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CONVERGENCE ANALYSIS OF THE ENERGY AND HELICITY PRESERVING SCHEME FOR AXISYMMETRIC FLOWS*

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Abstract. We give an error estimate for the energy and helicity preserving scheme (EHPS) in second order finite difference setting on axisymmetric incompressible flows with swirling velocity. This is accomplished by a weighted energy estimate, along with careful and nonstandard local truncation error analysis near the geometric singularity and a far field decay estimate for the stream function. A key ingredient in our a priori estimate is the permutation identities associated with the Jacobians, which are also a unique feature that distinguishes EHPS from standard finite difference schemes.

Key words. incompressible viscous flow, Navier–Stokes equation, pole singularity, conservative scheme, Jacobian, permutation identity, geometric singularity

AMS subject classifications. 65M06, 65M12, 65M15, 76D05, 35Q30

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1. Introduction.

Axisymmetric flow is an important subject in fluid dynamics and has become standard textbook material (e.g., [2]) as a starting point of theoretical study for complicated flow patterns. Although the number of independent spatial variables is reduced by symmetry, some of the essential features and complexities of generic three-dimensional (3D) flows remain. For example, when the swirling velocity is nonzero, there is a vorticity stretching term present. This is widely believed to account for possible singularity formation for Navier–Stokes and Euler flows. For general smooth initial data, it is well known that the solution remains smooth for a short time in Euler [8] and Navier–Stokes flows [9]. A fundamental regularity result concerning the solution of the Navier–Stokes equation (NSE) is given in the pioneering work of Caffarelli, Kohn, and Nirenberg [3]: The 1D Hausdorff measure of the singular set is zero. As a consequence, the only possible singularity for axisymmetric Navier–Stokes flows would be on the axis of rotation. This result has motivated subsequent research activities concerning the regularity of axisymmetric solutions of the NSE. Some regularity and partial regularity results for axisymmetric Euler and Navier–Stokes flows can be found, for example, in [4] and the references therein. To date, the regularity of the Navier–Stokes and Euler flows, whether axisymmetric or not, remains a challenging open problem. For a comprehensive review of the regularity of the NSE, see [10] and the references therein.

Due to the subtle regularity issue, the numerical simulation of axisymmetric flows is also a challenging subject for computational fluid dynamicists. The earliest attempt at a numerical search for potential singularities of axisymmetric flows dates back to the 90s [5, 6]. In a recent work [11], the authors have developed a class of energy and helicity preserving schemes (EHPS) for incompressible Navier–Stokes and MHD

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equations. There the authors extended the vorticity-stream formulation of axisymmetric flows given in [5] and proposed a generalized vorticity-stream formulation for 3D Navier–Stokes and MHD flows with coordinate symmetry. In the case of axisymmetric flows, the major difference between EHPS and the formulation in [5] is the expression and numerical discretization of the nonlinear terms. It is shown in [11] that all the nonlinear terms in the Navier–Stokes and MHD equation, including convection, vorticity stretching, geometric source, Lorentz force, and electro-motive force, can be written as Jacobians. Associated with the Jacobians is a set of permutation identities which leads naturally to the conservation laws for first and second moments. The primary feature of the EHPS is the numerical realization of these conservation laws. In addition to preserving physically relevant quantities, the discrete form of conservation laws provides numerical advantages as well. In particular, the conservation of energy automatically enforces nonlinear stability of EHPS. For 2D flows, EHPS is equivalent to the energy and enstrophy preserving scheme of Arakawa [1], who first pointed out the importance of discrete conservation laws in long time numerical simulations.

Other than the Jacobian approach, most of the energy conserving finite difference schemes for standard flows (without geometric singularity) are based on discretization of the fluid equation in primitive variables. A well-known trick that dates back to the 70s is to take the average of conservative and nonconservative discretizations of convection term (Piacsek and Williams [16]). In [14], Morinishi et al. further explored and compared various combinations among conservative, nonconservative, and rotation forms of the convection term. More recently in [18], Verstappen and Veldman proposed a discretization for the convection term that resulted in a skew-symmetric difference operator and therefore the conservation of energy could be achieved.

A potential difficulty associated with axisymmetric flows is the appearance of a $\frac{1}{r}$ factor which becomes infinite at the axis of rotation, and therefore sensitive to inconsistent or low order numerical treatment near this "pole singularity." In [11], the authors proposed a second order finite difference scheme and handled the pole singularity by shifting the grids a half-grid length away from the origin. Remarkably, the permutation identities and therefore the energy and helicity identities remain valid in this case. There are alternative numerical treatments proposed in literatures (e.g., [6]) to handle this coordinate singularity. However, rigorous justifications for various pole conditions are yet to be established.

The purpose of this paper is to give a rigorous error estimate of EHPS for axisymmetric flows. To focus on the pole singularity and avoid complication caused by physical boundary conditions, we consider here only the whole space problem with the swirling components of velocity and vorticity decaying fast enough at infinity. The error analysis of numerical methods for NSE with nonslip physical boundary condition has been well studied. We refer the works of Hou and Wetton [7] and Wang and Liu [19] to interested readers. Our proof is based on a weighted energy estimate along with a careful and detailed pointwise local truncation error analysis. A major ingredient in our energy estimate is the permutation identities associated with the Jacobians (4.17). These identities are key to the energy and helicity preserving property of EHPS for general symmetric flows. Here the same identities enable us to obtain a priori estimate even in the presence of the pole singularity; see section 5 for details. To our knowledge, this is the first rigorous convergence proof for finite difference schemes devised for axisymmetric flows.

In our pointwise local truncation error estimate, a fundamental issue is the identification of smooth flows in the vicinity of the pole. Using a symmetry argument, it can be shown [12] that if the swirling component is even in r (or more precisely, is the restriction of an even function on $r > 0$), the vector field is in fact singular. See Example 1 in section 2 for details. This is an easily overlooked mistake that even appeared in some research papers targeted at numerical search for potential formation of finite time singularities. In addition to the regularity issue at the axis of symmetry, a refined decay estimate for the stream function also plays an important role in our analysis. In general, the stream function only decays as $O((x^2 + r^2)^{-1})$ at infinity. Accordingly, we have selected an appropriate combination of weight functions that constitute an r-homogeneous norm. As a result, the slow decay of the stream function is compensated by the fast decay of velocity and vorticity. Overall, we obtained a second order error estimate on axisymmetric flows.

The rest of this paper is organized as follows: In section 2, we give a brief review of the regularity results developed in [12], including the characterization of pole regularity for general axisymmetric solenoidal vector fields and solutions of the axisymmetric NSE (2.2). In section 3, we formulate a regularity assumption on the solution of NSE at infinity. We basically assume that the swirling components of velocity and vorticity decay fast enough at infinity, and use this to analyze the decay rate of the stream function. In section 4, we briefly review the energy and helicity preserving property for EHPS and use it to prove our main theorem by energy estimate in section 5. The proof of some technical lemmas is given in the Appendix.

2. Generalized vorticity-stream formulation for axisymmetric flows.

In this section, we review the generalized vorticity-stream formulation of axisymmetric NSE

$$\begin{aligned} &\partial_t \boldsymbol{u} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} + \nabla p = -\nu \nabla \times \nabla \times \boldsymbol{u} \\ &\nabla \cdot \boldsymbol{u} = 0 \end{aligned} \tag{2.1}$$

and related regularity issues.

Denoting by the x-axis the axis of symmetry, the axisymmetric NSE in the cylindrical coordinate system $x = x$, $y = r\cos\theta$, $z = r\sin\theta$ can be written as [11]

$$\begin{aligned} &u_t + \tfrac{1}{r^2} J\,(ru, r\psi) = \nu(\nabla^2 - \tfrac{1}{r^2})u\,, \\ &\omega_t + J\left(\tfrac{\omega}{r}, r\psi\right) = \nu(\nabla^2 - \tfrac{1}{r^2})\omega + J\left(\tfrac{u}{r}, ru\right)\,, \\ &\omega = -(\nabla^2 - \tfrac{1}{r^2})\psi\,, \end{aligned} \tag{2.2}$$

where $J(a, b) = (\partial_x a)(\partial_r b) - (\partial_r a)(\partial_x b)$.

In (2.2), $u(t; x, r)$, $\omega(t; x, r)$, and $\psi(t; x, r)$ represent the swirling components of velocity, vorticity, and stream function, respectively. The quantity $r\psi$ is also known as Stokes' stream function and the formal correspondence between the solutions of (2.1) and (2.2) is given by

$$\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) = \frac{\partial_r(r\psi)}{r}\boldsymbol{e}_x - \partial_x\psi\boldsymbol{e}_r + u\boldsymbol{e}_\theta, \tag{2.3}$$

where $\boldsymbol{e}_x$, $\boldsymbol{e}_r$, and $\boldsymbol{e}_\theta$ are the unit vectors in the x, r, and θ directions, respectively. The vorticity-stream formulation (2.2) has appeared in [5] with an alternative expression for the nonlinear terms. In [11], the authors have generalized the vorticity formulation to general symmetric flows with the nonlinear terms recast in Jacobians as in (2.2) and proposed a class of EHPS based on discretizing (2.2). In sections 4 and 5, we will review EHPS for (2.2) and give a rigorous error estimate in second order finite difference setting. The error bound certainly depends on the regularity of the solution to (2.2). Although (2.2) can be derived formally from (2.1), the equivalence between the two expressions in terms of regularity of solutions is not quite obvious. An essential prerequisite to our analysis is to characterize the proper meaning of "smoothness" of solutions to (2.2). This turns out to be a subtle issue.

Example 1. Take

$$u(x,r) = r^2 e^{-r}, \qquad \omega = \psi \equiv 0. \tag{2.4}$$

It is easy to verify that (2.4) is an *exact* stationary solution of the Euler equation ($\nu = 0$ in (2.2)). Note that $u = O(r^2)$ near the axis and $\partial_r^2 u(x, 0^+) \neq 0$. Similar functions can be found in literatures as initial data in numerical search for finite time singularities. Although $u \in C^\infty(R \times \overline{R^+})$, the following regularity lemma for general axisymmetric solenoidal vector fields shows that $\boldsymbol{u} = u\boldsymbol{e}_\theta$ is not even in $C^2(R^3, R^3)$.

LEMMA 1 (see [12]). *Denote the axisymmetric divergence free subspace of C^k vector fields by*

$$\mathcal{C}_s^k \overset{def}{=} \{\boldsymbol{u} \in C^k(R^3, R^3), \quad \partial_\theta u_x = \partial_\theta u_r = \partial_\theta u_\theta = 0, \quad \nabla \cdot \boldsymbol{u} = 0\}. \tag{2.5}$$

Then

(a) *for any $\boldsymbol{u} \in \mathcal{C}_s^k$, there exists a unique (u, ψ) such that*

$$\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) = \frac{\partial_r(r\psi)}{r}\boldsymbol{e}_x - \partial_x \psi \boldsymbol{e}_r + u\boldsymbol{e}_\theta, \qquad r > 0, \tag{2.6}$$

with

$$u(x,r) \in C^k(R \times \overline{R^+}), \quad \partial_r^{2\ell} u(x, 0^+) = 0 \text{ for } 0 \le 2\ell \le k, \tag{2.7}$$

and

$$\psi(x,r) \in C^{k+1}(R \times \overline{R^+}), \quad \partial_r^{2\ell} \psi(x, 0^+) = 0 \text{ for } 0 \le 2\ell \le k+1. \tag{2.8}$$

(b) *If (u, ψ) satisfies* (2.7), (2.8) *and $\boldsymbol{u}$ is given by* (2.6) *for $r > 0$, then $\boldsymbol{u} \in \mathcal{C}_s^k$ with a removable singularity at $r = 0$.*

Here in (2.5) and throughout this paper, the subscripts of u are used to denote components rather than partial derivatives. The proof of Lemma 1 is based on the observation that $\boldsymbol{e}_\theta$ changes direction across the axis of symmetry; therefore $u = u_\theta$ must admit an odd extension in order to compensate for this discontinuity. The details can be found in [12].

For simplicity of presentation, we recast Lemma 1 as follows.

