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CHARACTERIZATION AND REGULARITY FOR AXISYMMETRIC SOLENOIDAL VECTOR FIELDS WITH APPLICATION TO NAVIER–STOKES EQUATION*

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Abstract. We consider the vorticity-stream formulation of axisymmetric incompressible flows and its equivalence with the primitive formulation. It is shown that, to characterize the regularity of a divergence free axisymmetric vector field in terms of the swirling components, an extra set of pole conditions is necessary to give a full description of the regularity. In addition, smooth solutions up to the axis of rotation give rise to smooth solutions of primitive formulation in the case of the Navier–Stokes equation, but not the Euler equation. We also establish a proper weak formulation and show its equivalence to Leray's formulation.

Key words. axisymmetric flow, Navier–Stokes equation, Euler equation, pole condition, pole singularity, Leray solution

AMS subject classifications. 35Q30, 76D03, 76D05, 65M06

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1. Introduction.

Axisymmetric flow is an important subject in fluid dynamics and has become standard textbook material as a starting point of theoretical study for complicated flow patterns. By means of Stoke's stream function ϕ [1], an axisymmetric divergence free vector field can be efficiently represented by two scalar components:

$$\boldsymbol{u} = \frac{\partial_r \phi}{r}\boldsymbol{e}_x - \frac{\partial_x \phi}{r}\boldsymbol{e}_r + u\boldsymbol{e}_\theta. \tag{1.1}$$

With the vector identities

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = (\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \nabla\left(\frac{|\boldsymbol{u}|^2}{2}\right) \tag{1.2}$$

and

$$\nabla^2\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times\nabla\times\boldsymbol{u}, \tag{1.3}$$

one can recast the Navier–Stokes equation as

$$\begin{aligned} &\partial_t\boldsymbol{u} + (\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \nabla\tilde{p} = -\nu\nabla\times\nabla\times\boldsymbol{u} \\ &\nabla\cdot\boldsymbol{u} = 0, \end{aligned} \tag{1.4}$$

where $\tilde{p} = p + \frac{|\boldsymbol{u}|^2}{2}$. Taking the swirling component from (1.4) and from the curl of (1.4), one can eliminate the pressure term to get two scalar convection diffusion equations:

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$$\begin{aligned} &\partial_t u + u_x \partial_x u + u_r \partial_r u + \tfrac{u_r}{r} u = \nu \mathscr{L} u, \\ &\partial_t \omega + u_x \partial_x \omega + u_r \partial_r \omega - \tfrac{u_r}{r} \omega = \tfrac{1}{r} \partial_x (u^2) + \nu \mathscr{L} \omega. \end{aligned} \tag{1.5}$$

The system is closed by the vorticity-stream function relation $\omega = -\mathscr{L}\psi$, $u_x = \dfrac{\partial_r(r\psi)}{r}$, and $u_r = -\partial_x \psi$. Here $\psi = \frac{\phi}{r}$ and $\mathscr{L}u = \partial_r^2 u + \frac{\partial_r u}{r} + \partial_x^2 u - \frac{u}{r^2}$.

This representation (1.5) has several advantages over the primitive formulation (1.4). It needs only two dependent variables ψ and u defined on $(x,r) \in (R \times R^+)$, and it is free from Lagrangian multipliers and is automatically divergence free. These advantages are particularly favorable in numerical computations.

A natural question is whether (1.5) is actually equivalent to the primitive formulation (1.4), and in which solution classes are they equivalent? In this paper, we have systematically investigated this issue for both classical and weak solutions. We start in section 2 with the characterization of smoothness of axisymmetric divergence free vector fields. It is shown that an additional pole condition of the form

$$\partial_r^j u(x, 0^+) = 0, \quad \partial_r^j \psi(x, 0^+) = 0 \quad \text{for } j \text{ even} \tag{1.6}$$

is essential to characterize the smoothness of the vector field (1.1) in classical spaces (see Lemma 2 for details). The construction of Sobolev spaces and the counterpart of (1.6) are established in 2.2. We then apply this pole condition to derive regularity and equivalence results in various solution spaces in section 3. Firstly, we show in section 3.1 that there exists $C^k(R \times \overline{R^+})$ solutions of the Euler equation with a genuine singularity on the axis of rotation. In addition, this pole singularity will persist in time. In contrast, we show in section 3.2 that if the solution to (1.5) is in $C^k(R \times \overline{R^+})$, then the pole condition (1.6) is automatically satisfied. Next, we consider weak formulation of (1.5) and study its relation with the Leray's weak solution in section 3.3. We end this paper by showing that, when appropriately formulated, the weak solutions to (1.5) are exactly the axisymmetric weak solutions obtained via Leray's construction [11].

2. Function spaces for axisymmetric solenoidal vector fields.

2.1. Classical spaces and the pole condition. In this section, we establish basic regularity results for axisymmetric vector fields. We will show that the swirling component of a smooth axisymmetric vector field has vanishing even order derivatives in the radial direction at the axis of rotation. This is done in Lemma 2 by a symmetry argument.

Throughout this paper, we will be using the cylindrical coordinate system

$$x = x, \quad y = r\cos\theta, \quad z = r\sin\theta, \tag{2.1}$$

where the x-axis is the axis of rotation. A vector field $\boldsymbol{u}$ is said to be axisymmetric if $\partial_\theta u_x = \partial_\theta u_r = \partial_\theta u_\theta = 0$. Here and throughout this paper, the subscripts of u are used to denote components rather than partial derivatives.

The three basic differential operators in the cylindrical coordinate system are given by

$$\nabla u = (\partial_x u)\boldsymbol{e}_x + (\partial_r u)\boldsymbol{e}_r + \left(\frac{1}{r}\partial_\theta u\right)\boldsymbol{e}_\theta, \tag{2.2}$$

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\left(\partial_x(r u_x) + \partial_r(r u_r) + \partial_\theta u_\theta\right), \tag{2.3}$$

$$\nabla \times \boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_r & r\boldsymbol{e}_\theta \\ \partial_x & \partial_r & \partial_\theta \\ u_x & u_r & r u_\theta \end{vmatrix}. \tag{2.4}$$

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Here $\boldsymbol{e}_x$, $\boldsymbol{e}_r$, and $\boldsymbol{e}_\theta$ are the unit vectors in the x, r, and θ directions, respectively.

LEMMA 1. *Let* $\boldsymbol{u} = u_x\boldsymbol{e}_x + u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta \in C^k(R^3, R^3)$, $k \geq 0$, *then for any fixed* $\theta \in [0, \pi)$, $u_x(\cdot,\cdot,\theta), u_r(\cdot,\cdot,\theta), u_\theta(\cdot,\cdot,\theta) \in C^k(R \times \overline{R^+})$. *Moreover,*

$$\partial_r^j u_x(x, 0^+, \theta) = (-1)^j \partial_r^j u_x(x, 0^+, \theta + \pi), \qquad 0 \leq j \leq k, \tag{2.5}$$
$$\partial_r^j u_r(x, 0^+, \theta) = (-1)^{j+1} \partial_r^j u_r(x, 0^+, \theta + \pi), \qquad 0 \leq j \leq k, \tag{2.6}$$
$$\partial_r^j u_\theta(x, 0^+, \theta) = (-1)^{j+1} \partial_r^j u_\theta(x, 0^+, \theta + \pi), \qquad 0 \leq j \leq k. \tag{2.7}$$

Proof. Let $\boldsymbol{u} = u_x(x,r,\theta)\boldsymbol{e}_x + u_r(x,r,\theta)\boldsymbol{e}_r + u_\theta(x,r,\theta)\boldsymbol{e}_\theta$. Note that $\boldsymbol{e}_x$ is a smooth vector field, while $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are discontinuous at the axis of rotation. More specifically, on the cross section $z = 0$, $y > 0$, we have

$$\boldsymbol{e}_x(x, y, z=0) = \boldsymbol{e}_x(x, r=|y|, \theta=0), \qquad \boldsymbol{e}_x(x, -y, z=0) = \boldsymbol{e}_x(x, r=|y|, \theta=\pi), \tag{2.8}$$
$$\boldsymbol{e}_y(x, y, z=0) = \boldsymbol{e}_r(x, r=|y|, \theta=0), \qquad \boldsymbol{e}_y(x, -y, z=0) = -\boldsymbol{e}_r(x, r=|y|, \theta=\pi), \tag{2.9}$$
$$\boldsymbol{e}_z(x, y, z=0) = \boldsymbol{e}_\theta(x, r=|y|, \theta=0), \qquad \boldsymbol{e}_z(x, -y, z=0) = -\boldsymbol{e}_\theta(x, r=|y|, \theta=\pi). \tag{2.10}$$

Consequently,

$$u_x(x, y, z=0) = u_x(x, r=|y|, \theta=0), \qquad u_x(x, -y, z=0) = u_x(x, r=|y|, \theta=\pi), \tag{2.11}$$
$$u_y(x, y, z=0) = u_r(x, r=|y|, \theta=0), \qquad u_y(x, -y, z=0) = -u_x(x, r=|y|, \theta=\pi), \tag{2.12}$$
$$u_z(x, y, z=0) = u_\theta(x, r=|y|, \theta=0), \qquad u_z(x, -y, z=0) = -u_\theta(x, r=|y|, \theta=\pi). \tag{2.13}$$

Taking the limit $y \to 0^+$, it follows that (2.5–2.7) hold with $\theta = 0$. The above argument can be easily modified to prove for any other $\theta \in [0, 2\pi)$. ∎

If $\boldsymbol{u}$ is axisymmetric, we immediately have the following direct consequence.

COROLLARY 1. *Let* $\boldsymbol{u} \in C^k(R^3, R^3)$ *be an axisymmetric vector field,* $\boldsymbol{u} = u_x(x,r)\boldsymbol{e}_x + u_r(x,r)\boldsymbol{e}_r + u_\theta(x,r)\boldsymbol{e}_\theta$. *Then* $u_x, u_r, u_\theta \in C^k(R \times \overline{R^+})$ *and*

$$\partial_r^{2\ell+1} u_x(x, 0^+) = 0, \qquad 1 \leq 2\ell + 1 \leq k, \tag{2.14}$$
$$\partial_r^{2m} u_r(x, 0^+) = \partial_r^{2m} u_\theta(x, 0^+) = 0, \qquad 0 \leq 2m \leq k. \tag{2.15}$$

Denote by $\mathcal{C}_s^k$ the axisymmetric divergence free subspace of C^k vector fields as follows.

DEFINITION 1.

$$\mathcal{C}_s^k(R^3, R^3) = \{\boldsymbol{u} \in C^k(R^3, R^3), \quad \partial_\theta u_x = \partial_\theta u_r = \partial_\theta u_\theta = 0, \quad \nabla \cdot \boldsymbol{u} = 0\}. \tag{2.16}$$

We have the following representation and regularity result for $\mathcal{C}_s^k$.

LEMMA 2.

