (\mathcal{K}_{22} - \mathcal{K}_{11})(\boldsymbol{y},\boldsymbol{z},\partial_{xy}\theta) \Big)\, d\boldsymbol{y}\, d\boldsymbol{z}\,,
\end{aligned}$$

as $h \to 0$. It follows from the Sobolev's embedding lemma that

$$\left| \int \nabla^\perp\theta\, \partial_t \boldsymbol{u}_h\, d\boldsymbol{x} \right| \le C \|\nabla^\perp\theta\|_{W^{1,\infty}(\mathbb{R}^2)} \le C\|\nabla^\perp\theta\|_{H^3(\mathbb{R}^2)}\,.$$

Taking the limit in the above estimate, we have

$$\left| \int \nabla^\perp\theta\, \partial_t \boldsymbol{u}\, d\boldsymbol{x} \right| \le C\|\nabla^\perp\theta\|_{H^3(\mathbb{R}^2)}\,. \tag{32}$$

Since $\boldsymbol{u}$ is divergence free, (32) implies that

$$\|\partial_t \boldsymbol{u}\|_{L^\infty\left([0,T),H^{-3}_{\rm loc}(\mathbb{R}^2)\right)} \le C\,. \tag{33}$$

Consequently, one has $\boldsymbol{u}(\boldsymbol{x},t) \in \mathrm{Lip}\big([0,T), H^{-3}_{\rm loc}(\mathbb{R}^2)\big)$, and so $\boldsymbol{u}(\boldsymbol{x},t)$ is a desired classical weak solution.

13. Convergence theorem

Now we can summarize the above analysis into a theorem.

Theorem 1. *Suppose $\omega_0 \ge 0$, compactly supported, and bounded in $M(\mathbb{R}^2) \cap H^{-1}_{\rm loc}(\mathbb{R}^2)$. Let ω_h be the vorticity measures generated by the point vortex method. Then subsequence ω_h converges weakly to $\omega \in M(\mathbb{R}^2)$. Furthermore, let $\boldsymbol{u} = K * \omega$. Then $\boldsymbol{u} \in L^\infty\big([0,T), L^2_{\rm loc}(\mathbb{R}^2)\big) \cap \mathrm{Lip}\big([0,T), H^{-3}_{\rm loc}(\mathbb{R}^2)\big)$ and is a classical weak solution to the Euler equation with the same initial data.*

Acknowledgments

The research of the first author was supported in part by NSF grant DMS-9805621. The research of the second author was supported in part by NSF grants DMS-9600137 and DMS-9971978, DOE Grant DE-FG02-88ER-25053, Hong Kong RGC Earmarked Research Grant CUHK-4219-99P, and the Zheng-Ge-Ru Foundation.

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