LEMMA 1$'$.

$$\mathcal{C}_s^k = \{u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) \,|\, u \in C_s^k(R \times \overline{R^+}), \, \psi \in C_s^{k+1}(R \times \overline{R^+})\}, \tag{2.9}$$

where

$$C_s^k\left(R \times \overline{R^+}\right) \overset{def}{=} \left\{f(x,r) \in C^k\left(R \times \overline{R^+}\right), \quad \partial_r^{2j} f(x, 0^+) = 0, \, 0 \le 2j \le k\right\}. \tag{2.10}$$

From Lemma 1 and Example 1, it is clear that the proper meaning of the smooth solution to (2.2) should be supplemented by the pole conditions (2.7), (2.8). In the case of NSE ($\nu > 0$), our main concern in this paper, (2.2) is an elliptic-parabolic system on a semibounded region ($r > 0$). From standard PDE theory, we need to assign one and only one boundary condition for each of the variables ψ, u, and ω. An obvious choice is the zeroth order part of the pole conditions (2.7), (2.8):

$$\psi(x,0) = u(x,0) = \omega(x,0) = 0. \tag{2.11}$$

It is therefore a natural question to ask whether a smooth solution of (2.2), (2.11) in the class

$$\begin{aligned} &\psi(t;x,r) \in C^1\left(0,T;C^{k+1}(R\times\overline{R^+})\right),\\ &u(t;x,r) \in C^1\left(0,T;C^{k}(R\times\overline{R^+})\right),\\ &\omega(t;x,r) \in C^1\left(0,T;C^{k-1}(R\times\overline{R^+})\right)\end{aligned} \tag{2.12}$$

will give rise to a smooth solution of (2.2). In other words, is the pole condition (2.7), (2.8) automatically satisfied if only the zeroth order part (2.11) is imposed?

The answer to this question is affirmative.

THEOREM 1 (see [12]).

(a) *If* $(\boldsymbol{u}, p)$ *is an axisymmetric solution to* (2.1) *with* $\boldsymbol{u} \in C^1(0,T;\mathcal{C}_s^k)$, $p \in C^0(0,T;C^{k-1}(R^3))$, *and* $k \geq 3$, *then there is a solution* (ψ,u,ω) *to* (2.2) *in the class*

$$\begin{aligned} &\psi(t;x,r) \in C^1\left(0,T;C_s^{k+1}(R\times\overline{R^+})\right),\\ &u(t;x,r) \in C^1\left(0,T;C_s^{k}(R\times\overline{R^+})\right),\\ &\omega(t;x,r) \in C^1\left(0,T;C_s^{k-1}(R\times\overline{R^+})\right),\end{aligned} \tag{2.13}$$

and $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla\times(\psi\boldsymbol{e}_\theta)$.

(b) *If* (ψ,u,ω) *is a solution to* (2.2), (2.11) *in the class* (2.12) *with* $k \geq 3$, *then* (ψ,u,ω) *is in the class* (2.13), $\boldsymbol{u} \stackrel{def}{=} u\boldsymbol{e}_\theta + \nabla\times(\psi\boldsymbol{e}_\theta) \in C^1(0,T;\mathcal{C}_s^k)$, *and there is an axisymmetric scalar function* $p \in C^0(0,T;C^{k-1}(R^3))$ *such that* $(\boldsymbol{u},p)$ *is a solution to* (2.1).

The proof of Theorem 1 can be found in [12]. We remark here that Theorem 1 not only establishes the equivalence between (2.1) and (2.2) for classical solutions; the fact that smooth solutions to (2.2) automatically satisfy the pole condition (2.13) is also crucial to our local truncation error analysis. See the appendix for details.

3. Regularity assumption on solutions of NSE at infinity. The focus of this paper is the convergence rate of EHPS in the presence of the pole singularity. To separate difficulties and avoid complications introduced by physical boundaries, we only consider the whole space problems with solutions decaying rapidly at infinity.

To be more specific, we restrict our attention to the case where the supports of the initial data $u(x,0)$ and $\omega(x,0)$ are essentially compact. Since (2.2) is a transport diffusion equation for u and ω with initially finite speed of propagation, we expect u and ω to be essentially compactly supported, at least for short time. In the case of *linear* transport diffusion equations, the solution together with its derivatives will then decay faster than polynomials at infinity for $t > 0$. Some rigorous results concerning the spatial decay rate for the solutions of axisymmetric flows can be found in [4] and the references therein. In particular, it is shown in [4] that both u and ω decay algebraically at infinity as long as this is the case initially. Here we make a stronger yet plausible assumption along this direction. The precise form of our assumption is formulated in terms of weighted norms and is less stringent than the analogy we draw from linear transport diffusion equations; see Assumption 1 below.

To quantify our assumption, we first introduce a family of r-homogeneous composite norms and corresponding function spaces which turn out to be natural for our pointwise energy estimate.

DEFINITION 1.

$$\|a\|_{\ell,\alpha,\beta} = \sum_{\ell_1+\ell_2=\ell} \| (1+r)^{\alpha}(1+|x|)^{\beta}|\partial_x^{\ell_1}\partial_r^{\ell_2}\left(\frac{a}{r}\right)| \|_{L^\infty(R\times\overline{R^+})}, \tag{3.1}$$

$$|||a|||_{k,\alpha,\beta} = \sum_{0\le\ell\le k} \|a\|_{k-\ell,\alpha-\ell,\beta}. \tag{3.2}$$

Note that the norms (3.1), (3.2) are well defined for functions in $C_s^k(R\times\overline{R^+})$ that decay properly at infinity. We denote them by

$$C_s^{k,\alpha,\beta} = \left\{ a(x,r) \in C_s^k\left(R\times\overline{R^+}\right), |||a|||_{k,\alpha,\beta} < \infty \right\}. \tag{3.3}$$

In section 5, we will show that EHPS is second order accurate provided the solution satisfies

$$\begin{cases} (\psi,\omega) \in C^1\left(0,T; C_s^{4,\alpha+\frac{7}{2},\beta} \cap C_s^{4,2\alpha+2,2\beta}\right), \\ u \in C^1\left(0,T; C_s^{4,2\alpha+2,2\beta} \cap C_s^{1,2,0}\right), \end{cases} \qquad \alpha > \frac{1}{2},\ \beta > \frac{1}{4}. \tag{3.4}$$

In view of (3.4), we formulate our regularity assumption as follows.

Assumption 1.

$$(\psi,\omega) \in C^1\left(0,T; C_s^{4,\gamma,\delta}\right), \quad u \in C^1\left(0,T; C_s^{4,5,\delta}\right), \quad \gamma > 4,\ \delta > \frac{1}{2}. \tag{3.5}$$

Although we expect u,ω and their derivatives to decay faster than any polynomial at infinity, the same expectation is not realizable for ψ. As we will see, generically ψ only decays like $O((x^2+r^2)^{-1})$ at infinity. Nevertheless, we will show that Assumption 1 is still realizable if ω decays fast enough.

To analyze the decay rate of ψ, we start with the integral expression for ψ. From the vorticity-stream relation

$$\nabla\times\nabla\times\boldsymbol{\psi} = \boldsymbol{\omega}$$

and the identification

$$\psi(x,r) = \psi_z(x,y,z)|_{y=r,z=0}, \qquad \omega(x,r) = \omega_z(x,y,z)|_{y=r,z=0},$$

one can derive the following integral formula for ψ [17]:

$$\psi(x,r) = \int_0^\infty \int_{-\infty}^\infty \omega(x',r')K(x-x',r,r')dx'dr', \tag{3.6}$$

where

$$\begin{aligned} K(x-x',r,r') &= r'\frac{1}{4\pi}\int_0^{2\pi} \frac{\cos\theta}{\sqrt{(x-x')^2+(r-r'\cos\theta)^2+(r'\sin\theta)^2}}d\theta \\ &= r'^2\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{r\cos^2\theta}{\rho_+\rho_-(\rho_++\rho_-)}d\theta \end{aligned} \tag{3.7}$$

and

$$\rho_\pm^2 = (x-x')^2 + (r \pm r'\cos\theta)^2 + (r'\sin\theta)^2.$$

As a consequence, we have the following far field estimate for K.

Lemma 2.

$$|\partial_x^\ell \partial_r^m K(x-x',r,r')| \le C_{\ell,m}(x',r')\left(\sqrt{x^2+r^2}\right)^{-2-\ell-m} \quad as \quad x^2+r^2 \to \infty.$$

Proof. We will derive a far field estimate for the integrand in (3.7). We first consider a typical term

$$\lim_{x^2+r^2\to\infty} |\partial_x^\ell \partial_r^m \rho|$$

with

$$\rho^2 = (x-x_0)^2 + (r-r_0)^2 + c_0^2,$$

where x_0, r_0, and c_0 are some constants.

With the change of variables

$$\begin{aligned} r - r_0 &= \sigma\cos\lambda, \\ x - x_0 &= \sigma\sin\lambda, \end{aligned}$$

we can rewrite the x and r derivatives by

$$\begin{aligned} \partial_r\rho &= \partial_r\sqrt{\sigma^2+c_0^2} = (\partial_r\sigma)\partial_\sigma\sqrt{\sigma^2+c_0^2} + (\partial_r\lambda)\partial_\lambda\sqrt{\sigma^2+c_0^2} = \tfrac{\sigma}{\rho}\cos\lambda, \\ \partial_x\rho &= \partial_x\sqrt{\sigma^2+c_0^2} = (\partial_x\sigma)\partial_\sigma\sqrt{\sigma^2+c_0^2} + (\partial_x\lambda)\partial_\lambda\sqrt{\sigma^2+c_0^2} = \tfrac{\sigma}{\rho}\sin\lambda. \end{aligned}$$

Therefore by induction

$$\partial_x^\ell \partial_r^m \rho = P^{\ell,m}(\cos\lambda,\sin\lambda)Q^{\ell,m}(\sigma,\rho),$$

where $P^{\ell,m}(\cos\lambda,\sin\lambda)$ is a polynomial of degree $\ell+m$ in its arguments and $Q^{\ell,m}(\sigma,\rho)$ a rational function of σ and ρ of degree $1-\ell-m$. By degree of a rational function we mean the degree of the numerator subtracting the degree of the denominator.

Since $\sigma = O(\sqrt{x^2+r^2})$ and $\rho = O(\sqrt{x^2+r^2})$, we conclude that

$$|\partial_x^\ell \partial_r^m \rho| = O\left(\sqrt{x^2+r^2}^{\,1-\ell-m}\right).$$

We can now apply the argument above and Leibniz's rule to get

$$\partial_x^\ell \partial_r^m \frac{r}{\rho_+\rho_-(\rho_++\rho_-)} = \sum_j^{J_{\ell,m}} \tilde{P}_j^{\ell,m}(\cos\lambda_+,\sin\lambda_+,\cos\lambda_-,\sin\lambda_-)\tilde{Q}_j^{\ell,m}(\sigma_+,\rho_+,\sigma_-,\rho_-,r),$$

where $J_{\ell,m}$ is a finite integer, $\sigma_\pm$ and $\rho_\pm$ are defined by

$$\begin{aligned} r \pm r'\cos\theta &= \sigma_\pm\cos\lambda_\pm, \\ x - x_0 &= \sigma_\pm\sin\lambda_\pm, \end{aligned}$$

and $\tilde{P}_j^{\ell,m}$, $\tilde{Q}_j^{\ell,m}$ are polynomials and rational functions of degrees $\ell+m$, $-2-\ell-m$ in their arguments, respectively. The lemma follows by integrating θ over $(0,\frac{\pi}{2})$ in (3.7). ∎

We close this section by noting that ψ exhibits slow decay rate at infinity as a consequence of (3.6) and Lemma 2. More precisely, $\psi(x,r) \sim O((x^2+r^2)^{-1})$ in general. This may seem to raise the question whether Assumption 1 is realizable at all.

Indeed, using a similar calculation as in the proof of Lemma 2, one can derive the following.

PROPOSITION 1. *If $\gamma+\delta < k+2$ and $\omega \in C_s^{k,\gamma',\delta'}$ for sufficiently large γ' and δ', then $\psi \in C_s^{k,\gamma,\delta}$.*

As a consequence, we see that the range of γ and δ in (3.5) is not void provided ω decays fast enough at infinity. This justifies Assumption 1.