(a) *For any* $\boldsymbol{u} \in \mathcal{C}_s^k(R^3, R^3)$, $k \geq 0$, *there exists a unique* (u, ψ), *such that*

$$\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) = \frac{\partial_r(r\psi)}{r}\boldsymbol{e}_x - \partial_x\psi\boldsymbol{e}_r + u\boldsymbol{e}_\theta, \qquad r > 0, \tag{2.17}$$

with

$$u(x, r) \in C^k(R \times \overline{R^+}), \quad \partial_r^{2\ell} u(x, 0^+) = 0 \text{ for } 0 \leq 2\ell \leq k, \tag{2.18}$$

and

$$\psi(x, r) \in C^{k+1}(R \times \overline{R^+}), \quad \partial_r^{2m} \psi(x, 0^+) = 0 \text{ for } 0 \leq 2m \leq k + 1. \tag{2.19}$$

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(b) *If* (u, ψ) *satisfies* (2.18) *and* (2.19) *and* $\boldsymbol{u}$ *is given by* (2.17) *for* $r > 0$*, then* $\boldsymbol{u} \in \mathcal{C}_s^k(R^3, R^3)$ *with a removable singularity at* $r = 0$.

Proof. *Part* (a). Since $\boldsymbol{u}$ is axisymmetric, we can write $\boldsymbol{u} = u_x(x,r)\boldsymbol{e}_x + u_r(x,r)\boldsymbol{e}_r + u_\theta(x,r)\boldsymbol{e}_\theta$ for $r > 0$. Rename u_θ by u, and (2.18) follows from Corollary 1.

Next, we derive the representation (2.17). Since $\boldsymbol{u}$ is divergence free, (2.3) gives

$$\partial_x(ru_x) + \partial_r(ru_r) = 0.$$

We know from standard argument that there exists a potential $\phi(x,r)$, known as Stokes' stream function, such that

$$\partial_x \phi = -ru_r, \quad \partial_r \phi = ru_x. \tag{2.20}$$

On the cross section $z = 0$, $y > 0$, we have

$$\begin{aligned} u_x(x,r) &= u_x(x, y = r, z = 0), \quad u_r(x,r) = u_y(x, y = r, z = 0), \\ u_\theta(x,r) &= u_z(x, y = r, z = 0). \end{aligned} \tag{2.21}$$

From (2.20) and (2.21), it is clear that $\phi(x,r) \in C^1(R \times \overline{R^+}) \cap C^{k+1}(R \times R^+)$. Since $\partial_x \phi(x, 0^+) = 0$, we may, without loss of generality, assume that $\phi(x, 0^+) = 0$. This also determines ϕ uniquely. Next, we define

$$\psi(x,r) = \frac{\phi(x,r)}{r}, \qquad r > 0. \tag{2.22}$$

It is easy to see that $\psi(x,r) \in C^{k+1}(R \times R^+)$, $\psi(x, 0^+) = \partial_r \phi(x, 0^+) = 0$, and (2.17) follows for $r > 0$.

Moreover, $\lim_{r \to 0^+} \partial_r^j \psi(x,r) = \lim_{r \to 0^+} \partial_r^j \frac{\phi(x,r)}{r}$. It follows from straightforward calculation with l'Hospital's rule and (2.20) that

$$\partial_r^j \psi(x, 0^+) = \frac{j}{j+1} \partial_r^{j-1} u_x(x, 0^+) \tag{2.23}$$

and therefore $\psi(x,r) \in C^{k+1}(R \times \overline{R^+})$. In addition, (2.19) follows from (2.14) and (2.23).

Part (b). Conversely, we now show the regularity of $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta)$ when (u, ψ) satisfies (2.18) and (2.19). Since $\boldsymbol{u}$ is axisymmetric, it suffices to check the continuity of the derivatives of $\boldsymbol{u}$ on a cross section, say $\theta = 0$, or $z = 0$, $y \geq 0$.

It is clear from (2.17) and (2.21) that $u_x(x,y,0)$, $u_y(x,y,0)$, and $u_z(x,y,0)$ have continuous x derivatives up to order k on $y \geq 0$. It remains to estimate the y-, z-, and mixed derivatives.

From

$$\partial_y = \cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta, \tag{2.24}$$

$$\partial_z = \sin\theta \partial_r + \frac{\cos\theta}{r} \partial_\theta \tag{2.25}$$

we can derive the following.

PROPOSITION 1.

(i)

$$\partial_y^j F(x,r,\theta) = \cos^j\theta \, \partial_r^j F(x,r,\theta) + \sin\theta \, G(x,r,\theta), \tag{2.26}$$

where G *consists of the derivatives of* F.

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(ii)

$$\partial_z^{2m}(f(x,r)\cos\theta) = y\sum_{\ell=0}^{m} a_{\ell,m} z^{2\ell}\left(\frac{1}{r}\partial_r\right)^{\ell+m}\left(\frac{f}{r}\right), \tag{2.27}$$

$$\partial_z^{2m+1}(f(x,r)\cos\theta) = y\sum_{\ell=0}^{m} b_{\ell,m} z^{2\ell+1}\left(\frac{1}{r}\partial_r\right)^{\ell+m+1}\left(\frac{f}{r}\right), \tag{2.28}$$

$$\partial_z^{2m}(g(x,r)\sin\theta) = \sum_{\ell=0}^{m} c_{\ell,m} z^{2\ell+1}\left(\frac{1}{r}\partial_r\right)^{\ell+m}\left(\frac{g}{r}\right), \tag{2.29}$$

$$\partial_z^{2m-1}(g(x,r)\sin\theta) = \sum_{\ell=0}^{m} d_{\ell,m} z^{2\ell}\left(\frac{1}{r}\partial_r\right)^{\ell+m-1}\left(\frac{g}{r}\right), \tag{2.30}$$

for some constants $a_{\ell,m}$, $b_{\ell,m}$, $c_{\ell,m}$, *and* $d_{\ell,m}$.

Proof. Part (i) follows straightforwardly from (2.24) and the following identity:

$$\left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)(F_1+\sin\theta G_1) = (\cos\theta\,\partial_r F_1)+\sin\theta\left(\cos\theta\,\partial_r G_1 - \frac{\partial_\theta(F_1+\sin\theta\, G_1)}{r}\right). \tag{2.31}$$

For part (ii), equations (2.27)–(2.30) result from substituting $\cos\theta = \frac{y}{r}$, $\sin\theta = \frac{z}{r}$, followed by straightforward calculations. We omit the details. ∎

Now we proceed to show that all the mixed derivatives of orders up to k are also continuous on $y \geq 0$. For simplicity of presentation, we consider mixed derivatives performed in the following order $\partial_y^j\partial_z^q\partial_x^i$. We start with $\partial_y^j\partial_z^q\partial_x^i u_x$ and analyze for q even and odd separately.

When $q = 2m+1$, we derive from (2.25) and (2.29) that

$$\begin{aligned} &\partial_y^j\partial_z^{2m+1}\partial_x^i u_x(x,y,0)\\ =\;& \partial_y^j\partial_z^{2m}(\sin\theta\,\partial_r\partial_x^i u_x(x,r))\big|_{\theta=0,r=y}\\ =\;& \partial_y^j\left(\sum_{\ell=0}^{m} c_{\ell,m} z^{2\ell+1}\left(\frac{1}{r}\partial_r\right)^{\ell+m}\left(\frac{\partial_r\partial_x^i u_x(x,r)}{r}\right)\right)\big|_{z=0,r=y}\\ =\;& 0. \end{aligned} \tag{2.32}$$

Next, when $q = 2m$, it follows from (2.25), (2.26), (2.30), (2.20), and (2.22) that

$$\begin{aligned} &\partial_y^j\partial_z^{2m}\partial_x^i u_x(x,y,0)\\ =\;& \partial_y^j\partial_z^{2m-1}(\sin\theta\,\partial_r\partial_x^i u_x(x,r))|_{\theta=0,r=y}\\ =\;& \left(\partial_r^j\partial_z^{2m-1}(\sin\theta\,\partial_r\partial_x^i u_x) + \sin\theta\, G\right)|_{\theta=0,r=y}\\ =\;& \partial_r^j\sum_{\ell=0}^{m} d_{\ell,m}(r\sin\theta)^{2\ell}\left(\frac{1}{r}\partial_r\right)^{\ell+m-1}\left(\frac{\partial_r\partial_x^i u_x(x,r)}{r}\right)|_{\theta=0,r=y}\\ =\;& d_{0,m}\partial_r^j\left(\tfrac{1}{r}\partial_r\right)^m\partial_x^i u_x(x,r)|_{r=y}\\ =\;& d_{0,m}\partial_r^j\left(\tfrac{1}{r}\partial_r\right)^{m+1}\left(r\partial_x^i\psi(x,r)\right)|_{r=y}. \end{aligned} \tag{2.33}$$

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From Lemma 2 and Taylor's theorem, we have

$$\psi(x,r) = a_1(x)r + a_3(x)r^3 + \cdots + a_{2m-1}(x)r^{2m-1} + R_{2m+1}(\psi),$$

where

$$a_\ell(x) = \frac{1}{\ell!}\partial_r^\ell \psi(x,0^+)$$

and

$$R_{2m+1}(\psi) = \int_0^r \partial_s^{2m+1}\psi(x,s)\frac{(r-s)^{2m}}{(2m)!}ds.$$

From direct calculation, we have

$$\left(\frac{1}{r}\partial_r\right)^{m+1}\left(r\partial_x^i\psi(x,r)\right) = \left(\frac{1}{r}\partial_r\right)^{m+1}\left(r\partial_x^i R_{2m+1}(\psi)\right).$$

In addition, for $j \geq 1$, we can write

$$R_{2m+1}(\psi) = a_{2m+1}(x)r^{2m+1} + \cdots + a_{2m+2n+1}(x)r^{2m+2n+1} + R_{2m+j+1}(\psi),$$

where n is the largest integer, such that $2n < j$. The remainder term R_{2m+j+1} satisfies

$$\begin{aligned} &\partial_r^\ell R_{2m+j+1}(\psi)(x,0^+) = 0, \quad 0 \leq \ell \leq 2m+j, \\ &\partial_r^{2m+j+1}R_{2m+j+1}(\psi)(x,0^+) = \partial_r^{2m+j+1}\psi(x,0^+). \end{aligned} \tag{2.34}$$

Thus, for $j \geq 0$, we have

$$\begin{aligned} \partial_r^j\left(\frac{1}{r}\partial_r\right)^{m+1}(r\partial_x^i\psi(x,r)) &= \partial_r^j\left(\frac{1}{r}\partial_r\right)^{m+1}(r\partial_x^i R_{2m+j+1}(\psi)) \\ &= \sum_{\ell=0}^{m+j+1} C_{\ell,m}\frac{\partial_r^\ell\partial_x^i R_{2m+j+1}(\psi)}{r^{2m+1-\ell+j}} \end{aligned} \tag{2.35}$$

for some constants $C_{\ell,m}$.