4. Energy and helicity preserving scheme. In this section, we outline the derivation of the discrete energy and helicity identities for EHPS. A key ingredient in the derivation is the reformulation of nonlinear terms into Jacobians. The details can be found in [11].

We introduce the standard notations:

$$D_x\phi(x,r) = \frac{\phi(x+\frac{\Delta x}{2},r)-\phi(x-\frac{\Delta x}{2},r)}{\Delta x}, \quad D_r\phi(x,r) = \frac{\phi(x,r+\frac{\Delta r}{2})-\phi(x,r-\frac{\Delta r}{2})}{\Delta r},$$

$$\tilde{D}_x\phi(x,r) = \frac{\phi(x+\Delta x,r)-\phi(x-\Delta x,r)}{2\Delta x}, \quad \tilde{D}_r\phi(x,r) = \frac{\phi(x,r+\Delta r)-\phi(x,r-\Delta r)}{2\Delta r},$$

and

$$\tilde{\nabla}_h = (\tilde{D}_x, \tilde{D}_r), \quad \tilde{\nabla}_h^{\perp} = (-\tilde{D}_r, \tilde{D}_x).$$

The finite difference approximation of ∇^2 and the Jacobians are given by

$$\nabla_h^2\psi = D_x\left(D_x\psi\right) + \frac{1}{r}\left(D_r(rD_r\psi)\right)$$

and

$$J_h(f,g) = \frac{1}{3}\left\{\tilde{\nabla}_h^{\perp} f\cdot\tilde{\nabla}_h g + \tilde{\nabla}_h^{\perp}\cdot(f\tilde{\nabla}_h g) + \tilde{\nabla}_h\cdot(g\tilde{\nabla}_h^{\perp} f)\right\}. \tag{4.1}$$

Altogether, the second order finite difference version of EHPS is

$$\begin{aligned}
&\partial_t u_h + \tfrac{1}{r^2}J_h\left(ru_h, r\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})u_h,\\
&\partial_t\omega_h + J_h\left(\tfrac{\omega_h}{r}, r\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})\omega_h + J_h\left(\tfrac{u_h}{r}, ru_h\right),\\
&\omega_h = (-\nabla_h^2 + \tfrac{1}{r^2})\psi_h.
\end{aligned} \tag{4.2}$$

To derive the discrete energy and helicity identity, we first introduce the discrete analogue of weighted inner products

$$\langle a,b\rangle_h = \sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(rab)_{i,j}\,\Delta x\Delta r, \tag{4.3}$$

$$\begin{aligned}
[a,b]_h = \Bigg(&\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\left(r(D_x a)(D_x b)\right)_{i-\frac{1}{2},j}\\
&+\sum_{i=-\infty}^{\infty}{\sum_{j=1}^{\infty}}'\left(r(D_r a)(D_r b)\right)_{i,j-\frac{1}{2}}\Bigg)\Delta x\Delta r + \langle \tfrac{a}{r}, \tfrac{b}{r}\rangle_h,
\end{aligned} \tag{4.4}$$

and the corresponding norms

$$\|a\|_{0,h}^2 = \langle a, a\rangle_h, \qquad \|a\|_{1,h}^2 = [a, a]_h, \tag{4.5}$$

where the grids have been shifted [13] to avoid placing the grid points on the axis of rotation:

$$x_i = i\Delta x, \quad i = 0, \pm 1, \pm 2, \dots, \quad r_j = \left(j - \frac{1}{2}\right)\Delta r, \quad j = 1, 2, \dots, \tag{4.6}$$

and

$$\sum_{j=1}^{\infty}{}' f_{j-\frac{1}{2}} = \frac{1}{2} f_{\frac{1}{2}} + \sum_{j=2}^{\infty} f_{j-\frac{1}{2}}. \tag{4.7}$$

The evaluation of the $\tilde{D}_r$ and ∇_h^2 terms in (4.2) at $j = 1$ involves the dependent variables u_h, ψ_h, ω_h and the stretching factor $h_3 = |\nabla\theta|^{-1} = r$ at the ghost points $j = 0$. In view of Lemma 1, we impose the following reflection boundary condition across the axis of rotation:

$$u_h(i, 0) = -u_h(i, 1), \quad \psi_h(i, 0) = -\psi_h(i, 1), \quad \omega_h(i, 0) = -\omega_h(i, 1). \tag{4.8}$$

Furthermore, we take even extension for the coordinate stretching factor $h_3 = |\nabla\theta|^{-1} = r$ which appears in the evaluation of the Jacobians at $j = 1$:

$$h_3(i, 0) = h_3(i, 1). \tag{4.9}$$

We will show in the remaining sections that the extensions (4.8) and (4.9) indeed give rise to a discrete version of energy and helicity identity and optimal local truncation error. As a consequence, second order accuracy of EHPS is justified for axisymmetric flows.

Remark 1. At first glance, the extension (4.9) may seem to contradict (4.6) on the ghost points $j = 0$. A less ambiguous restatement of (4.9) is to incorporate it into (4.2) as

(4.10)

$$\partial_t u_h + \tfrac{1}{r^2} J_h\left(|r|u_h, |r|\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})u_h,$$

$$\partial_t \omega_h + J_h\left(\tfrac{\omega_h}{|r|}, |r|\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})\omega_h + J_h\left(\tfrac{u_h}{|r|}, |r|u_h\right) \qquad \text{on } (x_i, r_j), \quad j \geq 1,$$

$$\omega_h = (-\nabla_h^2 + \tfrac{1}{r^2})\psi_h.$$

The following identities are essential to the discrete energy and helicity identity and the error estimate.

LEMMA 3. *Suppose* (a, b, c) *satisfies the reflection boundary condition*

$$a(i, 0) = -a(i, 1), \quad b(i, 0) = -b(i, 1), \quad c(i, 0) = -c(i, 1)$$

and define

$$T_h(a, b, c) := \frac{1}{3} \sum_{i=-\infty}^{\infty} \sum_{j=1}^{\infty} \left(c\tilde{\nabla}_h^{\perp} a \cdot \tilde{\nabla}_h b + a\tilde{\nabla}_h^{\perp} b \cdot \tilde{\nabla}_h c + b\tilde{\nabla}_h^{\perp} c \cdot \tilde{\nabla}_h a\right)_{i,j} \Delta x \Delta r. \tag{4.11}$$

Then

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty} c_{i,j} J_h(a,b)_{i,j}\Delta x\Delta r = T_h(a,b,c), \tag{4.12}$$

and

$$\left\langle a, \left(-\nabla_h^2 + \frac{1}{r^2}\right) b \right\rangle_h = [a,b]_h. \tag{4.13}$$

Proof. We first derive (4.12). In view of (4.1) and (4.11), it suffices to show that

$$\sum_j \sum_i c\tilde{\nabla}_h^{\perp}\cdot(a\tilde{\nabla}_h b) = -\sum_{i,j} a\tilde{\nabla}_h^{\perp} c\cdot\tilde{\nabla}_h b, \tag{4.14}$$

$$\sum_i \sum_j c\tilde{\nabla}_h\cdot(b\tilde{\nabla}_h^{\perp} a) = -\sum_{i,j} b\tilde{\nabla}_h c\cdot\tilde{\nabla}_h^{\perp} a \tag{4.15}$$

or, since there is no boundary terms in the x direction, simply

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(f\tilde{D}_r g)_{i,j} = -\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(g\tilde{D}_r f)_{i,j} \tag{4.16}$$

with $f = c$ and $g = b\tilde{D}_x a - a\tilde{D}_x b$.

Using the summation-by-parts identity (see, for example, [15] or [11]), it is straightforward to verify that

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(f\tilde{D}_r g)_{i,j} = -\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}(g\tilde{D}_r f)_{i,j} - \sum_{i=-\infty}^{\infty}(f_{i,0}g_{i,1} + g_{i,0}f_{i,1}).$$

In the derivation of the discrete energy and helicity identities (see (4.18)–(4.20) below), a typical triplet (a,b,c) is given by, say, $a = r\psi_h$, $b = ru_h$, and $c = \frac{u_h}{r}$. From the reflection boundary condition (4.8) and (4.9), we see that

$$f_{i,0} = -f_{i,1}, \quad g_{i,0} = g_{i,1}.$$

This gives (4.16), and therefore (4.14), (4.15), and (4.12).

Next we derive (4.13). From the identity

$$\sum_{j=1}^{\infty} f_j(g_{j+\frac{1}{2}} - g_{j-\frac{1}{2}}) = -{\sum_{j=1}^{\infty}}'(f_j - f_{j-1})g_{j-\frac{1}{2}} - \frac{1}{2}(f_1 + f_0)g_{\frac{1}{2}}$$

and $r_{\frac{1}{2}} = 0$, it is easy to show that

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty} a_{i,j}D_r(rD_r b)_{i,j} = -\sum_{i=-\infty}^{\infty}{\sum_{j=1}^{\infty}}'(D_r a)_{i,j-\frac{1}{2}} r_{j-\frac{1}{2}}(D_r b)_{i,j-\frac{1}{2}}\,.$$

Therefore (4.13) follows. ∎

From (4.11), we can easily derive the permutation identities

$$T_h(a,b,c) = T_h(b,c,a) = T_h(c,a,b), \qquad T_h(a,b,c) = -T_h(b,a,c)\,. \tag{4.17}$$

Moreover, from (4.12), (4.13), it follows that

$$\langle \upsilon, \partial_t u_h\rangle_h + T_h(ru_h, r\psi_h, \tfrac{\upsilon}{r}) = \nu\langle \upsilon, (\nabla_h^2 - \tfrac{1}{r^2})u_h\rangle_h,$$

$$[\varphi, \partial_t \psi_h]_h + T_h(\tfrac{\omega_h}{r}, r\psi_h, r\varphi) = \nu\langle \varphi, (\nabla_h^2 - \tfrac{1}{r^2})\omega_h\rangle_h + T_h(\tfrac{u_h}{r}, ru_h, r\varphi), \tag{4.18}$$

$$\langle \xi, \omega_h\rangle_h = [\xi, \psi_h]_h$$

for all υ, φ, and ξ satisfying

$$\upsilon(i,0) = -\upsilon(i,1), \quad \varphi(i,0) = -\varphi(i,1), \quad \xi(i,0) = -\xi(i,1).$$

As a direct consequence of the permutation identity (4.17), we take $(\upsilon, \varphi) = (u_h, \psi_h)$ in (4.18) and recover the discrete energy identity

$$\frac{d}{dt}\frac{1}{2}(\langle u_h, u_h\rangle_h + [\psi_h, \psi_h]_h) + \nu([u_h, u_h]_h + \langle \omega_h, \omega_h\rangle_h) = 0. \tag{4.19}$$

Similarly, the discrete helicity identity

$$\frac{d}{dt}\langle u_h, \omega_h\rangle_h + \nu\left([u_h, \omega_h]_h - \left\langle \omega_h, \left(\nabla_h^2 - \frac{1}{r^2}\right)u_h\right\rangle_h\right) = 0 \tag{4.20}$$

follows by taking $(\upsilon, \varphi) = (\omega_h, u_h)$ in (4.18).

Remark 2. In the presence of physical boundaries, the no-slip boundary condition gives

$$\boldsymbol{u}\cdot\boldsymbol{n} = \partial_\tau(r\psi) = 0, \quad \boldsymbol{u}\cdot\boldsymbol{\tau} = \partial_n(r\psi) = 0, \quad \boldsymbol{u}\cdot\boldsymbol{e}_\theta = u = 0, \tag{4.21}$$

where $\boldsymbol{\tau} = \boldsymbol{n}\times\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\theta$ is the unit vector in θ direction. When the cross section Ω is simply connected, (4.21) reads as follows:

$$u = 0, \quad \psi = 0, \quad \partial_n(r\psi) = 0 \quad \text{on} \quad \partial\Omega\,. \tag{4.22}$$

It can be shown that the energy and helicity identities (4.19), (4.20) remain valid in the presence of physical boundary conditions [11]. The numerical realization of the no-slip condition (4.22) introduced in [11] is second order accurate and seems to be new even for usual 2D flows. The convergence proof for this new boundary condition will be reported elsewhere.