From (2.34), (2.35), and l'Hospital's rule we conclude that

$$\partial_r^j\left(\frac{1}{r}\partial_r\right)^{m+1}(r\partial_x^i\psi)(x,0^+) = \left(\sum_{\ell=0}^{m+1}\sum_{p=0}^{j}\frac{C_{\ell,m}}{(2m+1-\ell+j-p)!}\right)\partial_r^{2m+1+j}\partial_x^i\psi(x,0^+). \tag{2.36}$$

Since $\psi \in C^{k+1}(R \times \overline{R^+})$, it follows from (2.33), (2.36), and (2.32) that $\partial_y^j\partial_z^q\partial_x^i u_x(x,y,0)$ is continuous and bounded up to $y = 0^+$ for $j+q+i \leq k$.

Next, we consider the mixed derivatives of u_y and u_z. It suffices to calculate $\partial_y^j\partial_z^q\partial_x^i(f(x,r)\cos\theta + g(x,r)\sin\theta)|_{\theta=0,r=y}$, where f and g are either $\pm\partial_x\psi$ or $\pm u$.

When $q = 2m$, it follows from (2.27) and (2.29) that

$$\begin{aligned} &\partial_y^j\partial_z^{2m}\partial_x^i(f(x,r)\cos\theta + g(x,r)\sin\theta)|_{\theta=0,r=y} \\ = \ &\partial_y^j\partial_z^{2m}(\partial_x^i f(x,r)\cos\theta + \partial_x^i g(x,r)\sin\theta)|_{\theta=0,r=y} \\ = \ &a_{0,m}\partial_r^j\left(r\left(\tfrac{1}{r}\partial_r\right)^m\left(\tfrac{\partial_x^i f}{r}\right)\right)|_{r=y}. \end{aligned}$$

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From (2.18–2.19), both $-\partial_x\psi(x,r)$ and $u(x,r)$ have local expansions of the form

$$b_1(x)r + b_3(x)r^3 + \cdots + b_{2m-1}(x)r^{2m-1} + R_{2m+1}.$$

Following the same argument above, we can show that both $\partial_y^j\partial_z^{2m}\partial_x^i u_y$ and $\partial_y^j\partial_z^{2m}\partial_x^i u_z$ are continuous and bounded up to $y = 0^+$ for $j + 2m + i \le k$. The calculations for $\partial_y^j\partial_z^{2m+1}\partial_x^i u_y$ and $\partial_y^j\partial_z^{2m+1}\partial_x^i u_z$ are similar. This completes the proof of part (b) of Lemma 2. ◻

In view of Lemma 2, we now introduce the following function space.

Definition 2.

$$C_s^k(R \times \overline{R^+}) = \{f(x,r) \in C^k(R \times \overline{R^+}), \quad \partial_r^{2j} f(x,0^+) = 0, \quad 0 \le 2j \le k\}.$$

We can recast Lemma 2 as follows.

Lemma 2′. For $k \ge 0$,

$$\mathcal{C}_s^k(R^3,R^3) = \{u\boldsymbol{e}_\theta + \nabla \times (\psi\boldsymbol{e}_\theta) \,|\, u \in C_s^k(R \times \overline{R^+}), \quad \psi \in C_s^{k+1}(R \times \overline{R^+})\}. \tag{2.37}$$

In the following sections, we will construct natural Sobolev spaces for axisymmetric divergence free vector fields, derive the counter part of Lemma 2 in these Sobolev spaces, and establish various regularity and equivalence results. These results rely heavily on the expression and pole condition in Lemma 2. We list below a few related lemmas which will be used in later sections.

Lemma 3. *Suppose $\boldsymbol{u} \in \mathcal{C}_s^k(R^3,R^3)$ is given by the representation $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi\boldsymbol{e}_\theta)$ with $u \in C_s^k(R \times \overline{R^+})$ and $\psi \in C_s^{k+1}(R \times \overline{R^+})$. Then $(\nabla\times)^\ell \boldsymbol{u} \in \mathcal{C}_s^{k-\ell}(R^3,R^3)$ and*

$$(\nabla\times)^{2m}\boldsymbol{u} = (-1)^m\Big((\mathscr{L}^m u)\boldsymbol{e}_\theta + \nabla \times \big((\mathscr{L}^m\psi)\boldsymbol{e}_\theta\big)\Big) \quad \text{if} \quad 2m \le k,$$

$$(\nabla\times)^{2m+1}\boldsymbol{u} = (-1)^{m+1}(\mathscr{L}^{m+1}\psi)\boldsymbol{e}_\theta + (-1)^m\nabla \times \big((\mathscr{L}^m u)\boldsymbol{e}_\theta\big) \quad \text{if} \quad 2m+1 \le k,$$

where

$$\mathscr{L} := \nabla^2 - \frac{1}{r^2} = \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_x^2\right) - \frac{1}{r^2}.$$

Moreover,

$$\mathscr{L}^m u \in C_s^{k-2m}(R \times \overline{R^+}) \quad \text{if} \quad 2m \le k,$$

$$\mathscr{L}^{m+1}\psi \in C_s^{k-1-2m}(R \times \overline{R^+}) \quad \text{if} \quad 2m+1 \le k.$$

Proof. For any $\phi \in C_s^i(R \times \overline{R^+})$, we have $\phi\,\boldsymbol{e}_\theta \in \mathcal{C}_s^i$ from Lemma 2 (b). With a straightforward calculation using (2.4), it is easy to verify that for $i \ge 2$,

$$\nabla \times \nabla \times (\phi\,\boldsymbol{e}_\theta) = -(\mathscr{L}\phi)\boldsymbol{e}_\theta. \tag{2.38}$$

On the other hand, it is clear that

$$\nabla \times \nabla \times (\phi\,\boldsymbol{e}_\theta) \in \mathcal{C}_s^{i-2}$$

and therefore from Lemma 2 (a),

$$\mathscr{L}\phi \in C_s^{i-2}(R \times \overline{R^+}). \tag{2.39}$$

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The lemma then follows from (2.38) and (2.39). ◻

LEMMA 4. *If $v \in C^k(R \times \overline{R^+})$ and $v(x, 0^+) = 0$, then*

$$\lim_{r \to 0^+} j\partial_r^{j-1} \left(\frac{v(x,r)}{r} \right) = \partial_r^j v(x, 0^+), \qquad 1 \le j \le k. \tag{2.40}$$

Proof. Since $v \in C^k(R \times \overline{R^+})$, we have

$$v(x,r) = a_1(x)r + a_2(x)r^2 + \cdots + a_{k-1}(x)r^{k-1} + R_k(v) \tag{2.41}$$

from Taylor's theorem. Here

$$a_\ell(x) = \frac{1}{\ell!}\partial_r^\ell v(x, 0^+),$$

$$R_k(v) = \int_0^r \partial_r^k v(x,s) \frac{(r-s)^{k-1}}{(k-1)!} ds$$

and

$$\partial_r^\ell R_k(v)(x, 0^+) = 0, \quad 0 \le \ell \le k-1, \qquad \partial_r^k R_k(v)(x, 0^+) = \partial_r^k v(x, 0^+). \tag{2.42}$$

From (2.41), it follows that

$$\partial_r^{k-1} \left(\frac{v(x,r)}{r} \right) = \partial_r^{k-1} \left(\frac{R_k(v)}{r} \right) = \sum_{\ell=0}^{k-1} C_{k-1}^\ell (-1)^\ell \ell! \frac{\partial_r^{k-\ell-1} R_k(v)}{r^{\ell+1}}. \tag{2.43}$$

The assertion (2.40) is obvious for $j < k$. For $j = k$, from (2.42), (2.43), and l'Hospital's rule, we can easily derive

$$\lim_{r \to 0^+} \partial_r^{k-1} \left(\frac{v(x,r)}{r} \right) = \left(\sum_{\ell=0}^{k-1} C_{k-1}^\ell (-1)^\ell \frac{1}{\ell+1} \right) \partial_r^k v(x, 0^+) = \frac{1}{k} \partial_r^k v(x, 0^+).$$

This completes the proof of Lemma 4. ◻

LEMMA 5. *If $v \in C^{2m}(R \times \overline{R^+}) \cap C_s^{2m-2}(R \times \overline{R^+})$, then*

$$\partial_r^{2m-2} \mathscr{L} v(\cdot, 0^+) \equiv 0 \text{ if and only if } \partial_r^{2m} v(\cdot, 0^+) \equiv 0. \tag{2.44}$$

Proof. From

$$\left(\nabla^2 - \frac{1}{r^2} \right) v = \left(\partial_x^2 v + \partial_r^2 v + \partial_r \left(\frac{v}{r} \right) \right),$$

we have

$$\partial_r^{2m-2} \mathscr{L} v = \left(\partial_x^2 \partial_r^{2m-2} v + \partial_r^{2m} v + \partial_r^{2m-1} \left(\frac{v}{r} \right) \right).$$

Since $v \in C_s^{2m-2}(R \times \overline{R^+})$, it follows from Lemma 4 that

$$\partial_r^{2m-2} \mathscr{L} v(x, 0^+) = \frac{2m+1}{2m} \partial_r^{2m} v(x, 0^+)$$

and the assertion follows. ◻

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2.2. Sobolev spaces. In this section, we will construct a family of Sobolev spaces $H_s^k(R \times R^+)$ and show a counterpart for (2.37) in these Sobolev spaces: A weak solenoidal axisymmetric vector field admits the representation (2.17) with $u(x,r)$ and $\psi(x,r)$ in H_s^k. Moreover, both u and ψ, together with certain even order derivatives, have vanishing traces on $r = 0^+$.

We start with the following identity for general solenoidal vector fields.

LEMMA 6. *If* $\boldsymbol{u} \in C^k(R^3, R^3) \cap H^k(R^3, R^3)$ *and* $\nabla \cdot \boldsymbol{u} = 0$, $k \geq 0$, *then*

$$\|\boldsymbol{u}\|^2_{H^k(R^3,R^3)} = \sum_{\ell=0}^{k} \|(\nabla\times)^\ell \boldsymbol{u}\|^2_{L^2(R^3,R^3)}. \tag{2.45}$$

Proof. We prove (2.45) for ℓ even and odd separately.