5. Energy estimate and the main theorem. In this section, we proceed with the main theorem of the error estimate. We denote by (ψ_h, u_h, ω_h) the numerical solution satisfying

$$\partial_t u_h + \tfrac{1}{r^2}J_h(ru_h, r\psi_h) = \nu(\nabla_h^2 - \tfrac{1}{r^2})u_h,$$

$$\partial_t\omega_h + J_h\left(\tfrac{\omega_h}{r}, r\psi_h\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})\omega_h + J_h\left(\tfrac{u_h}{r}, ru_h\right), \tag{5.1}$$

$$\omega_h = (-\nabla_h^2 + \tfrac{1}{r^2})\psi_h,$$

and (ψ, u, ω) the exact solution to (2.2),

$$\partial_t u + \tfrac{1}{r^2}J_h(ru, r\psi) = \nu(\nabla_h^2 - \tfrac{1}{r^2})u + \mathcal{E}_1,$$

$$\partial_t\omega + J_h\left(\tfrac{\omega}{r}, r\psi\right) = \nu(\nabla_h^2 - \tfrac{1}{r^2})\omega + J_h\left(\tfrac{u}{r}, ru\right) + \mathcal{E}_2, \tag{5.2}$$

$$\omega = (-\nabla_h^2 + \tfrac{1}{r^2})\psi + \mathcal{E}_3,$$

where the local truncation errors $\mathcal{E}_j$ can be derived by subtracting (2.2) from (5.2):

$$\begin{aligned} &\mathcal{E}_1 = \tfrac{1}{r^2}(J_h - J)(ru, r\psi) - \nu(\nabla_h^2 - \nabla^2)u, \\ &\mathcal{E}_2 = (J_h - J)\left(\tfrac{\omega}{r}, r\psi\right) - \nu(\nabla_h^2 - \nabla^2)\omega - (J_h - J)\left(\tfrac{u}{r}, ru\right), \\ &\mathcal{E}_3 = (\nabla_h^2 - \nabla^2)\psi. \end{aligned} \tag{5.3}$$

From (5.1) and (5.2), we see that

$$\partial_t(u - u_h) + \frac{1}{r^2}\left(J_h(ru, r\psi) - J_h(ru_h, r\psi_h)\right) = \nu\left(\nabla_h^2 - \frac{1}{r^2}\right)(u - u_h) + \mathcal{E}_1, \tag{5.4}$$

$$\begin{aligned} &\partial_t(\omega - \omega_h) + \left(J_h\left(\tfrac{\omega}{r}, r\psi\right) - J_h\left(\tfrac{\omega_h}{r}, r\psi_h\right)\right) \\ = \;& \nu(\nabla_h^2 - \tfrac{1}{r^2})(\omega - \omega_h) + \left(J_h\left(\tfrac{u}{r}, ru\right) - J_h\left(\tfrac{u_h}{r}, ru_h\right)\right) + \mathcal{E}_2, \end{aligned} \tag{5.5}$$

$$(\omega - \omega_h) = \left(-\nabla_h^2 + \frac{1}{r^2}\right)(\psi - \psi_h) + \mathcal{E}_3. \tag{5.6}$$

Lemmas 4 and 5 below are key to our error estimate. The permutation identities (4.17) associated with EHPS result in exact cancellation among the nonlinear terms and lead to an exact identity (5.7). The estimates for the trilinear form in (5.13), (5.14) then furnish necessary inequalities for our a priori error estimate. The proof for Lemma 5 and the local truncation error analysis, Lemma 6, is given in the appendix.

LEMMA 4.

$$\begin{aligned} &\tfrac{1}{2}\partial_t(\|u - u_h\|_{0,h}^2 + \|\psi - \psi_h\|_{1,h}^2) + \nu(\|u - u_h\|_{1,h}^2 + \|\omega - \omega_h\|_{0,h}^2) \\ &= \langle u - u_h, \mathcal{E}_1\rangle_h + \langle \psi - \psi_h, \mathcal{E}_2 - \partial_t\mathcal{E}_3\rangle_h + \nu\langle \omega - \omega_h, \mathcal{E}_3\rangle_h - T_h\left(\tfrac{u-u_h}{r}, r(u - u_h), r\psi\right) \\ &\quad -T_h\left(r(\psi - \psi_h), \tfrac{(\omega-\omega_h)}{r}, r\psi\right) + T_h\left(r(\psi - \psi_h), \tfrac{u}{r}, r(u - u_h)\right). \end{aligned} \tag{5.7}$$

Proof. We take the weighted inner product of $u - u_h$ with (5.4) to get

$$\begin{aligned} &\tfrac{1}{2}\partial_t\|u - u_h\|_{0,h}^2 + \langle u - u_h, \tfrac{1}{r^2}\left(J_h(ru, r\psi) - J_h(ru_h, r\psi_h)\right)\rangle_h \\ &= \nu\langle u - u_h, (\nabla_h^2 - \tfrac{1}{r^2})(u - u_h)\rangle_h + \langle u - u_h, \mathcal{E}_1\rangle_h\,. \end{aligned} \tag{5.8}$$

The second term on the left-hand side of (5.8) can be rewritten as

$$\begin{aligned} &\langle u - u_h, \tfrac{1}{r^2}\left(J_h(ru, r\psi) - J_h(ru_h, r\psi_h)\right)\rangle_h \\ &= T_h\left(\tfrac{u-u_h}{r}, ru, r\psi\right) - T_h\left(\tfrac{u-u_h}{r}, ru_h, r\psi_h\right) \\ &= -T_h\left(\tfrac{u-u_h}{r}, r(u - u_h), r(\psi - \psi_h)\right) + T_h\left(\tfrac{u-u_h}{r}, r(u - u_h), r\psi\right) \\ &\qquad + T_h\left(\tfrac{u-u_h}{r}, ru, r(\psi - \psi_h)\right). \end{aligned} \tag{5.9}$$

In addition, from (4.13) we have

$$\nu\left\langle u - u_h, \left(\nabla_h^2 - \frac{1}{r^2}\right)(u - u_h)\right\rangle_h = -\nu[u - u_h, u - u_h]_h = -\nu\|u - u_h\|_{1,h}^2\,.$$

Thus

$$\frac{1}{2}\partial_t\|u-u_h\|_{0,h}^2 - T_h\left(\tfrac{u-u_h}{r}, r(u-u_h), r(\psi-\psi_h)\right) + \nu\|u-u_h\|_{1,h}^2 \tag{5.10}$$
$$= \langle u-u_h, \mathcal{E}_1\rangle_h - T_h\left(\tfrac{u-u_h}{r}, r(u-u_h), r\psi\right) - T_h\left(\tfrac{u-u_h}{r}, ru, r(\psi-\psi_h)\right).$$

Similarly, we take the weighted inner product of $\psi-\psi_h$ with (5.5) and proceed as (5.9)–(5.10) to get

$$\begin{aligned} &\tfrac{1}{2}\partial_t\|\psi-\psi_h\|_{1,h}^2 + T_h\left(r(\psi-\psi_h), \tfrac{(\omega-\omega_h)}{r}, r\psi\right) \\ &= -T_h\left(r(\psi-\psi_h), \tfrac{(u-u_h)}{r}, r(u-u_h)\right) + T_h\left(r(\psi-\psi_h), \tfrac{u}{r}, r(u-u_h)\right) \\ &\quad + T_h\left(r(\psi-\psi_h), \tfrac{(u-u_h)}{r}, ru\right) + \langle\psi-\psi_h, \mathcal{E}_2-\partial_t\mathcal{E}_3\rangle_h \\ &\quad + \nu\langle(\psi-\psi_h), (\nabla_h^2 - \tfrac{1}{r^2})(\omega-\omega_h)\rangle_h . \end{aligned} \tag{5.11}$$

Next, we apply (4.13) twice to get

(5.12)
$$\begin{aligned} \nu\left\langle(\psi-\psi_h), \left(\nabla_h^2 - \frac{1}{r^2}\right)(\omega-\omega_h)\right\rangle_h &= \nu\left\langle\left(\nabla_h^2 - \frac{1}{r^2}\right)(\psi-\psi_h), \omega-\omega_h\right\rangle_h \\ &= -\nu\|\omega-\omega_h\|_{0,h}^2 + \nu\langle\omega-\omega_h, \mathcal{E}_3\rangle_h, \end{aligned}$$

and (5.7) follows. This completes the proof of this lemma. □

We now proceed with the estimate for the trilinear form T_h.

LEMMA 5. *For a, b, and $c \in C_s^2(R\times\overline{R^+})$, we have*

$$|T_h\left(ra, rb, \frac{c}{r}\right)| \le C\|a\|_{1,h}\|b\|_{1,h}|\!|\!|c|\!|\!|_{1,2,0} \tag{5.13}$$

and

$$|T_h\left(\frac{a}{r}, rb, rc\right)| \le C\|a\|_{0,h}\|b\|_{1,h}|\!|\!|c|\!|\!|_{2,2,0}. \tag{5.14}$$

Proof. See section A.1. □

From Lemmas 4 and 5, we can therefore derive

(5.15)
$$\begin{aligned} &\tfrac{1}{2}\partial_t(\|u-u_h\|_{0,h}^2 + \|\psi-\psi_h\|_{1,h}^2) + \nu(\|u-u_h\|_{1,h}^2 + \|\omega-\omega_h\|_{0,h}^2) \\ \le\ & |\langle u-u_h, \mathcal{E}_1\rangle_h| + |\langle\psi-\psi_h, \mathcal{E}_2-\partial_t\mathcal{E}_3\rangle_h| + \nu|\langle\omega-\omega_h, \mathcal{E}_3\rangle_h| \\ &+ C\|u-u_h\|_{0,h}\|u-u_h\|_{1,h}|\!|\!|\psi|\!|\!|_{2,2,0} + C\|\omega-\omega_h\|_{0,h}\|\psi-\psi_h\|_{1,h}|\!|\!|\psi|\!|\!|_{2,2,0} \\ &+ C\|\psi-\psi_h\|_{1,h}\|u-u_h\|_{1,h}|\!|\!|u|\!|\!|_{1,2,0}. \end{aligned}$$

Since

$$\|\frac{a}{r}\|_{0,h} \le \|a\|_{1,h},$$

we can further estimate the first few terms on the right-hand side of (5.15) by

$$|\langle u-u_h, \mathcal{E}_1\rangle_h| = |\left\langle\frac{u-u_h}{r}, r\mathcal{E}_1\right\rangle_h| \le \frac{\nu}{4}\|u-u_h\|_{1,h}^2 + \frac{1}{\nu}\|r\mathcal{E}_1\|_{0,h}^2,$$
$$|\langle\psi-\psi_h, \mathcal{E}_2-\partial_t\mathcal{E}_3\rangle_h| \le \|\psi-\psi_h\|_{1,h}^2 + \|r(\mathcal{E}_2-\partial_t\mathcal{E}_3)\|_{0,h}^2,$$

and

$$|\langle \omega - \omega_h, \mathcal{E}_3 \rangle_h| \le \frac{1}{2}\|\omega - \omega_h\|_{0,h}^2 + \frac{1}{2}\|\mathcal{E}_3\|_{0,h}^2.$$

Applying Hölder's inequality to the remaining terms of (5.15), we have derived the following proposition.

PROPOSITION 2.

$$\begin{aligned} &\tfrac{1}{2}\partial_t(\|u-u_h\|_{0,h}^2 + \|\psi-\psi_h\|_{1,h}^2) + \tfrac{\nu}{4}(\|u-u_h\|_{1,h}^2 + \|\omega-\omega_h\|_{0,h}^2) \\ &\le \|\psi-\psi_h\|_{1,h}^2 + \tfrac{C}{\nu}\|r\mathcal{E}_1\|_{0,h}^2 + \|r\mathcal{E}_2\|_{0,h}^2 + \|r\partial_t\mathcal{E}_3\|_{0,h}^2 \\ &\quad + \nu\|\mathcal{E}_3\|_{0,h}^2 + \tfrac{C}{\nu}\|u-u_h\|_{0,h}^2 \|\!|\psi|\!\|_{2,2,0}^2 \\ &\quad + \tfrac{C}{\nu}\|\psi-\psi_h\|_{1,h}^2 \|\!|\psi|\!\|_{2,2,0}^2 + \tfrac{C}{\nu}\|\psi-\psi_h\|_{1,h}^2 \|\!|u|\!\|_{1,2,0}^2. \end{aligned} \tag{5.16}$$

With Proposition 2, it remains to estimate $\|r\mathcal{E}_1\|_{0,h}$, $\|r\mathcal{E}_2\|_{0,h}$, $\|r\partial_t\mathcal{E}_3\|_{0,h}$, and $\|\mathcal{E}_3\|_{0,h}$. We summarize the results in the following lemma.