Since $\nabla \cdot \boldsymbol{u} = 0$, it follows that $\nabla \times \nabla \times \boldsymbol{u} = -\nabla^2 \boldsymbol{u}$. Thus, if $\ell = 2m$ is even, we can write

$$\|(\nabla\times)^{2m}\boldsymbol{u}\|_{L^2(R^3,R^3)} = \|(\nabla^2)^m \boldsymbol{u}\|_{L^2(R^3,R^3)}. \tag{2.46}$$

When $m = 1$ and $u \in C^k(R^3)$, we can integrate by parts to get

$$\int_{R^3} |\nabla^2 u|^2 = \int_{R^3} \left(\sum_{i_1=1}^{3} \partial^2_{i_1} u \right)^2 = \int_{R^3} \sum_{i_1,i_2=1}^{3} \partial^2_{i_1} u \partial^2_{i_2} u = \int_{R^3} \sum_{i_1,i_2=1}^{3} (\partial_{i_1}\partial_{i_2} u)^2.$$

Similarly, when $m = 2$,

$$\begin{aligned} \int_{R^3} |(\nabla^2)^2 u|^2 &= \int_{R^3} \left(\left(\sum_{i=1}^{3} \partial_i^2 \right)^2 u \right)^2 = \int_{R^3} \sum_{i_1,i_2,i_3,i_4=1}^{3} (\partial^2_{i_1}\partial^2_{i_2} u)(\partial^2_{i_3}\partial^2_{i_4} u) \\ &= \sum_{i_1,i_2,i_3,i_4=1}^{3} \int_{R^3} (\partial_{i_1}\partial_{i_2}\partial_{i_3}\partial_{i_4} u)^2. \end{aligned}$$

It is, therefore, easy to see that

$$\int_{R^3} |(\nabla^2)^m u|^2 = \sum_{i_1,\ldots,i_{2m}=1}^{3} \int_{R^3} (\partial_{i_1} \cdots \partial_{i_{2m}} u)^2$$

and consequently for $\boldsymbol{u} \in C^k(R^3, R^3)$, $2m \leq k$,

$$\|(\nabla^2)^m \boldsymbol{u}\|^2_{L^2(R^3,R^3)} = \sum_{i_1,\ldots,i_{2m}=1}^{3} \|(\partial_{i_1} \cdots \partial_{i_{2m}})\boldsymbol{u}\|^2_{L^2(R^3,R^3)}. \tag{2.47}$$

On the other hand, if $\ell = 2m + 1$ is odd, we first write

$$(\nabla\times)^{2m+1}\boldsymbol{u} = \nabla \times (-(\nabla^2))^m \boldsymbol{u} = (-1)^m \nabla \times (\nabla^2)^m \boldsymbol{u},$$

then apply the identity

$$\|\nabla \boldsymbol{v}\|^2_{L^2(R^3,R^{3\times 3})} = \|\nabla \times \boldsymbol{v}\|^2_{L^2(R^3,R^3)} + \|\nabla \cdot \boldsymbol{v}\|^2_{L^2(R^3)}$$

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to get

$$\|(\nabla\times)^{2m+1}\boldsymbol{u}\|_{L^2(R^3,R^3)} = \|\nabla\times(\nabla^2)^m\boldsymbol{u}\|_{L^2(R^3,R^3)} = \|(\nabla^2)^m\nabla\boldsymbol{u}\|_{L^2(R^3,R^3)} \tag{2.48}$$

and from (2.47),

$$\begin{aligned}\|(\nabla^2)^m\nabla\boldsymbol{u}\|^2_{L^2(R^3,R^3)} &= \sum_{i,j=1}^{3}\|(\nabla^2)^m\partial_i u_j\|^2_{L^2(R^3)}\\ &= \sum_{i,j=1}^{3}\sum_{i_1,\dots,i_{2m}=1}^{3}\int_{R^3}(\partial_{i_1}\cdots\partial_{i_{2m}}\partial_i u_j)^2\\ &= \sum_{i_1,\dots,i_{2m+1}=1}^{3}\|\left(\partial_{i_1}\cdots\partial_{i_{2m+1}}\right)\boldsymbol{u}\|^2_{L^2(R^3,R^3)}.\end{aligned} \tag{2.49}$$

It follows from (2.46), (2.47), (2.48), and (2.49) that

$$\|\boldsymbol{u}\|^2_{H^k(R^3,R^3)} = \sum_{\ell=0}^{k}\sum_{i_1,\dots,i_\ell=1}^{3}\|\partial_{i_1}\cdots\partial_{i_\ell}\boldsymbol{u}\|^2_{L^2(R^3,R^3)} = \sum_{\ell=0}^{k}\|(\nabla\times)^\ell\boldsymbol{u}\|^2_{L^2(R^3,R^3)}.$$

This completes the proof of Lemma 6. ∎

In Lemma 7 below, we will derive an equivalent representation of the Sobolev norms for axisymmetric solenoidal vector fields. We first introduce the following weighted Sobolev space for axisymmetric solenoidal vector fields. Let $a, b \in C^0(R\times\overline{R^+})$, and we define the weighted L^2 inner product and norm

$$\langle a,b\rangle = \int_{-\infty}^{\infty}\int_0^{\infty} a(x,r)b(x,r)\,r dx dr, \qquad \|a\|_0^2 = \langle a,a\rangle, \tag{2.50}$$

and for $a, b \in C^1_s(R\times\overline{R^+})$, we define the weighted H^1 inner product and norm

$$[a,b] = \langle\partial_x a,\partial_x b\rangle + \langle\partial_r a,\partial_r b\rangle + \left\langle\frac{a}{r},\frac{b}{r}\right\rangle, \qquad |a|_1^2 = [a,a], \tag{2.51}$$

and we define

$$\|a\|_1^2 = \|a\|_0^2 + |a|_1^2. \tag{2.52}$$

When $a \in C^1_s(R\times\overline{R^+})$ and $b \in C^1_s(R\times\overline{R^+})\cap C^2(R\times R^+)$, we also have the following identity from integration by parts:

$$\langle a,\mathscr{L}b\rangle = -[a,b].$$

If $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla\times(\psi\boldsymbol{e}_\theta)$, with $u \in C^0(R\times\overline{R^+})$ and $\psi \in C^1_s(R\times\overline{R^+})$, it is easy to see that

$$\|\boldsymbol{u}\|^2_{L^2(R^3,R^3)} = \|u\|_0^2 + |\psi|_1^2. \tag{2.53}$$

Higher order Sobolev norms can be defined similarly in terms of u and ψ.

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DEFINITION 3. *For $a \in C_s^k(R \times \overline{R^+})$ and $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) \in \mathcal{C}_s^k(R^3, R^3)$, we define*

$$\begin{aligned}
\|a\|^2_{H_s^{2m}(R\times R^+)} &:= \sum_{\ell=0}^{m-1} \|\mathscr{L}^\ell a\|_1^2 + \|\mathscr{L}^m a\|_0^2, && 2m \le k,\\
\|a\|^2_{H_s^{2m+1}(R\times R^+)} &:= \sum_{\ell=0}^{m} \|\mathscr{L}^\ell a\|_1^2, && 2m+1 \le k,\\
\|\boldsymbol{u}\|^2_{\mathcal{H}_s^{2m}(R\times R^+,R^3)} &:= |\psi|_1^2 + \sum_{\ell=0}^{m-1} \|\mathscr{L}^\ell u\|_1^2 + \sum_{\ell=1}^{m} \|\mathscr{L}^\ell \psi\|_1^2 + \|\mathscr{L}^m u\|_0^2, && 2m \le k,\\
\|\boldsymbol{u}\|^2_{\mathcal{H}_s^{2m+1}(R\times R^+,R^3)} &:= |\psi|_1^2 + \sum_{\ell=0}^{m} \|\mathscr{L}^\ell u\|_1^2 + \sum_{\ell=1}^{m} \|\mathscr{L}^\ell \psi\|_1^2 + \|\mathscr{L}^{m+1} \psi\|_0^2, && 2m+1 \le k.
\end{aligned}$$

When $k = 0$, we denote $\|a\|_{L_s^2(R\times R^+)} = \|a\|_{H_s^0(R\times R^+)}$ and $\|\boldsymbol{u}\|_{\mathcal{L}_s^2(R\times R^+,R^3)} = \|\boldsymbol{u}\|_{\mathcal{H}_s^0(R\times R^+,R^3)}$ by convention. In view of Lemma 2, Lemma 3, Lemma 6, and (2.53), we have proved the following.

LEMMA 7. *If $\boldsymbol{u} \in \mathcal{C}_s^k(R^3, R^3)$, $k \ge 0$, then*

$$\|\boldsymbol{u}\|_{H^k(R^3,R^3)} = \|\boldsymbol{u}\|_{\mathcal{H}_s^k(R\times R^+,R^3)}.$$

We can now define the Sobolev spaces for axisymmetric solenoidal vector fields following standard procedure. Denote by C_0 the space of compactly supported functions, and we define the following.

DEFINITION 4.

$$\begin{aligned}
L_s^2(R\times R^+) &:= \textit{completion of } C_s^0(R\times \overline{R^+}) \cap C_0(R\times \overline{R^+}), \textit{ with respect to } \|\cdot\|_0,\\
\hat{H}_s^1(R\times R^+) &:= \textit{completion of } C_s^1(R\times \overline{R^+}) \cap C_0(R\times \overline{R^+}), \textit{ with respect to } |\cdot|_1,\\
H_s^k(R\times R^+) &:= \textit{completion of } C_s^k(R\times \overline{R^+}) \cap C_0(R\times \overline{R^+}), \textit{ with respect to } \|\cdot\|_{H_s^k(R\times R^+)},\\
\mathcal{H}_s^k(R\times R^+, R^3) &:= \textit{completion of } \mathcal{C}_s^k(R^3,R^3) \cap C_0(R^3,R^3), \textit{ with respect to } \|\cdot\|_{\mathcal{H}_s^k(R\times R^+,R^3)}.
\end{aligned}$$

With the spaces introduced above, it is easy to see that a necessary and sufficient condition for $a \in H_s^k(R\times R^+)$, $k \ge 1$ is

$$\begin{cases} \mathscr{L}^\ell a \in H_s^1(R\times R^+) & \text{for all } 0 \le 2\ell \le k-1;\\ \mathscr{L}^m a \in L_s^2(R\times R^+) & \text{for all } 2 \le 2m \le k. \end{cases}$$

As a consequence, we have the following characterization for the divergence free Sobolev spaces $\mathcal{H}_s^k(R\times R^+, R^3)$.

LEMMA 8. *The following statements are equivalent:*

1. $\boldsymbol{u} \in \mathcal{H}_s^k(R\times R^+, R^3)$,
2. $\boldsymbol{u} \in H^k(R^3,R^3)$, $\nabla\cdot\boldsymbol{u} = 0$ *and $\boldsymbol{u}$ is axisymmetric,*
3. $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla\times(\psi\boldsymbol{e}_\theta)$, *with $u \in H_s^k(R\times R^+)$, $\psi \in \hat{H}_s^1(R\times R^+)$ and, if $k \ge 1$, $\mathscr{L}\psi \in H_s^{k-1}(R\times R^+)$.*

When the above statements hold, we have

$$\|\boldsymbol{u}\|_{H^k(R^3,R^3)} = \|\boldsymbol{u}\|_{\mathcal{H}_s^k(R\times R^+,R^3)}. \tag{2.54}$$

Lemma 8 follows from Lemma 3, Lemma 7, and standard density argument. We omit the details.