LEMMA 6. *Let $(\psi, u, \omega) \in C^1(0,T;C_s^4)$ be a solution of the axisymmetric NSE (2.2) and $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ be defined by (5.2). Then we have the following pointwise local truncation error estimate for $\alpha, \beta \in R$:*

$$r|\mathcal{E}_1| \le C\frac{\Delta x^2 + \Delta r^2}{(1+r)^{2\alpha}(1+|x|)^{2\beta}}\left(\|\!|\psi|\!\|_{4,\alpha+\frac{7}{2},\beta}\|\!|u|\!\|_{4,\alpha+\frac{7}{2},\beta} + \|\!|u|\!\|_{4,2\alpha+2,2\beta}\right), \tag{5.17}$$

$$r|\mathcal{E}_2| \le C\frac{\Delta x^2 + \Delta r^2}{(1+r)^{2\alpha}(1+|x|)^{2\beta}}\left(\|\!|\psi|\!\|_{4,\alpha+\frac{7}{2},\beta}\|\!|\omega|\!\|_{4,\alpha+\frac{7}{2},\beta} + \|\!|u|\!\|_{4,\alpha+\frac{7}{2},\beta}^2 + \|\!|\omega|\!\|_{4,2\alpha+2,2\beta}\right), \tag{5.18}$$

$$r|\partial_t\mathcal{E}_3| \le C\frac{\Delta x^2 + \Delta r^2}{(1+r)^{2\alpha}(1+|x|)^{2\beta}}\|\!|\partial_t\psi|\!\|_{4,2\alpha+2,2\beta}, \tag{5.19}$$

and

$$|\mathcal{E}_3| \le C\frac{\Delta x^2 + \Delta r^2}{r(1+r)^{2\alpha}(1+|x|)^{2\beta}}\|\!|\psi|\!\|_{4,2\alpha+2,2\beta}. \tag{5.20}$$

Proof. See section A.2. ☐

From Lemma 4 to 6, our main result follows.

THEOREM 2. *Let (ψ, u, ω) be a solution of the axisymmetric NSE (2.2) satisfying*

$$(\psi,\omega) \in C^1\left(0,T;C_s^{4,\gamma,\delta}\right), \quad u \in C^1\left(0,T;C_s^{4,5,\delta}\right), \qquad \gamma > 4,\ \delta > \frac{1}{2}. \tag{5.21}$$

Then

$$\begin{aligned} &\sup_{[0,T]}\left(\|u-u_h\|_{0,h}^2 + \|\psi-\psi_h\|_{1,h}^2\right) \\ &\qquad + \int_0^T (\|u-u_h\|_{1,h}^2 + \|\omega-\omega_h\|_{0,h}^2)dt \le C(\Delta x^4 + \Delta r^4)|\log \Delta r|, \end{aligned} \tag{5.22}$$

where $C = C(\psi, u, \nu, T)$.

Proof. From Lemma 6, we have

$$\begin{aligned}&\|r\mathcal{E}_1\|_{0,h}^2+\|r\mathcal{E}_2\|_{0,h}^2+\|r\partial_t\mathcal{E}_3\|_{0,h}^2\\ &\le C(\Delta x^4+\Delta r^4)\left(\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{r_j\Delta r\Delta x}{(1+r_j)^{4\alpha}(1+|x_i|)^{4\beta}}\right)\\ &\qquad\times\left(\|\!|(\psi,u,\omega)|\!\|_{4,\alpha+\frac{7}{2},\beta}^4+\|\!|(u,\omega,\partial_t\psi)|\!\|_{4,2\alpha+2,2\beta}^2\right).\end{aligned}$$

Similarly,

$$\|\mathcal{E}_3\|_{0,h}^2\le C(\Delta x^4+\Delta r^4)\left(\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{\Delta r\Delta x}{r_j(1+r_j)^{4\alpha}(1+|x_i|)^{4\beta}}\right)\|\!|\psi|\!\|_{4,2\alpha+2,2\beta}^2.$$

Since

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{r_j\Delta r\Delta x}{(1+r_j)^{4\alpha}(1+|x_i|)^{4\beta}}\le C\qquad\text{for }\alpha>\frac{1}{2},\ \beta>\frac{1}{4}$$

and

$$\sum_{i=-\infty}^{\infty}\sum_{j=1}^{\infty}\frac{\Delta r\Delta x}{r_j(1+r_j)^{4\alpha}(1+|x_i|)^{4\beta}}\le C|\log\Delta r|\qquad\text{for }\alpha>0,\ \beta>\frac{1}{4},$$

it follows that

(5.23)
$$\|r\mathcal{E}_1\|_{0,h}^2+\|r\mathcal{E}_2\|_{0,h}^2+\|r\partial_t\mathcal{E}_3\|_{0,h}^2\le C(\Delta x^4+\Delta r^4)\left(\|\!|(\psi,u,\omega)|\!\|_{4,\gamma,\delta}^4+\|\!|(u,\omega,\partial_t\psi)|\!\|_{4,\gamma,\delta}^2\right)$$

and

$$\|\mathcal{E}_3\|_{0,h}^2\le C(\Delta x^4+\Delta r^4)|\log\Delta r|\|\!|\psi|\!\|_{4,\gamma,\delta}^2 \tag{5.24}$$

provided $\gamma>4$, $\delta>\frac{1}{2}$.

Under assumption (5.21), we have, in particular, $\psi\in C_s^{2,2,0}$, $u\in C_s^{1,2,0}$. It follows from Proposition 2 and (5.23), (5.24) that

$$\begin{aligned}&\tfrac{1}{2}\partial_t(\|u-u_h\|_{0,h}^2+\|\psi-\psi_h\|_{1,h}^2)+\tfrac{\nu}{4}(\|u-u_h\|_{1,h}^2+\|\omega-\omega_h\|_{0,h}^2)\\ \le\ &C\|u-u_h\|_{0,h}^2+C\|\psi-\psi_h\|_{1,h}^2+C(\Delta x^4+\Delta r^4)|\log\Delta r|.\end{aligned}$$

The error estimate (5.22) then follows from Gronwall's inequality. ∎

6. Conclusion. The importance and subtlety of the pole singularity has been a major difficulty in theoretical analysis and algorithm design for axisymmetric flows. The numerical analysis near the pole singularity is much more complicated than that of standard smooth flows. The principal ingredients of our error analysis are as follows:

(a) The fact that smooth solutions to (2.2) automatically satisfy the pole condition and thus belong to the class (2.13). This symmetry property plays an essential role in the local truncation error analysis.

(b) Proper formulation and discretization of the nonlinear terms. Here the Jacobian formulation along with the distinctive discretization (4.1) result in exact cancellation among the nonlinear terms in the energy estimate and therefore lead to conservation identities in discrete setting.

These ingredients may also serve as a guideline of algorithm design for axisymmetric flows.

In addition, the slow decay of the stream function at infinity poses extra technical difficulties in analyzing the whole space problem. This difficulty is carefully resolved by choosing a properly weighted r-homogeneous norm (3.2). On the one hand, (3.2) takes into account the local behavior of the swirling components near the pole singularity. On the other hand, it incorporates free parameters so that the slow decay of the stream function can be properly compensated by tuning the parameters through careful analysis.

Appendix A. Proof of technical lemmas.

A.1. Estimate for the trilinear form T_h—proof of Lemma 5. We start with the following basic identities.

PROPOSITION 3. *Define*

$$(\widetilde{A}_x f)_{i,j} = \frac{1}{2}(f_{i+1,j} + f_{i-1,j}), \qquad (\widetilde{A}_r f)_{i,j} = \frac{1}{2}(f_{i,j+1} + f_{i,j-1}).$$

Then the following estimates hold for $j \geq 1$:

$$|\widetilde{D}_r(ra)| \leq C|\widetilde{A}_r a| + Cr|\widetilde{D}_r a|, \tag{A.1}$$

$$|\widetilde{D}_r(\frac{a}{r})| \leq C\frac{|\widetilde{A}_r a|}{r^2} + C\frac{|\widetilde{D}_r a|}{r}, \tag{A.2}$$

$$|\widetilde{A}_r(ra)| \leq Cr\widetilde{A}_r|a|, \tag{A.3}$$

$$|\Delta r \widetilde{D}_r a| \leq \widetilde{A}_r|a|, \quad |\Delta x \widetilde{D}_x a| \leq \widetilde{A}_x|a|. \tag{A.4}$$

Remark 3. As in Remark 1, the stretching factor r in the arguments of the left-hand side of (A.1)–(A.3) satisfy the even extension (4.9). A more precise statement for, say, (A.1) is given by

$$|\widetilde{D}_r(|r|a)|_{i,j} \leq C|\widetilde{A}_r a|_{i,j} + Cr_j|\widetilde{D}_r a|_{i,j}, \qquad j \geq 1.$$

For simplicity of presentation, we will adopt the expression as in (A.1)–(A.3) through the rest of the paper.

Proof of Proposition 3. It is easy to verify that

$$\widetilde{D}_r(fg) = (\widetilde{A}_r f)(\widetilde{D}_r g) + (\widetilde{A}_r g)(\widetilde{D}_r f), \quad \widetilde{D}_x(fg) = (\widetilde{A}_x f)(\widetilde{D}_x g) + (\widetilde{A}_x g)(\widetilde{D}_x f).$$

A straightforward calculation shows that

$$(\tilde{A}_r|r|)_j \leq Cr_j, \quad |\,\tilde{D}_r|r|\,|_j \leq C$$

and

$$\tilde{A}_r\left(\frac{1}{|r|}\right)_j \leq C\frac{1}{r_j}, \quad |\tilde{D}_r\left(\frac{1}{|r|}\right)|_j \leq C\frac{1}{r_j^2}$$

for $j \geq 1$. The estimates (A.1)–(A.3) then follow. The proof for (A.4) is also straightforward. ◻

Proof of Lemma 5. We begin with the proof of (5.13). We expand the left-hand side as

$$T_h\left(ra, rb, \tfrac{c}{r}\right) = \tfrac{1}{3}\Big(\langle \tfrac{c}{r^2}, \tilde{\nabla}_h^{\perp}(ra)\cdot\tilde{\nabla}_h(rb)\rangle_h + \langle a, \tilde{\nabla}_h^{\perp}(rb)\cdot\tilde{\nabla}_h(\tfrac{c}{r})\rangle_h + \langle b, \tilde{\nabla}_h^{\perp}(\tfrac{c}{r})\cdot\tilde{\nabla}_h(ra)\rangle_h\Big) = \tfrac{1}{3}(I_1 + I_2 + I_3)$$

and estimate the I_j's term by term. First, we have

$$|I_1| = |\left\langle \frac{c}{r^2}, \tilde{\nabla}_h^{\perp}(ra)\cdot\tilde{\nabla}_h(rb)\right\rangle_h| = |\left\langle c, -\frac{1}{r}\tilde{D}_r(ra)\tilde{D}_x(b) + \tilde{D}_x(a)\frac{1}{r}\tilde{D}_r(rb)\right\rangle_h|;$$

therefore the estimate

$$|I_1| \leq C\left\langle |c|, \left(|\frac{\tilde{A}_r(a)}{r}| + |\tilde{D}_r(a)|\right)|\tilde{D}_x(b)| + \left(|\frac{\tilde{A}_r(b)}{r}| + |\tilde{D}_r(b)|\right)|\tilde{D}_x(a)|\right\rangle_h \leq C\|a\|_{1,h}\|b\|_{1,h}\|c\|_{0,1,0}$$

follows from (A.1), Hölder's inequality, and the inequality $|c| = |r\frac{c}{r}| \leq \|c\|_{0,1,0}$.

Second, we have

$$\begin{aligned}|I_2| &\leq C\langle |a|, |\tfrac{\tilde{A}_r(b)}{r}| + |\tilde{D}_r(b)||\tilde{D}_x(c)|\rangle_h + C\langle |a|, |\tilde{D}_r(c)||\tilde{D}_x(b)|\rangle_h \\ &\quad + C\langle \tfrac{|a|}{r}, |A_r(c)||\tilde{D}_x(b)|\rangle_h \\ &= C\langle \tfrac{|a|}{r}, |\tfrac{\tilde{A}_r(b)}{r}| + |\tilde{D}_r(b)||r\tilde{D}_x(c)|\rangle_h + C\langle \tfrac{|a|}{r}, |r\tilde{D}_r(c)||\tilde{D}_x(b)|\rangle_h \\ &\quad + C\langle \tfrac{|a|}{r}, |A_r(c)||\tilde{D}_x(b)|\rangle_h \\ &\leq C\|a\|_{1,h}\|b\|_{1,h}(\|c\|_{0,1,0} + \|c\|_{1,2,0}) \leq C\|a\|_{1,h}\|b\|_{1,h}\|c\|_{1,2,0}.\end{aligned}$$

The estimate for I_3 is similar and (5.13) follows.

Next we proceed with (5.14). Since

$$|T_h\left(\frac{a}{r}, rb, rc\right)| = |\left\langle a, \frac{1}{r^2}J_h(rb, rc)\right\rangle_h| \leq \|a\|_{0,h}\|\frac{1}{r^2}J_h(rb, rc)\|_{0,h},$$

it suffices to give a pointwise estimate for the integrand $J_h(rb, rc)$ as follows:

(A.5)

$$\begin{aligned}-3J_h(rb, rc) &= \widetilde{D}_r(rb)\widetilde{D}_x(rc) - \widetilde{D}_x(rb)\widetilde{D}_r(rc) + \widetilde{D}_r\big(rb\widetilde{D}_x(rc)\big) - \widetilde{D}_x\big(rb\widetilde{D}_r(rc)\big) \\ &\quad + \widetilde{D}_x\big(rc\widetilde{D}_r(rb)\big) - \widetilde{D}_r\big(rc\widetilde{D}_x(rb)\big) \\ &= \widetilde{D}_r(rb)(I + \widetilde{A}_r)\widetilde{D}_x(rc) - \widetilde{D}_x(rb)(I + \widetilde{A}_x)\widetilde{D}_r(rc) \\ &\quad + (\widetilde{A}_r - \widetilde{A}_x)(rb)\widetilde{D}_r\widetilde{D}_x(rc) + (\tilde{A}_x - \tilde{A}_r)(rc)\tilde{D}_x\tilde{D}_r(rb) \\ &\quad + \widetilde{D}_x(rc)\widetilde{A}_x\widetilde{D}_r(rb) - \widetilde{D}_r(rc)\widetilde{A}_r\widetilde{D}_x(rb) \\ &= \widetilde{D}_r(rb)(I + \widetilde{A}_r)\widetilde{D}_x(rc) - \widetilde{D}_x(rb)(I + \widetilde{A}_x)\widetilde{D}_r(rc) \\ &\quad + (\widetilde{A}_r - \widetilde{A}_x)(rb)\widetilde{D}_r\widetilde{D}_x(rc) \\ &\quad + \tfrac{1}{2}\Delta x^2\widetilde{D}_r\widetilde{D}_x(rb)D_x^2(rc) - \tfrac{1}{2}\Delta r^2\widetilde{D}_r\widetilde{D}_x(rb)D_r^2(rc) \\ &\quad + \widetilde{D}_x(rc)\widetilde{A}_x\widetilde{D}_r(rb) - \widetilde{D}_r(rc)\widetilde{A}_r\widetilde{D}_x(rb).\end{aligned}$$

Here I is the identity operator and we have used the identities

$$\widetilde{A}_x = \frac{1}{2}\Delta x^2 D_x^2 + I, \quad \widetilde{A}_r = \frac{1}{2}\Delta r^2 D_r^2 + I$$

in the second equality of (A.5).

From (A.1), the first two terms on the right-hand side of (A.5) can be estimated by

$$|\widetilde{D}_r(rb)(I+\widetilde{A}_r)\widetilde{D}_x(rc)| \le Cr^2\left(|\widetilde{D}_r b| + \frac{|\widetilde{A}_r b|}{r}\right)\|\partial_x c\|_{L^\infty}, \tag{A.6}$$

$$|\widetilde{D}_x(rb)(I+\widetilde{A}_x)\widetilde{D}_r(rc)| \le Cr^2|\widetilde{D}_x b|\|\partial_r c + \frac{c}{r}\|_{L^\infty} \le Cr^2|\widetilde{D}_x b|(\|c\|_{0,0,0} + \|c\|_{1,1,0}). \tag{A.7}$$

From (A.3) and (A.4), we can similarly estimate the remaining terms in (A.5):

$$\begin{aligned}|(\widetilde{A}_r - \widetilde{A}_x)(rb)\widetilde{D}_r\widetilde{D}_x(rc)| &\le Cr^2\frac{(\widetilde{A}_r+\widetilde{A}_x)|b|}{r}\|\partial_x\partial_r(rc)\|_{L^\infty} \\ &\le Cr^2\frac{(\widetilde{A}_r+\widetilde{A}_x)|b|}{r}(\|c\|_{1,1,0} + \|c\|_{2,2,0}),\end{aligned} \tag{A.8}$$

$$|\frac{1}{2}\Delta x^2\widetilde{D}_r\widetilde{D}_x(rb)D_x^2(rc)| \le C\frac{(\Delta x)^2}{\Delta r}|\widetilde{A}_r(r\widetilde{D}_x(b))D_x^2(rc)| \le Cr^2\frac{\Delta r}{r}\widetilde{A}_r|\widetilde{D}_x b|\|c\|_{2,2,0}, \tag{A.9}$$

$$|\frac{1}{2}\Delta r^2\widetilde{D}_r\widetilde{D}_x(rb)D_r^2(rc)| \le C\Delta r|\widetilde{A}_r\widetilde{D}_x(rb)|\|\partial_r^2(rc)\|_{L^\infty} \le Cr^2\frac{\Delta r}{r}\widetilde{A}_r|\widetilde{D}_x b|\|c\|_{2,2,0}, \tag{A.10}$$

$$|\widetilde{A}_x\widetilde{D}_r(rb)\widetilde{D}_x(rc)| \le Cr^2|\widetilde{A}_x\left(\frac{1}{r}\widetilde{D}_r(rb)\right)|\|\partial_x c\|_{L^\infty} \le Cr^2\widetilde{A}_x\left(|\widetilde{D}_r b| + \frac{1}{r}\widetilde{A}_r|b|\right)\|c\|_{1,1,0}, \tag{A.11}$$

and

$$|\widetilde{A}_r\widetilde{D}_x(rb)\widetilde{D}_r(rc)| \le Cr^2\widetilde{A}_r|\widetilde{D}_x b|\|c\|_{1,1,0}. \tag{A.12}$$

From (A.6)–(A.12), we can estimate the weighted L^2 norm of $\frac{1}{r^2}J_h(rb, rc)$ by

$$\|\frac{1}{r^2}J_h(rb,rc)\|_{0,h} \le C\|\left(|\widetilde{D}_x b| + |\widetilde{D}_r b| + \frac{|b|}{r}\right)\|_{0,h}\|c\|_{2,2,0} \le C\|b\|_{1,h}\|c\|_{2,2,0}$$

and (5.14) follows. ∎

A.2. Local truncation error analysis—proof of Lemma 6. In this subsection, we proceed with the local truncation error estimate. All the assertions in Lemmas 7 to 10 are pointwise estimates on the grid points (x_i, r_j), $j \ge 1$. For brevity, we omit the indices (i, j) whenever it is obvious.

We start with the estimates of the diffusion terms in (5.3).

LEMMA 7. *If $a \in C_s^4(R \times \overline{R^+})$ and $\alpha_0, \beta_0 \in R$, we have*

$$r|(\nabla_h^2 - \nabla^2)a| \le C\left(\Delta x^2 + \Delta r^2\right)\frac{1}{(1+r)^{\alpha_0}(1+|x|)^{\beta_0}}\|a\|_{4,\alpha_0+2,\beta_0} \tag{A.13}$$

and

$$|(\nabla_h^2 - \nabla^2)a| \le C\left(\Delta x^2 + \Delta r^2\right)\frac{1}{r}\frac{1}{(1+r)^{\alpha_0}(1+|x|)^{\beta_0}}\|a\|_{4,\alpha_0+2,\beta_0}. \tag{A.14}$$

Proof. Since $a \in C_s^4(R \times \overline{R^+})$, the odd extension of a given by

$$\tilde{a}(x,r) = \begin{cases} a(x,r), & \text{if } r \geq 0, \\ -a(x,-r), & \text{if } r < 0, \end{cases}$$

is in $C^4(R^2)$. It follows that

$$\begin{aligned} \nabla_h^2 a &= \left(D_x^2 + D_r^2 + \frac{\widetilde{D}_r}{r} \right) a \\ &= \nabla^2 a + \frac{1}{12}\Delta x^2 \partial_x^4 a|_{(\xi,r)} + \Delta r^2 \left(\frac{1}{12}\partial_r^4 a|_{(x,\eta_1)} + \frac{1}{6}\frac{1}{r}(\partial_r^3 a)_{(x,\eta_2)} \right) \end{aligned} \tag{A.15}$$

is valid for all $j \geq 1$ with $\xi \in (x - \Delta x, x + \Delta x)$ and $\eta_1, \eta_2 \in (r - \Delta r, r + \Delta r)$.

Thus

$$\begin{aligned} & r|(\nabla_h^2 - \nabla^2)a| \\ &\leq C\left(\Delta x^2 + \Delta r^2\right)\left(r|\partial_x^4(r\tfrac{a}{r})|_{(\xi,r)}| + r|\partial_r^4(r\tfrac{a}{r})|_{(x,\eta_1)}| + |\partial_r^3(r\tfrac{a}{r})|_{(x,\eta_2)}| \right) \\ &\leq C\left(\Delta x^2 + \Delta r^2\right)\left(\tfrac{r\|a\|_{4,\alpha_0+2,\beta_0}}{(1+r)^{\alpha_0+1}(1+|\xi|)^{\beta_0}} + \tfrac{r(\|a\|_{4,\alpha_0+2,\beta_0} + \|a\|_{3,\alpha_0+1,\beta_0})}{(1+\eta_1)^{\alpha_0+1}(1+|x|)^{\beta_0}} \right. \\ &\qquad\qquad \left. + \tfrac{\|a\|_{3,\alpha_0+1,\beta_0} + \|a\|_{2,\alpha_0,\beta_0}}{(1+\eta_2)^{\alpha_0}(1+|x|)^{\beta_0}} \right) \\ &\leq C\left(\Delta x^2 + \Delta r^2\right) \tfrac{1}{(1+r)^{\alpha_0}(1+|x|)^{\beta_0}} \|a\|_{4,\alpha_0+2,\beta_0}. \end{aligned}$$

This gives (A.13), together with (A.14) as a direct consequence. ∎

Next we proceed with the estimates for the Jacobians, starting with their typical factors.

LEMMA 8. *For $a \in C_s^4(R \times \overline{R^+})$, $\alpha, \beta \in R$, we have*

$$\widetilde{D}_x\left(\frac{a}{r}\right) = \partial_x\left(\frac{a}{r}\right) + O(1)\Delta x^2 \frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha,\beta}, \tag{A.16}$$

$$\widetilde{D}_x(ra) = \partial_x(ra) + O(1)r^2\Delta x^2 \frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha,\beta}, \tag{A.17}$$

$$\widetilde{D}_r\left(\frac{a}{r}\right) = \partial_r\left(\frac{a}{r}\right) + O(1)\frac{\Delta r^2}{r^3} \frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha+3,\beta}, \tag{A.18}$$

$$\widetilde{D}_r(ra) = \partial_r(ra) + O(1)\frac{\Delta r^2}{r} \frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha+3,\beta}. \tag{A.19}$$

Proof. We begin with (A.16) and (A.17).