Finally, the counterpart of (2.18) and (2.19), for $\boldsymbol{u} \in \mathcal{H}_s^k(R\times R^+, R^3)$, is given the following trace Lemma and Corollary.

LEMMA 9. *If $v \in \hat{H}_s^1(R\times R^+)$, then the trace of v on $r = 0$ vanishes.*

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Proof. For any $v \in C^1\left(R \times \overline{R^+}\right) \cap C_0\left(R \times \overline{R^+}\right)$, we have

$$\int_R |v(x,0)|^2\,dx = -2\int\int_{R\times R^+} v\partial_r v\,dx\,dr \le \int\int_{R\times R^+}\left(\frac{v^2}{r^2} + (\partial_r v)^2\right) r\,dx\,dr \le \|v\|_1^2.$$

Since $v(x,0) = 0$ for $v \in C_s^1(R \times \overline{R^+})$, the lemma follows from standard density argument. ∎

Using the same density argument, we have the following.

COROLLARY 2.

(i) *If $v \in H_s^k(R \times R^+)$, then the trace of $\mathscr{L}^\ell \partial_x^n v$ on $r = 0$ vanishes provided $2\ell + n \le k - 1$.*

(ii) *If $v \in H_s^k(R \times R^+)$, then the trace of $\partial_r^{2\ell} v$ on $r = 0$ vanishes provided $2\ell \le k - 1$.*

Example 1. Take $\boldsymbol{u} = u\boldsymbol{e}_\theta$ with $u = r^2 e^{-r}$. Note that $u = O(r^2)$ near the axis. Similar functions can be found in literature as initial data in numerical search for finite time singularities. Although $u \in C^\infty(R \times \overline{R^+})$ and $\boldsymbol{u}$ may appear to be a smooth vector field, it is easy to verify that $\mathscr{L}u(x, 0^+) \ne 0$. Thus, from Lemma 2, Lemma 8, and Lemma 9, $\boldsymbol{u}$ is neither in $C^2(R^3, R^3)$ nor in $H^3(R^3, R^3)$.

3. Axisymmetric Navier–Stokes equations and equivalence results.

The axisymmetric Navier–Stokes equation (1.5) can be formally derived from (1.4). From Lemma 2, a smooth solution of (1.4) gives rise to a smooth solution of (1.5). However, it is not clear whether smooth solutions of (1.5) also give rise to smooth solutions of (1.4). For example, take $\nu = 0$ in (1.5) and consider the Euler equation

$$\begin{aligned} &\partial_t u + u_x\partial_x u + u_r\partial_r u + \tfrac{u_r}{r}u = 0, \\ &\partial_t \omega + u_x\partial_x \omega + u_r\partial_r \omega - \tfrac{u_r}{r}\omega = \tfrac{1}{r}\partial_x(u^2), \\ &\omega = -\mathscr{L}\psi. \end{aligned} \tag{3.1}$$

It is easy to see that

$$\begin{cases} \boldsymbol{u} = u\boldsymbol{e}_\theta, \quad u(t,x,r) = f(r), \\ \omega = \psi \equiv 0 \end{cases} \tag{3.2}$$

gives rise to an *exact* stationary solution to (3.1) for any function $f(r) \in C^k(R \times \overline{R^+})$, including the one given in Example 1. In other words, it is possible to have a solution in the class

$$\begin{aligned} &\psi(t;x,r) \in C^1\left(0,T;C^{k+1}\left(R\times\overline{R^+}\right)\right), \\ &u(t;x,r) \in C^1\left(0,T;C^{k}\left(R\times\overline{R^+}\right)\right), \\ &\omega(t;x,r) \in C^1\left(0,T;C^{k-1}\left(R\times\overline{R^+}\right)\right) \end{aligned} \tag{3.3}$$

with a genuine singularity on $r = 0$ as described in Example 1. This singularity is invisible to the $C^k(R \times \overline{R^+})$ norm. In addition, it may well persist in time. In section 3.1, we will show that the persistence of the pole singularity is indeed generic for the Euler equation.

3.1. Propagation and persistence of pole singularity.

In Euler (3.1), both u and ω transport with the velocity $(u_x, u_r) = (\partial_r\psi + \frac{\psi}{r}, -\partial_x\psi)$. The equation for ψ

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is elliptic and one needs to impose one boundary condition for ψ. This is naturally given by

$$\psi(x,0)=0 \tag{3.4}$$

in view of Lemma 2. Consequently, the r component of the velocity field $u_r=-\partial_x\psi$ vanishes on the boundary $r=0$ and turns it into a characteristic boundary. As a result, the values of both u and ω on $r=0^+$ are completely determined by the value of initial data on $r=0^+$ and the dynamics. Neither u nor ω should be imposed on $r=0$. In the following theorem, we will show that the pole singularity will propagate and remain on the boundary $r=0$. Moreover, we will show that the order of singularity will persist in time as illustrated in the special example mentioned above.

THEOREM 1. *Let* (ψ,u,ω) *be a solution to the axisymmetric Euler equation* (3.1) *in the class*

$$\begin{aligned}&\psi(t;x,r)\in C^0\left([0,T);C^{k+1}\left(R\times\overline{R^+}\right)\right),\\&u(t;x,r)\in C^0\left([0,T);C^{k}\left(R\times\overline{R^+}\right)\right),\\&\omega(t;x,r)\in C^0\left([0,T);C^{k-1}\left(R\times\overline{R^+}\right)\right)\end{aligned} \tag{3.5}$$

with $k\geq 2$ *and*

$$\boldsymbol{u}=\nabla\times(\psi\boldsymbol{e}_\theta)+u\boldsymbol{e}_\theta.$$

Then for $0<t<T$, $0\leq j\leq k$,

$$\boldsymbol{u}(t,\cdot)\in\mathcal{C}_s^j(R^3,R^3)\ \text{if and only if}\ \boldsymbol{u}(0,\cdot)\in\mathcal{C}_s^j(R^3,R^3). \tag{3.6}$$

Proof. From Lemma 3 and (3.5), it suffices to show that, for $0<t\leq T$, $0\leq j\leq k$,

$$\begin{cases}\partial_r^{2\ell}u(t;\cdot,0^+)\equiv 0 & \text{for all } 2\ell\leq j,\\ \partial_r^{2n}\psi(t;\cdot,0^+)\equiv 0 & \text{for all } 2n\leq j+1\end{cases}\quad\text{if and only if}\quad\begin{cases}\partial_r^{2\ell}u(0,\cdot,0^+)\equiv 0 & \text{for all } 2\ell\leq j,\\ \partial_r^{2n}\psi(0,\cdot,0^+)\equiv 0 & \text{for all } 2n\leq j+1.\end{cases} \tag{3.7}$$

We will prove (3.7) by induction on j using Lemma 10 below. We first prove the case $j=0$ in part (i) of Lemma 10. The induction from $j=2m$ to $j=2m+1$ and from $j=2m+1$ to $j=2m+2$ are given by parts (ii) and (iii) of Lemma 10, respectively. ∎

LEMMA 10.

(i) *If* (3.5) *holds and*

$$\psi\in C^0\left([0,T),C_s^1\left(R\times\overline{R^+}\right)\right), \tag{3.8}$$

then for $0<t\leq T$,

$$u(t,\cdot,\cdot)\in C_s^0(R\times\overline{R^+})\quad\text{if and only if}\quad u(0,\cdot,\cdot)\in C_s^0(R\times\overline{R^+}).$$

(ii) *If* $2m+1\leq k$, (3.5) *holds and*

$$\psi\in C^0\big([0,T),C_s^{2m}(R\times\overline{R^+})\big),\quad u\in C^0\big([0,T),C_s^{2m}(R\times\overline{R^+})\big), \tag{3.9}$$

then for $0<t\leq T$,

$$\psi(t,\cdot,\cdot)\in C_s^{2m+2}(R\times\overline{R^+})\quad\text{if and only if}\quad\psi(0,\cdot,\cdot)\in C_s^{2m+2}(R\times\overline{R^+}).$$

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(iii) *If* $2m+2 \le k$, (3.5) *holds and*

$$\psi \in C^0\big([0,T), C_s^{2m+2}(R\times \overline{R^+})\big), \quad u \in C^0\big([0,T), C_s^{2m}(R\times \overline{R^+})\big), \tag{3.10}$$

then for $0 < t \le T$,

$$u(t,\cdot,\cdot) \in C_s^{2m+2}(R\times \overline{R^+}) \quad \textit{if and only if} \quad u(0,\cdot,\cdot) \in C_s^{2m+2}(R\times \overline{R^+}).$$

Proof. Part (i). From the boundary condition (3.4) we know that $u_r(t,x,0^+) = 0$. From Lemma 4, we also have $\lim_{r\to 0^+} \frac{u_r}{r} = -\partial_r\partial_x\psi(t,x,0^+)$ and $u_x(t,x,0^+) = 2(\partial_r\psi|_{r=0^+})$. Therefore, the first equation of (3.1) on $r = 0^+$ reads

$$\partial_t u + 2(\partial_r\psi|_{r=0^+})\partial_x u - (\partial_r\partial_x\psi|_{r=0^+})u = 0.$$

This is a first order linear hyperbolic equation with continuous coefficients in $(t,x) \in (0,T)\times R$ for $u(t,x,0^+)$. Hence, for $0 < t \le T$,

$$u(t,\cdot,0^+) \equiv 0 \text{ if and only if } u(0,\cdot,0^+) \equiv 0.$$

Part (ii). From Lemma 5 we see that

$$\omega \in C^0\big([0,T), C_s^{2m-2}(R\times \overline{R^+})\big). \tag{3.11}$$