Since

$$(\widetilde{D}_x - \partial_x)f = \frac{\Delta x^2}{6}\partial_x^3 f_{|(\xi,r)}, \quad \xi \in (x - \Delta x, x + \Delta x),$$

it follows that

$$\left|(\widetilde{D}_x - \partial_x)\left(\frac{a}{r}\right)\right| = \frac{\Delta x^2}{6}\left|\partial_x^3\left(\frac{a}{r}\right)\right|_{|(\xi,r)} \leq C\Delta x^2 \frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha,\beta}$$

and

$$|(\widetilde{D}_x - \partial_x)(ra)| = \frac{\Delta x^2}{6}\left|\partial_x^3\left(r^2\frac{a}{r}\right)_{|(\xi,r)}\right| \le Cr^2\Delta x^2\frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha,\beta}.$$

For (A.18) and (A.19), the estimate is more complicated due to our reflection boundary condition (4.8) and (4.9). We estimate for $j > 1$ and $j = 1$ separately.

When $j > 1$, we have

$$(\widetilde{D}_r - \partial_r)f = \frac{1}{6}\Delta r^2\partial_r^3 f|_{(x,\eta)}, \quad \eta \in (r - \Delta r, r + \Delta r).$$

Therefore

$$\left|(\widetilde{D}_r - \partial_r)\left(\frac{a}{r}\right)\right| = \frac{\Delta r^2}{6}\left|\partial_r^3\left(\frac{a}{r}\right)\right|_{(x,\eta)} \le C\frac{\Delta r^2}{r^3}\frac{1}{(1+r)^\alpha(1+|x|)^\beta}\|a\|_{3,\alpha+3,\beta}$$

and

$$\begin{aligned}|(\widetilde{D}_r - \partial_r)(ra)| &\le C\Delta r^2\left|\partial_r^3(r^2\tfrac{a}{r})\right|_{(x,\eta)} \\ &\le C\tfrac{\Delta r^2}{r}\tfrac{1}{(1+r)^\alpha(1+|x|)^\beta}(\|a\|_{3,\alpha+3,\beta} + \|a\|_{2,\alpha+2,\beta} + \|a\|_{1,\alpha+1,\beta}).\end{aligned}$$

When $j = 1$, we have

$$\left|\partial_r\left(\frac{a}{r}\right)\right|_{j=1} = C\frac{\Delta r^2}{r_1^3}r_1\left|\partial_r\left(\frac{a}{r}\right)\right|_{j=1} \le C\frac{\Delta r^2}{r_1^3}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{1,\alpha+1,\beta}.$$

In addition, since $r_1 = \frac{\Delta r}{2}$, we apply (4.9) to get

$$\left|\widetilde{D}_r\left(\frac{a}{r}\right)\right|_{j=1} = \left|\frac{\frac{a_2}{r_2}+\frac{a_1}{r_1}}{2\Delta r}\right| = \left|C\frac{\Delta r^2}{r_1^3}\left(\frac{a_2}{r_2}+\frac{a_1}{r_1}\right)\right| \le C\frac{\Delta r^2}{r_1^3}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{0,\alpha,\beta},$$

and (A.18) follows.

(A.19) can be proved similarly, as follows:

$$\widetilde{D}_r(ra)_{j=1} = \frac{\frac{3}{2}\Delta r a_2 + \frac{1}{2}\Delta r a_1}{2\Delta r} = \frac{3}{4}a_2 + \frac{1}{4}a_1,$$

$$|a_1| \le C\frac{\Delta r^2}{r_1}\left|\frac{a_1}{r_1}\right| \le C\frac{\Delta r^2}{r_1}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{0,\alpha,\beta},$$

and

$$|a_2| \le C\frac{\Delta r^2}{r_1}\left|\frac{a_2}{r_2}\right| \le C\frac{\Delta r^2}{r_1}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{0,\alpha,\beta}.$$

Therefore

$$\left|\widetilde{D}_r(ra)\right|_{j=1} \le C\frac{\Delta r^2}{r_1}\frac{1}{(1+r_1)^\alpha(1+|x|)^\beta}\|a\|_{0,\alpha,\beta}.$$

In addition,

$$|\partial_r(ra)|_{j=1} \le \left(r^2\left|\partial_r\left(\frac{a}{r}\right)\right| + 2r\left|\frac{a}{r}\right|\right)_{j=1} \le C\frac{\Delta r^2}{r_1}\frac{\|a\|_{1,\alpha+1,\beta}+\|a\|_{0,\alpha,\beta}}{(1+r_1)^\alpha(1+|x|)^\beta},$$

and (A.19) follows. ∎

We now continue with the pointwise estimate for the Jacobi terms $\frac{1}{r}|J_h(ra,rb) - J(ra,rb)|$ and $r|J_h\left(\frac{a}{r},rb\right) - J\left(\frac{a}{r},rb\right)|$. Since

$$\frac{3}{r^2}J_h(ra,rb) = \widetilde{D}_x(\tfrac{a}{r})\widetilde{D}_r(rb) - \widetilde{D}_r(ra)\widetilde{D}_x(\tfrac{b}{r}) + \widetilde{D}_x\left(\tfrac{a}{r}\widetilde{D}_r(rb) - \tfrac{b}{r}\widetilde{D}_r(ra)\right) + \tfrac{1}{r^2}\widetilde{D}_r\left(r^2 b\widetilde{D}_x a - r^2 a\widetilde{D}_x b\right), \tag{A.20}$$

$$3J_h\left(\tfrac{a}{r},rb\right) = \widetilde{D}_x(\tfrac{a}{r})\widetilde{D}_r(rb) - \widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb) + \widetilde{D}_x\left(\tfrac{a}{r}\widetilde{D}_r(rb) - rb\widetilde{D}_r(\tfrac{a}{r})\right) + \widetilde{D}_r\left(b\widetilde{D}_x a - a\widetilde{D}_x b\right), \tag{A.21}$$

it suffices to estimate the terms in (A.20) and (A.21) individually. We summarize them as the following lemma.

LEMMA 9. *If $a, b \in C_s^4(R\times\overline{R^+})$ and $\alpha_1,\alpha_2,\beta_1,\beta_2 \in R$, then*

$$r|\widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb) - \partial_r(\tfrac{a}{r})\partial_x(rb)| + \tfrac{1}{r}|\widetilde{D}_r(rb)\widetilde{D}_x(ra) - \partial_r(rb)\partial_x(ra)| \le C(\Delta x^2+\Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{3,\alpha_2+\frac{5}{2},\beta_2}, \tag{A.22}$$

$$r|\widetilde{D}_x(\tfrac{a}{r}\widetilde{D}_r(rb)) - \partial_x(\tfrac{a}{r}\partial_r(rb))| + r|\widetilde{D}_x(ra\widetilde{D}_r(\tfrac{b}{r})) - \partial_x(ra\partial_r(\tfrac{b}{r}))| \le C(\Delta x^2+\Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}, \tag{A.23}$$

$$r|\widetilde{D}_r(a\widetilde{D}_x b) - \partial_r(a\partial_x b)| + \tfrac{1}{r}|\widetilde{D}_r(r^2a\widetilde{D}_x b) - \partial_r(r^2a\partial_x b)| \le C(\Delta x^2+\Delta r^2)\tfrac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}. \tag{A.24}$$

Proof. Since (A.16)–(A.19) are valid for any α, $\beta \in R$, we have

$$\widetilde{D}_x\left(\frac{a}{r}\right) = \partial_x\left(\frac{a}{r}\right) + O(1)\Delta x^2\frac{1}{(1+r)^{\alpha_1+\lambda}(1+|x|)^{\beta_1}}\|a\|_{3,\alpha_1+\lambda,\beta_1}, \tag{A.25}$$

$$\widetilde{D}_x(ra) = \partial_x(ra) + O(1)r^2\Delta x^2\frac{1}{(1+r)^{\alpha_1+\lambda}(1+|x|)^{\beta_1}}\|a\|_{3,\alpha_1+\lambda,\beta_2}, \tag{A.26}$$

$$\widetilde{D}_r\left(\frac{a}{r}\right) = \partial_r\left(\frac{a}{r}\right) + O(1)\frac{\Delta r^2}{r^3}\frac{1}{(1+r)^{\alpha_1+\lambda}(1+|x|)^{\beta_1}}\|a\|_{3,\alpha_1+\lambda+3,\beta_1}, \tag{A.27}$$

$$\widetilde{D}_r(ra) = \partial_r(ra) + O(1)\frac{\Delta r^2}{r}\frac{1}{(1+r)^{\alpha_1+\lambda}(1+|x|)^{\beta_1}}\|a\|_{3,\alpha_1+\lambda+3,\beta_1}, \tag{A.28}$$

and

$$\widetilde{D}_x\left(\frac{b}{r}\right) = \partial_x\left(\frac{b}{r}\right) + O(1)\Delta x^2\frac{1}{(1+r)^{\alpha_2+\mu}(1+|x|)^{\beta_2}}\|b\|_{3,\alpha_2+\mu,\beta_2}, \tag{A.29}$$

$$\widetilde{D}_x(rb) = \partial_x(rb) + O(1)r^2\Delta x^2\frac{1}{(1+r)^{\alpha_2+\mu}(1+|x|)^{\beta_2}}\|b\|_{3,\alpha_2+\mu,\beta_2}, \tag{A.30}$$

$$\widetilde{D}_r\left(\frac{b}{r}\right) = \partial_r\left(\frac{b}{r}\right) + O(1)\frac{\Delta r^2}{r^3}\frac{1}{(1+r)^{\alpha_2+\mu}(1+|x|)^{\beta_2}}\|b\|_{3,\alpha_2+\mu+3,\beta_2}, \tag{A.31}$$

$$\widetilde{D}_r(rb) = \partial_r(rb) + O(1)\frac{\Delta r^2}{r}\frac{1}{(1+r)^{\alpha_2+\mu}(1+|x|)^{\beta_2}}\|b\|_{3,\alpha_2+\mu+3,\beta_2} \tag{A.32}$$

for any λ, $\mu \in R$. We apply (A.27), (A.30) with $\lambda = -\frac{1}{2}$, $\mu = \frac{5}{2}$ to get

$$\begin{aligned} & r|\widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb) - \partial_r(\tfrac{a}{r})\partial_x(rb)| \\ = \; & r|\widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb) - \partial_r(\tfrac{a}{r})\widetilde{D}_x(rb) + \partial_r(\tfrac{a}{r})\widetilde{D}_x(rb) - \partial_r(\tfrac{a}{r})\partial_x(rb)| \\ = \; & O(|\widetilde{D}_x(rb)|)\tfrac{\Delta r^2}{r^2}\tfrac{\|\!|a|\!\|_{3,\alpha_1+\frac{5}{2},\beta_1}}{(1+r)^{\alpha_1-\frac{1}{2}}(1+|x|)^{\beta_1}} + O(|\partial_r(\tfrac{a}{r})|)r^3\Delta x^2\tfrac{\|\!|b|\!\|_{3,\alpha_2+\frac{5}{2},\beta_2}}{(1+r)^{\alpha_2+\frac{5}{2}}(1+|x|)^{\beta_2}}. \end{aligned}$$

Moreover, since

$$r^3|\partial_r\left(\frac{a}{r}\right)| \leq \frac{1}{(1+r)^{\alpha_1-\frac{5}{2}}(1+|x|)^{\beta_1}}\|a\|_{1,\alpha_1+\frac{1}{2},\beta_1}$$

and

$$|\widetilde{D}_x(rb)| = |\partial_x(rb)(\xi,r)| \leq r^2\frac{1}{(1+r)^{\alpha_2+\frac{1}{2}}(1+|x|)^{\beta_2}}\|b\|_{1,\alpha_2+\frac{1}{2},\beta_2},$$

it follows that

$$\begin{aligned} & r|\widetilde{D}_r(\tfrac{a}{r})\widetilde{D}_x(rb) - \partial_r(\tfrac{a}{r})\partial_x(rb)| \\ \leq \; & C(\Delta x^2+\Delta r^2)\tfrac{\|\!|a|\!\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{1,\alpha_2+\frac{1}{2},\beta_2}+\|a\|_{1,\alpha_1+\frac{1}{2},\beta_1}\|\!|b|\!\|_{3,\alpha_2+\frac{5}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}} \\ \leq \; & C(\Delta x^2+\Delta r^2)\tfrac{\|\!|a|\!\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|\!|b|\!\|_{3,\alpha_2+\frac{5}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}. \end{aligned} \tag{A.33}$$

Similarly, from (A.32) and (A.25), we have

$$\begin{aligned} & r|\widetilde{D}_x(\tfrac{a}{r})\widetilde{D}_r(rb) - \partial_x(\tfrac{a}{r})\partial_r(rb)| \\ = \; & r|\widetilde{D}_x(\tfrac{a}{r})\widetilde{D}_r(rb) - \widetilde{D}_x(\tfrac{a}{r})\partial_r(rb) + \widetilde{D}_x(\tfrac{a}{r})\partial_r(rb) - \partial_x(\tfrac{a}{r})\partial_r(rb)| \\ = \; & O(|\widetilde{D}_x(\tfrac{a}{r})|)\Delta r^2\tfrac{\|\!|b|\!\|_{3,\alpha_2+\frac{5}{2},\beta_2}}{(1+r)^{\alpha_2-\frac{1}{2}}(1+|x|)^{\beta_2}} + O(|\partial_r(rb)|)r\Delta x^2\tfrac{\|\!|a|\!\|_{3,\alpha_1+\frac{5}{2},\beta_1}}{(1+r)^{\alpha_1+\frac{5}{2}}(1+|x|)^{\beta_1}} \\ \leq \; & C\Delta r^2\tfrac{\|\!|a|\!\|_{1,\alpha_1+\frac{1}{2},\beta_1}\|\!|b|\!\|_{3,\alpha_2+\frac{5}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}} + C\Delta x^2\tfrac{\|\!|a|\!\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|\!|b|\!\|_{1,\alpha_2+\frac{1}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}} \\ \leq \; & C(\Delta x^2+\Delta r^2)\tfrac{\|\!|a|\!\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|\!|b|\!\|_{3,\alpha_2+\frac{5}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}. \end{aligned} \tag{A.34}$$

The estimate (A.22) then follows from (A.33) and (A.34).

For (A.23), we have

$$\begin{aligned} & \widetilde{D}_x(f\widetilde{D}_r g) - \partial_x(f\partial_r g) \\ = \; & \widetilde{D}_x(f(\widetilde{D}_r - \partial_r)g) + (\widetilde{D}_x - \partial_x)(f\partial_r g) \\ = \; & \partial_x(f(\widetilde{D}_r - \partial_r)g)_{|(\xi_1,r)} + \tfrac{1}{6}\Delta x^2\partial_x^3(f\partial_r g)_{|(\xi_2,\eta)} \\ = \; & (\partial_x f)((\widetilde{D}_r - \partial_r)g)_{|(\xi_1,r)} + f((\widetilde{D}_r - \partial_r)\partial_x g)_{|(\xi_1,r)} + \tfrac{1}{6}\Delta x^2\partial_x^3(f\partial_r g)_{|(\xi_2,\eta)}. \end{aligned} \tag{A.35}$$

We proceed with individual terms in (A.35), taking $f = \frac{a}{r}$ and $g = rb$. From (A.30) with $\mu = -\frac{1}{2}$, we have

$$\begin{aligned} r\left|(\partial_x \tfrac{a}{r})(\widetilde{D}_r - \partial_r)(rb)\right| &\le C|\partial_x(\tfrac{a}{r})|\Delta r^2 \frac{1}{(1+r)^{\alpha_2-\frac{1}{2}}(1+|x|)^{\beta_2}} |\!|\!| b |\!|\!|_{3,\alpha_2+\frac{5}{2},\beta_2} \\ &\le C\Delta r^2 \frac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}} |\!|\!| a |\!|\!|_{1,\alpha_1+\frac{1}{2},\beta_1} |\!|\!| b |\!|\!|_{3,\alpha_2+\frac{5}{2},\beta_2}. \end{aligned}$$

Similarly, from (A.32)

$$\begin{aligned} & r\left|\tfrac{a}{r}(\widetilde{D}_r - \partial_r)\partial_x(rb)\right| \\ \le\ & C\Delta r^2 |\tfrac{a}{r}| \frac{1}{(1+r)^{\alpha_2+\frac{1}{2}}(1+|x|)^{\beta_2}} |\!|\!| \partial_x b |\!|\!|_{3,\alpha_2+\frac{7}{2},\beta_2} \\ \le\ & C\Delta r^2 \frac{1}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}} \|a\|_{0,\alpha_1-\frac{1}{2},\beta_1} \|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}, \\ & r\left|\Delta x^2 \partial_x^3(\tfrac{a}{r}\partial_r(rb))_{|(x,\eta)}\right| \\ \le\ & C\Delta x^2 \left| r\partial_x^3\left(\tfrac{a}{r}\right)\partial_r(rb) + r\left(\tfrac{a}{r}\right)\partial_x^3\partial_r(rb)\right| \\ \le\ & C\Delta x^2 \frac{\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{1,\alpha_2+\frac{1}{2},\beta_2} + \|a\|_{0,\alpha_1-\frac{1}{2},\beta_1}\|b\|_{3,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}} \\ \le\ & C\Delta x^2 \frac{|\!|\!| a |\!|\!|_{3,\alpha_1+\frac{5}{2},\beta_1} |\!|\!| b |\!|\!|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}. \end{aligned}$$

Therefore

$$r|\widetilde{D}_x\left(\frac{a}{r}\widetilde{D}_r(rb)\right) - \partial_x\left(\frac{a}{r}\partial_r(rb)\right)| \le C(\Delta x^2 + \Delta r^2)\frac{|\!|\!| a |\!|\!|_{3,\alpha_1+\frac{5}{2},\beta_1} |\!|\!| b |\!|\!|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}.$$

Using the same argument as above, one can derive

$$r|\widetilde{D}_x\left(ra\widetilde{D}_r\left(\frac{b}{r}\right)\right) - \partial_x\left(ra\partial_r\left(\frac{b}{r}\right)\right)| \le C(\Delta x^2 + \Delta r^2)\frac{|\!|\!| a |\!|\!|_{3,\alpha_1+\frac{5}{2},\beta_1} |\!|\!| b |\!|\!|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}$$

and therefore (A.23) is proved.

We continue with (A.24). For the first term, we can write

$$\widetilde{D}_r(a\widetilde{D}_x b) - \partial_r(a\partial_x b) = \widetilde{D}_r(a(\widetilde{D}_x - \partial_x)b) + (\widetilde{D}_r - \partial_r)(a\partial_x b).$$

Since $a, b \in C_s^4(R \times \overline{R^+})$, by extending a, b to odd functions across $r = 0$, we see that the extended $a\widetilde{D}_x b$ is in $C^4(R^2)$; thus

$$\begin{aligned} \widetilde{D}_r(a(\widetilde{D}_x - \partial_x)b) &= \partial_r(a(\widetilde{D}_x - \partial_x)b)|_{(x,\eta)} \\ &= \left(\partial_r a(\widetilde{D}_x - \partial_x)b + a(\widetilde{D}_x - \partial_x)(\partial_r b)\right)|_{(x,\eta)} \\ &= \frac{\Delta x^2}{6}\left(\partial_r a|_{(x,\eta)}\partial_x^3 b|_{(\xi_1,\eta)} + a|_{(x,\eta)}\partial_x^3\partial_r b|_{(\xi_2,\eta)}\right) \end{aligned}$$

and therefore

$$r|\widetilde{D}_r(a(\widetilde{D}_x - \partial_x)b)| \le C\Delta x^2 \frac{|\!|\!| a |\!|\!|_{1,\alpha_1+\frac{1}{2},\beta_1} |\!|\!| b |\!|\!|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}. \tag{A.36}$$

Similarly, the extended $a\partial_x b$ is in $C^3(R^2)$, and thus we have

$$r|(\widetilde{D}_r-\partial_r)(a\partial_x b)| = r\frac{\Delta r^2}{6}\partial_r^3(a\partial_x b)|_{(x,\eta)} \le C\Delta r^2 \frac{\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}. \tag{A.37}$$

From (A.36) and (A.37), we have the following estimate for the first term of (A.24):

$$r|\widetilde{D}_r(a\widetilde{D}_x b) - \partial_r(a\partial_x b)| \le C(\Delta x^2+\Delta r^2)\frac{\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}. \tag{A.38}$$

The second term in (A.24) can be treated similarly, as follows:

$$\frac{1}{r}\widetilde{D}_r(r^2a\widetilde{D}_x b) - \frac{1}{r}\partial_r(r^2a\partial_x b) = \frac{1}{r}\widetilde{D}_r(r^2a(\widetilde{D}_x-\partial_x)b) + \frac{1}{r}(\widetilde{D}_r-\partial_r)(r^2a\partial_x b). \tag{A.39}$$

Again, since the extensions of $r^2a(\widetilde{D}_x-\partial_x)b$ and $r^2a\partial_x b$ are both in $C^3(R^2)$, we can directly estimate these two terms by

$$\begin{aligned}\tfrac{1}{r}\widetilde{D}_r(r^2a(\widetilde{D}_x-\partial_x)b) &= \tfrac{1}{r}\partial_r(r^2a(\widetilde{D}_x-\partial_x)b)_{(x,\eta)}\\ &= \tfrac{1}{r}\left(\Big(\partial_r(r^2a)(\widetilde{D}_x-\partial_x)b\Big)_{(x,\eta)} + \Big(r^2a(\widetilde{D}_x-\partial_x)(\partial_r b)\Big)_{(x,\eta)}\right)\\ &= C\Delta x^2\left(\big((r\partial_r a+2a)\partial_x^3 b\big)_{(\xi_1,\eta)} + \big(ra\partial_x^3(\partial_r b)\big)_{(\xi_2,\eta)}\right)\end{aligned} \tag{A.40}$$

and

$$\frac{1}{r}(\widetilde{D}_r-\partial_r)(r^2a\partial_x b) = \frac{\Delta r^2}{r}\partial_r^3(r^2a\partial_x b)_{(x,\eta)} = \frac{\Delta r^2}{r}\partial_r^3\left(r^4\frac{a}{r}\partial_x(\frac{b}{r})\right)_{(x,\eta)}. \tag{A.41}$$

From (A.40) and (A.41), we have

$$\begin{aligned}&|\tfrac{1}{r}\widetilde{D}_r(r^2a(\widetilde{D}_x-\partial_x)b)|\\ \le\ & C\Delta x^2\tfrac{\|a\|_{1,\alpha_1+\frac{1}{2},\beta_1}\|b\|_{3,\alpha_2+\frac{5}{2},\beta_2}+\|a\|_{0,\alpha_1-\frac{1}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\\ \le\ & C\Delta x^2\tfrac{\|a\|_{1,\alpha_1+\frac{1}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}\end{aligned} \tag{A.42}$$

and

$$\frac{1}{r}\left|(\widetilde{D}_r-\partial_r)(r^2a\partial_x b)\right| \le C\Delta r^2\frac{\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}. \tag{A.43}$$

From (A.39), (A.42), and (A.43), we conclude that

$$\left|\frac{1}{r}\widetilde{D}_r(r^2a\widetilde{D}_x b) - \frac{1}{r}\partial_r(r^2a\partial_x b)\right| \le C(\Delta x^2+\Delta r^2)\frac{\|a\|_{3,\alpha_1+\frac{5}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}. \tag{A.44}$$

The estimates (A.38) and (A.44) imply (A.24). Thus the proof of Lemma 9 is completed. ∎

As a direct consequence of Lemma 9, we have the following pointwise estimate for the Jacobians.

LEMMA 10. *If* $a, b \in C_s^4(R \times \overline{R^+})$*, then*

$$\frac{1}{r}|J_h(ra, rb) - J(ra, rb)| \le C(\Delta x^2 + \Delta r^2)\frac{\|a\|_{4,\alpha_1+\frac{7}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}},$$

$$r\left|J_h\left(\frac{a}{r}, rb\right) - J\left(\frac{a}{r}, rb\right)\right| \le C(\Delta x^2 + \Delta r^2)\frac{\|a\|_{4,\alpha_1+\frac{7}{2},\beta_1}\|b\|_{4,\alpha_2+\frac{7}{2},\beta_2}}{(1+r)^{\alpha_1+\alpha_2}(1+|x|)^{\beta_1+\beta_2}}$$

for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$.

From (5.3), Lemma 7, and Lemma 10, we can easily derive (5.17)–(5.20). This completes the proof of Lemma 6. ∎

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