Let $v(t,x) = \partial_r^{2m}\omega(t,x,0^+)$, and we can derive a linear hyperbolic equation for $v(t,x)$ by applying ∂_r^{2m} to the terms in the second equation of (3.1) as follows:

$$\begin{aligned} \partial_r^{2m}(u_x\partial_x\omega)\big|_{r=0^+} &= \sum_{\ell=0}^{2m} C_{2m}^\ell \partial_r^\ell \left(\partial_r\psi + \frac{\psi}{r}\right)|_{r=0^+}\partial_x\partial_r^{2m-\ell}\omega|_{r=0^+} \\ &= \sum_{\ell=0}^{2m}\left(1+\frac{1}{\ell+1}\right) C_{2m}^\ell \partial_r^{\ell+1}\psi\big|_{r=0^+}\partial_x\partial_r^{2m-\ell}\omega\big|_{r=0^+}, \end{aligned} \tag{3.12}$$

$$\partial_r^{2m}(u_r\partial_r\omega)\big|_{r=0^+} = -\sum_{\ell=0}^{2m} C_{2m}^\ell \partial_r^\ell(\partial_x\psi)\big|_{r=0^+}\partial_r\partial_r^{2m-\ell}\omega\big|_{r=0^+}, \tag{3.13}$$

$$\begin{aligned} \partial_r^{2m}\left(-\frac{u_r}{r}\omega\right)|_{r=0^+} &= \sum_{\ell=0}^{2m} C_{2m}^\ell\partial_r^\ell\left(\partial_x\frac{\psi}{r}\right)|_{r=0^+}\partial_r^{2m-\ell}\omega|_{r=0^+} \\ &= \sum_{\ell=0}^{2m}\frac{1}{\ell+1}C_{2m}^\ell\partial_x\partial_r^{\ell+1}\psi|_{r=0^+}\partial_r^{2m-\ell}\omega|_{r=0^+}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \partial_r^{2m}\left(\frac{u}{r}\partial_x u\right)|_{r=0^+} &= \sum_{\ell=0}^{2m} C_{2m}^\ell\partial_r^\ell\left(\frac{u}{r}\right)|_{r=0^+}\partial_x\left(\partial_r^{2m-\ell}u\right)|_{r=0^+} \\ &= \sum_{\ell=0}^{2m}\frac{1}{\ell+1}C_{2m}^\ell\partial_r^{\ell+1}u|_{r=0^+}\partial_x\partial_r^{2m-\ell}u|_{r=0^+}. \end{aligned} \tag{3.15}$$

In (3.12)–(3.15), we have used Lemma 4 to get

$$\partial_r^\ell\left(\frac{\psi}{r}\right)|_{r=0^+} = \frac{1}{\ell+1}\partial_r^{\ell+1}\psi|_{r=0^+}. \tag{3.16}$$

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Next, from (3.9)

$$\partial_r^{2\ell}\psi|_{r=0^+} = 0, \quad \partial_r^{2\ell}u|_{r=0^+} = 0 \quad \text{for } \ell \le m \tag{3.17}$$

and from (3.11)

$$\partial_r^{2\ell}\omega|_{r=0^+} = 0 \quad \text{for } \ell \le m-1. \tag{3.18}$$

It follows that all the terms on the right-hand side of (3.12)–(3.15) vanish except $\ell = 0$ in (3.12, 3.14) and $\ell = 1$ in (3.13). In summary, we have

$$\partial_r^{2m}(u_x\partial_x\omega)\big|_{r=0^+} = 2(\partial_r\psi\big|_{r=0^+})\partial_x v, \tag{3.19}$$

$$\partial_r^{2m}(u_r\partial_r\omega)\big|_{r=0^+} = -2j(\partial_r\partial_x\psi\big|_{r=0^+})v, \tag{3.20}$$

$$\partial_r^{2m}\left(\frac{-u_r}{r}\omega\right)\big|_{r=0^+} = (\partial_r\partial_x\psi\big|_{r=0^+})v, \tag{3.21}$$

$$\partial_r^{2m}\left(\frac{u}{r}\partial_x u\right)\big|_{r=0^+} = 0. \tag{3.22}$$

Thus, we end up with a first order hyperbolic equation with smooth coefficients for v

$$\partial_t v + 2(\partial_r\psi\big|_{r=0^+})\partial_x v - (2m-1)(\partial_r\partial_x\psi\big|_{r=0^+})v = 0.$$

It follows that for $0 < t \le T$,

$$\partial_r^{2m}\omega(t,\cdot,0^+) \equiv 0 \text{ if and only if } \partial_r^{2m}\omega(0,\cdot,0^+) \equiv 0, \tag{3.23}$$

that is, in view of Lemma 5 and (3.9),

$$\partial_r^{2m+2}\psi(t,\cdot,0^+) \equiv 0 \text{ if and only if } \partial_r^{2m+2}\psi(0,\cdot,0^+) \equiv 0$$

for $0 < t \le T$.

Part (iii). Let $z(t,x) = \partial_r^{2m+2}u(t,x,0^+)$. Following a similar calculation as in part (ii), we have

$$\partial_r^{2m+2}(u_x\partial_x u)\big|_{r=0^+} = \sum_{\ell=0}^{2m+2} C_{2m+2}^{\ell}(\partial_r^{\ell}u_x)\big|_{r=0^+}\partial_x(\partial_r^{2m+2-\ell}u)\big|_{r=0^+} = 2(\partial_r\psi\big|_{r=0^+})\partial_x z \tag{3.24}$$

$$\begin{aligned}\partial_r^{2m+2}(u_r\partial_r u)\big|_{r=0^+} &= \sum_{\ell=0}^{2m+2} C_{2m+2}^{\ell}\left(\partial_r^{\ell}u_r\right)\big|_{r=0^+}\left(\partial_r^{2m+3-\ell}u\right)\big|_{r=0^+} \\ &= -(2m+2)(\partial_r\partial_x\psi\big|_{r=0^+})z\end{aligned} \tag{3.25}$$

and

$$\partial_r^{2m+2}\left(\frac{u_r}{r}u\right)\big|_{r=0^+} = \sum_{\ell=0}^{2m+2} C_{2m+2}^{\ell}\partial_r^{\ell}\left(\frac{-\partial_x\psi}{r}\right)\big|_{r=0^+}\partial_r^{2m+2-\ell}u = -(\partial_r\partial_x\psi\big|_{r=0^+})z. \tag{3.26}$$

We, therefore, obtain a first order linear hyperbolic equation with smooth coefficients for z

$$\partial_t z + 2(\partial_r\psi\big|_{r=0^+})\partial_x z - (2m+3)(\partial_r\partial_x\psi\big|_{r=0^+})z = 0.$$

Therefore, we have proved that for $0 < t \le T$,

$$\partial_r^{2m+2}u(t,\cdot,0^+) \equiv 0 \text{ if and only if } \partial_r^{2m+2}u(0,\cdot,0^+) \equiv 0.$$

This completes the proof of part (iii) and hence the proof of Theorem 1. ∎

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3.2. Classical solutions of axisymmetric Navier–Stokes equations. Theorem 1 reveals the subtlety of the pole singularity. In the case of the Navier–Stokes equation,

$$\begin{aligned} &\partial_t u + u_x \partial_x u + u_r \partial_r u + \tfrac{u_r}{r} u = \nu \mathscr{L} u, \\ &\partial_t \omega + u_x \partial_x \omega + u_r \partial_r \omega - \tfrac{u_r}{r} \omega = \tfrac{1}{r} \partial_x (u^2) + \nu \mathscr{L} \omega, \\ &\omega = -\mathscr{L} \psi, \end{aligned} \tag{3.27}$$

with $\nu > 0$, and we have an elliptic-parabolic system on a semibounded region $\{r > 0\}$. We expect a certain regularizing effect to take place. In the case where the swirling velocity u is zero, there exists a unique global smooth solution [10, 20]. However, with the swirl velocity, whether or not initially smooth data develops singularity in finite time, is still a major open problem. A fundamental regularity result concerning the solution of the Navier–Stokes equation is given in the pioneering work of Caffarelli, Kohn, and Nirenberg [3]: The one-dimensional Hausdorff measure of the singular set is zero. As a consequence, the only possible singularity for axisymmetric Navier–Stokes flows would be on the axis of rotation. Further results on partial regularity for axisymmetric flow can be found in [2, 17, 4, 9, 5]. A recent breakthrough concerning the subtle behavior of the axisymmetric Navier–Stokes equation can be found in [8].

In contrast to the case of the Euler equation, the equivalence theorem that we present below rules out the possibility of persistence of the pole singularity for solutions which are smooth up to the boundary $r = 0$. From standard PDE theory, we need to assign boundary values for (ψ, u, ω). The leading order pole conditions (2.18), (2.19) would suffice:

$$\psi(x, 0) = u(x, 0) = \omega(x, 0) = 0. \tag{3.28}$$

It is, therefore, a natural question to ask if a smooth solution of (3.27), (3.28) in the class

$$\begin{aligned} &\psi(t; x, r) \in C^1 \left(0, T; C^{k+1} \left(R \times \overline{R^+}\right)\right), \\ &u(t; x, r) \in C^1 \left(0, T; C^{k} \left(R \times \overline{R^+}\right)\right), \\ &\omega(t; x, r) \in C^1 \left(0, T; C^{k-1} \left(R \times \overline{R^+}\right)\right) \end{aligned} \tag{3.29}$$

will give rise to a smooth solution of (3.27) in the class

$$\begin{aligned} &\psi(t; x, r) \in C^1 \left(0, T; C_s^{k+1} \left(R \times \overline{R^+}\right)\right), \\ &u(t; x, r) \in C^1 \left(0, T; C_s^{k} \left(R \times \overline{R^+}\right)\right), \\ &\omega(t; x, r) \in C^1 \left(0, T; C_s^{k-1} \left(R \times \overline{R^+}\right)\right). \end{aligned} \tag{3.30}$$

In other words, are the pole conditions (2.18), (2.19) automatically satisfied if only (3.28) is imposed?

The answer to this question is affirmative. We will show in Theorem 2 that (3.30) and (3.29) are indeed equivalent for solutions of (3.27), (3.28). The proof is based on local Taylor expansion. We decompose the proof into several lemmas.

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LEMMA 11. *If* $2m \leq k-2$ *and*

$$\begin{aligned} &\psi \in C^{k+1}\left(R\times\overline{R^+}\right)\cap C_s^{2m}\left(R\times\overline{R^+}\right),\\ &u \in C^{k}\left(R\times\overline{R^+}\right)\cap C_s^{2m}\left(R\times\overline{R^+}\right),\\ &\omega \in C^{k-1}\left(R\times\overline{R^+}\right)\cap C_s^{2m}\left(R\times\overline{R^+}\right), \end{aligned} \tag{3.31}$$

then the nonlinear terms in (3.27)*, that is,*

$$u_x\partial_x u, \quad u_r\partial_r u, \quad \frac{u_r}{r}u, \tag{3.32}$$

and

$$u_x\partial_x\omega, \quad u_r\partial_r\omega, \quad \frac{u_r}{r}\omega, \quad \frac{1}{r}\partial_x(u^2) \tag{3.33}$$

are all in $C_s^{2m}(R\times\overline{R^+})$.

Proof. We start with the terms in (3.33). The proof is very similar to that of Lemma 10, where the identities (3.12)–(3.15) together with the condition (3.9) lead to (3.19)–(3.22). Here with the same identities (3.12)–(3.15) and the assumption (3.31), it is easy to see that for $j \leq m$,

$$\partial_r^{2j}(u_x\partial_x\omega)\big|_{r=0^+} = \partial_r^{2j}(u_r\partial_r\omega)\big|_{r=0^+} = \partial_r^{2j}\left(\frac{u_r}{r}\omega\right)\big|_{r=0^+} = \partial_r^{2j}\left(\frac{1}{r}\partial_x\left(u^2\right)\right)\big|_{r=0^+} = 0.$$

The procedure is quite similar, so we omit the details. In summary, we have shown that

$$u_x\partial_x\omega, \quad u_r\partial_r\omega, \quad \frac{u_r}{r}\omega, \quad \frac{1}{r}\partial_x(u^2) \in C_s^{2m}(R\times\overline{R^+}). \tag{3.34}$$

Similarly, with the same argument as in (3.12)–(3.18), we also have for $j \leq m$,

$$\begin{aligned} \partial_r^{2j}(u_x\partial_x u)\big|_{r=0^+} &= \sum_{\ell=0}^{2j} C_{2j}^{\ell}\left(\partial_r^{\ell}u_x\right)\big|_{r=0^+}\partial_x\left(\partial_r^{2j-\ell}u\right)\big|_{r=0^+},\\ \partial_r^{2j}(u_r\partial_r u)\big|_{r=0^+} &= \sum_{\ell=0}^{2j} C_{2j}^{\ell}\left(\partial_r^{\ell}u_r\right)\big|_{r=0^+}\left(\partial_r^{2j-\ell}u\right)\big|_{r=0^+},\\ \partial_r^{2j}\left(\tfrac{u_r}{r}u\right)\big|_{r=0^+} &= \sum_{\ell=0}^{2j} C_{2j}^{\ell}\partial_r^{\ell}\left(\frac{-\partial_x\psi}{r}\right)\bigg|_{r=0^+}\partial_r^{2j-\ell}u. \end{aligned}$$

From the assumption (3.31), it follows that for $j \leq m$,

$$\partial_r^{2j}(u_x\partial_x u)\bigg|_{r=0^+} = \partial_r^{2j}(u_r\partial_r u)\bigg|_{r=0^+} = \partial_r^{2j}\left(\frac{u_r}{r}u\right)\bigg|_{r=0^+} = 0.$$

Thus,

$$u_x\partial_x u, \quad u_r\partial_r u, \quad \frac{u_r}{r}u \in C_s^{2m}(R\times\overline{R^+}), \tag{3.35}$$

completing the proof of Lemma 11. ∎

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THEOREM 2. *Let* (ψ, u, ω) *be a solution to* (3.27), (3.28) *in the class* (3.29) *with* $k \geq 3$. *Then*

$$\begin{aligned} &\psi \in C_s^{k+1}(R \times \overline{R^+}), \\ &u \in C_s^{k}(R \times \overline{R^+}), \\ &\omega \in C_s^{k-1}(R \times \overline{R^+}) \end{aligned} \tag{3.36}$$

for $0 < t < T$.

Proof. Let j^* be the largest integer such that $2j^* \leq k-1$. We first show that on $0 < t < T$,

$$\begin{aligned} \partial_r^{2\ell}\psi(t, x, 0^+) &= 0, \\ \partial_r^{2\ell}u(t, x, 0^+) &= 0, \\ \partial_r^{2\ell}\omega(t, x, 0^+) &= 0 \end{aligned} \tag{3.37}$$

for $0 \leq \ell \leq j^*$.

This is done by induction on ℓ. When $\ell = 0$, (3.37) is given by the boundary condition (3.28). Suppose that (3.37) is verified for $\ell = j$ with $j+1 \leq j^*$. We apply $\partial_r^{2j-2}|_{(x,0^+)}$ on both sides of (3.27) and conclude that, in view of Lemma 11,

$$\begin{aligned} \nu\partial_r^{2j}\left(\nabla^2 - \tfrac{1}{r^2}\right)u(x, 0^+) &= 0, \\ \nu\partial_r^{2j}\left(\nabla^2 - \tfrac{1}{r^2}\right)\omega(x, 0^+) &= 0, \\ \partial_r^{2j}\left(\nabla^2 - \tfrac{1}{r^2}\right)\psi(x, 0^+) &= 0. \end{aligned}$$

Applying Lemma 5 to $\partial_r^{2j}\psi$, $\partial_r^{2j}u$, $\partial_r^{2j}\omega$, and one has $\partial_r^{2j+2}\psi(x, 0^+) = \partial_r^{2j+2}u(x, 0^+) = \partial_r^{2j+2}\omega(x, 0^+) = 0$; thus, (3.37) is verified for $\ell = j+1$.

We can continue the induction until (3.37) is verified for $\ell = j^*$ to get

$$\begin{aligned} &\psi \in C^{k+1}(R \times \overline{R^+}) \cap C_s^{2j^*}(R \times \overline{R^+}), \\ &u \in C^{k}(R \times \overline{R^+}) \cap C_s^{2j^*}(R \times \overline{R^+}), \\ &\omega \in C^{k-1}(R \times \overline{R^+}) \cap C_s^{2j^*}(R \times \overline{R^+}). \end{aligned} \tag{3.38}$$

To complete the proof, we proceed with k odd and even separately.

If k is odd, say $k = 2m+1$, then $j^* = m$ and (3.38) can be written as

$$\begin{aligned} &\psi \in C^{2m+2}(R \times \overline{R^+}) \cap C_s^{2m}(R \times \overline{R^+}), \quad u \in C_s^{2m+1}(R \times \overline{R^+}), \\ &\omega \in C_s^{2m}(R \times \overline{R^+}). \end{aligned} \tag{3.39}$$

Apply Lemma 5 to $\partial_r^{2m}\psi$, and one has that $\partial_r^{2m+2}\psi(x, 0) = 0$; therefore, $\psi \in C_s^{2m+2}(R \times \overline{R^+})$.

Similarly, if $k = 2n$, then $j^* = n-1$, and we have from (3.38)

$$\begin{aligned} &\psi \in C^{2n+1}(R \times \overline{R^+}) \cap C_s^{2n-2}(R \times \overline{R^+}), \\ &u \in C^{2n}(R \times \overline{R^+}) \cap C_s^{2n-2}(R \times \overline{R^+}), \\ &\omega \in C_s^{2n-1}(R \times \overline{R^+}). \end{aligned}$$

Since $2n-2 = k-2$, the assumption in Lemma 11 is satisfied. Therefore, we can continue the induction for u to get $\partial_r^{2n}u(x, 0^+) = 0$; thus, $u \in C_s^{2n}(R \times \overline{R^+})$.

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Finally, apply Lemma 5 to $\partial_r^{2n-2}\psi$, and we conclude that $\partial_r^{2n}\psi(x,0^+)=0$ and $\psi\in C^{2n+1}(R\times\overline{R^+})\cap C_s^{2n}(R\times\overline{R^+})=C_s^{2n+1}(R\times\overline{R^+})$. This completes the proof of Theorem 2. ◻

The equivalence of (1.4) and (3.27) in terms of regularity of classical solutions is given by the following.

THEOREM 3.

(I) *Suppose* $(\boldsymbol{u},\tilde{p})$ *is an axisymmetric solution to Navier–Stokes equation* (1.4) *with* $\boldsymbol{u}\in C^1\left(0,T;\mathcal{C}_s^k\right)$, $\tilde{p}\in C^0\left(0,T;C^{k-1}(R^3)\right)$, *and* $k\geq 3$. *Then there is a solution* (ψ,u,ω) *to* (3.27) *in the class*

$$\begin{aligned}&\psi(t,x,r)\in C^1\big(0,T;C_s^{k+1}(R\times\overline{R^+})\big),\\&u(t,x,r)\in C^1\big(0,T;C_s^{k}(R\times\overline{R^+})\big),\\&\omega(t,x,r)\in C^1\big(0,T;C_s^{k-1}(R\times\overline{R^+})\big)\end{aligned}$$

and $\boldsymbol{u}=u\boldsymbol{e}_\theta+\nabla\times(\psi\boldsymbol{e}_\theta)$.

(II) *Let* (ψ,u,ω) *be a solution to* (3.27), (3.28) *in the class*

$$\begin{aligned}&\psi(t,x,r)\in C^1\big(0,T;C^{k+1}(R\times\overline{R^+})\big),\\&u(t,x,r)\in C^1\big(0,T;C^{k}(R\times\overline{R^+})\big),\\&\omega(t,x,r)\in C^1\big(0,T;C^{k-1}(R\times\overline{R^+})\big)\end{aligned}$$

with $k\geq 3$. *Then*

$$\boldsymbol{u}=u\boldsymbol{e}_\theta+\nabla\times(\psi\boldsymbol{e}_\theta)\in C^1(0,T;\mathcal{C}_s^k),$$

and there is an axisymmetric scalar function $\tilde{p}\in C^0(0,T;C^{k-1}(R^3))$, *such that* $(\boldsymbol{u},\tilde{p})$ *is a solution to Navier–Stokes equation* (1.4).

Proof. *Part* (I). Since $\boldsymbol{u}\in C^1\left(0,T;\mathcal{C}_s^k\right)$ is a solution to (1.4), with $k\geq 3$, it follows that

$$\boldsymbol{\omega}=\nabla\times\boldsymbol{u}=\omega\boldsymbol{e}_\theta+\nabla\times(u\boldsymbol{e}_\theta)\in C^1\left(0,T;\mathcal{C}_s^{k-1}\right)$$

is also an axisymmetric solution to the Navier–Stokes equation in vorticity form

$$\partial_t\boldsymbol{\omega}+\nabla\times(\boldsymbol{\omega}\times\boldsymbol{u})=-\nu\nabla\times\nabla\times\boldsymbol{\omega}. \tag{3.40}$$

Next, we express each term of (3.40) in the cylindrical coordinate as

$$\partial_t\boldsymbol{\omega}=\partial_t\omega\boldsymbol{e}_\theta+\nabla\times(\partial_t u\boldsymbol{e}_\theta), \tag{3.41}$$

$$-\nabla\times\nabla\times\boldsymbol{\omega}=\left(\left(\nabla^2-\frac{1}{r^2}\right)\omega\right)\boldsymbol{e}_\theta+\nabla\times\left(\left(\nabla^2-\frac{1}{r^2}\right)u\boldsymbol{e}_\theta\right), \tag{3.42}$$

and

$$\nabla\times(\boldsymbol{\omega}\times\boldsymbol{u})=\left(J\left(\frac{\omega}{r},r\psi\right)-J\left(\frac{u}{r},ru\right)\right)\boldsymbol{e}_\theta+\nabla\times\left(\frac{1}{r^2}J\left(ru,r\psi\right)\boldsymbol{e}_\theta\right). \tag{3.43}$$

From (3.41)–(3.43), we can rewrite (3.40) as

$$a\boldsymbol{e}_\theta+\nabla\times(b\boldsymbol{e}_\theta)=\boldsymbol{0}, \tag{3.44}$$

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where

$$a = \partial_t \omega + J\left(\frac{\omega}{r}, r\psi\right) - J\left(\frac{u}{r}, ru\right) - \nu\Big(\nabla^2 - \frac{1}{r^2}\Big)\omega$$

and

$$b = \partial_t u + \frac{1}{r^2} J\left(ru, r\psi\right) - \nu\Big(\nabla^2 - \frac{1}{r^2}\Big)u.$$

From (3.44), it follows that $a(x,r) \equiv 0$ and $rb(x,r)$ is a constant. Since $b(x,0^+) = 0$ from Lemma 11 and Lemma 2, we conclude that $b(x,r) \equiv 0$ as well. Expanding the Jacobians in the two equations above, we get exactly (3.27). This completes the proof of part (I).

Part (II). From Theorem 2, we know that (ψ, u, ω) satisfies (3.36). Therefore, Lemma 2 applies and we have

$$\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta) \in C^1(0,T;\mathcal{C}_s^k).$$

Next we define $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. From (3.41)–(3.43), we see that $\boldsymbol{\omega}$ satisfies the Navier–Stokes equation in vorticity formulation (3.40). That is,

$$\nabla \times (\partial_t \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nu \nabla \times \boldsymbol{\omega}) = \mathbf{0}.$$

Thus, there exists a function $\tilde{p} : (0,T) \to C^{k-1}(R^3)$, such that

$$\partial_t \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nu \nabla \times \boldsymbol{\omega} = -\nabla \tilde{p}. \tag{3.45}$$

In other words, $(\boldsymbol{u}, \tilde{p})$ satisfies Navier–Stokes equation (1.4). Since $\boldsymbol{u} \in C^1(0,T;\mathcal{C}_s^k)$, it follows from (3.45) that $\nabla \tilde{p} \in C^0(0,T;\mathcal{C}_s^{k-2})$. In addition, we can further assign $\tilde{p}(t)$ on a reference point (x_0, r_0) so that $\tilde{p} \in C^0(0,T;C^{k-1}(R^3))$.

By construction, the left-hand side of (3.45) is axisymmetric and therefore so is $\nabla \tilde{p}$. In particular,

$$\partial_\theta(\nabla \tilde{p} \cdot \boldsymbol{e}_\theta) = \partial_\theta\Big(\frac{1}{r}\partial_\theta \tilde{p}\Big) = 0.$$

Therefore,

$$\tilde{p} = a(x,r)\theta + b(x,r).$$

Since $\tilde{p}$ is continuous and single-valued, we conclude that $a = 0$. In other words, $\tilde{p}$ is axisymmetric. This completes the proof of theorem. ∎

3.3. Weak formulation and Leray solution. The Navier–Stokes equation in vorticity formulation for axisymmetric flows (3.27) can be recast as following in terms of Jacobians [14]

$$\begin{aligned} &u_t + \tfrac{1}{r^2} J\left(ru, r\psi\right) = \nu \mathscr{L} u\,, \\ &\omega_t + J\left(\tfrac{\omega}{r}, r\psi\right) - J\left(\tfrac{u}{r}, ru\right) = \nu \mathscr{L}\omega\,, \\ &\omega = -\mathscr{L}\psi\,. \end{aligned} \tag{3.46}$$

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The expression of the nonlinear terms in (3.46) in terms of Jacobians are equivalent to the usual expression (1.5) for strong solutions. Accompanied with the Jacobians is a set of permutation identities which leads naturally to an energy and helicity preserving numerical scheme and plays a key role in the convergence proof of the scheme [14, 15].

We propose the following formulation for weak solution.

Find $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H_s^1)$, $\psi \in L^\infty(0,T;H_s^1)$, and $\omega \in L^2(0,T;L^2)$, such that

$$\begin{aligned} &\langle \partial_t u, v\rangle + \langle \tfrac{v}{r^2}, J(ru, r\psi)\rangle + \nu[u,v] = 0,\\ &[\partial_t \psi, \phi] + \langle \tfrac{\omega}{r^2}, J(r\psi, r\phi)\rangle - \langle \tfrac{u}{r^2}, J(ru, r\phi)\rangle + \nu\langle \omega, \mathscr{L}\phi\rangle = 0,\\ &\langle \omega, \xi\rangle = [\psi, \xi] \end{aligned} \tag{3.47}$$

for all $v \in H_s^1(R\times R^+)$, $\phi \in H_s^1 \cap H^2(R\times R^+)$, and $\xi \in H_s^1(R\times R^+)$.

Note that the viscous term in (3.47) is not treated the same way in standard variational formulation. In addition, only $u = 0$ and $\psi = 0$ are imposed on the boundary $r = 0$. One can regard (3.47) as a variational formulation of the fourth order PDE for ψ, where the boundary condition $\omega = 0$ is imposed *implicitly*. Although we have shown equivalence of the Navier–Stokes equation in vorticity-stream formulation and primitive formulation for the classic solutions which are smooth up to the boundary $r = 0$, it is still not clear a priori how (3.47) is related to the weak solutions of (1.4) as constructed in Leray's seminal work [11]. To answer this question, we will show in Theorem 4 that (3.47) can be recast in standard 3D notations as follows.

Find $\boldsymbol{u} \in L^\infty\left(0,T;L^2(R\times R^+,R^3)\right) \cap L^2\left(0,T;\mathcal{H}_s^1(R\times R^+,R^3)\right)$, such that

$$\langle \boldsymbol{v}, \partial_t \boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{u}\rangle + \nu\langle \nabla\times\boldsymbol{v}, \nabla\times\boldsymbol{u}\rangle = 0 \quad \text{for all } \boldsymbol{v} \in \mathcal{H}_s^1(R\times R^+,R^3). \tag{3.48}$$

Now we recall Leray's definition of weak solution as follows.

Find $\boldsymbol{u} \in L^\infty\left(0,T;L^2(R^3,R^3)\right) \cap L^2\left(0,T;H^1(R^3,R^3)\right)$

$$\langle \boldsymbol{v}, \partial_t \boldsymbol{u} + \boldsymbol{\omega}\times\boldsymbol{u}\rangle + \nu\langle \nabla\times\boldsymbol{v}, \nabla\times\boldsymbol{u}\rangle = 0 \quad \text{for all } \boldsymbol{v} \in C_0^1(R^3,R^3), \quad \nabla\cdot\boldsymbol{v} = 0. \tag{3.49}$$

Upon comparing (3.48) and (3.49), we see that the key point in establishing the equivalence result lies in a proper decomposition of a general divergence free test function into two parts; one is axisymmetric and the other has mean zero components. This is given by the following lemma.

LEMMA 12. *Let $\boldsymbol{v} \in C^1(R^3,R^3)$, $\nabla\cdot\boldsymbol{v} = 0$, then there exists a $\boldsymbol{v}^{sym} \in \mathcal{C}_s^1(R^3,R^3)$, with*

$$\overline{v_x}(x,r,\theta) = v_x^{sym}(x,r), \quad \overline{v_r}(x,r,\theta) = v_r^{sym}(x,r), \quad \overline{v_\theta}(x,r,\theta) = v_\theta^{sym}(x,r), \tag{3.50}$$

where

$$\overline{f}(x,r) = \frac{1}{2\pi}\int_0^{2\pi} f(x,r,\theta)d\theta.$$

Proof. Since $\boldsymbol{v} \in \mathcal{C}^1(R^3,R^3)$, $\nabla\cdot\boldsymbol{v} = 0$, there exists $\boldsymbol{\phi} = \phi_x\boldsymbol{e}_x + \phi_r\boldsymbol{e}_r + \phi_\theta\boldsymbol{e}_\theta \in C^2(R^3,R^3)$, such that $\nabla\times\boldsymbol{\phi} = \boldsymbol{v}$. We then define

$$\boldsymbol{v}^{sym} = \nabla\times(\overline{\phi_\theta}\boldsymbol{e}_\theta) + v_\theta^{sym}\boldsymbol{e}_\theta, \quad v_\theta^{sym} = \partial_x\overline{\phi_r} - \partial_r\overline{\phi_x}.$$

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It follows that $\boldsymbol{v}^{sym}$ is divergence free and satisfies (3.50). In addition, $\phi_x(\cdot,\cdot,\theta)$, $\phi_r(\cdot,\cdot,\theta), \phi_\theta(\cdot,\cdot,\theta) \in C^2(R\times\overline{R^+})$ for any fixed θ in view of Corollary 1. We, therefore, conclude from the bounded convergence theorem that

$$\lim_{r\to 0^+} \frac{1}{2\pi}\int_0^{2\pi} \partial_x^i\partial_r^j \begin{pmatrix} \phi_x(x,r,\theta) \\ \phi_r(x,r,\theta) \\ \phi_\theta(x,r,\theta) \end{pmatrix} = \frac{1}{2\pi}\int_0^{2\pi} \lim_{r\to 0^+} \partial_x^i\partial_r^j \begin{pmatrix} \phi_x(x,r,\theta) \\ \phi_r(x,r,\theta) \\ \phi_\theta(x,r,\theta) \end{pmatrix}, \quad 0 \le i+j \le 2. \tag{3.51}$$

In other words, $\overline{\phi_x}, \overline{\phi_r}, \overline{\phi_\theta} \in C^2(R\times\overline{R^+})$. Moreover, (2.14), (2.15) imply that $\overline{\phi_\theta} \in C^2_s(R\times\overline{R^+})$, $v_\theta^{sym} \in C^1_s(R\times\overline{R^+})$ and therefore $\boldsymbol{v}^{sym} \in \mathcal{C}^1_s$. ∎

We are now ready to show the following equivalence result.

THEOREM 4. *Let* $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla\times(\psi\boldsymbol{e}_\theta)$ *and* $\omega = \mathscr{L}\psi$. *The following three statements are all equivalent:*

(i) (ψ, u, ω) *is a weak solution* (3.47).

(ii) $\boldsymbol{u}$ *is a axisymmetric weak solution defined by* (3.48).

(iii) $\boldsymbol{u}$ *is a Leray weak solution as defined in* (3.49).

Proof. We first show that (i) and (ii) are equivalent. Let $\boldsymbol{u}$ be an axisymmetric weak solution (3.48) and let the test function be given by $\boldsymbol{v} = v\boldsymbol{e}_\theta + \nabla\times(\phi\boldsymbol{e}_\theta)$. Simple calculation gives

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \partial_t u, v\rangle + [\partial_t\psi, \phi], \tag{3.52}$$

$$\langle \nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}\rangle = \langle \omega, \mathscr{L}\phi\rangle + [u, v], \tag{3.53}$$

$$\langle \boldsymbol{\omega}\times\boldsymbol{u}, \boldsymbol{v}\rangle = \int_{R^3} \omega\boldsymbol{e}_\theta\times(\nabla\times(\psi\boldsymbol{e}_\theta))\cdot(\nabla\times(\phi\boldsymbol{e}_\theta)) - \int_{R^3} u\boldsymbol{e}_\theta\times(\nabla\times(u\boldsymbol{e}_\theta))\cdot(\nabla\times(\phi\boldsymbol{e}_\theta))$$
$$+ \int_{R^3} v\boldsymbol{e}_\theta\times(\nabla\times(u\boldsymbol{e}_\theta))\cdot(\nabla\times(\psi\boldsymbol{e}_\theta)).$$

In cylindrical coordinates, we can write

$$\int_{R^3} a\boldsymbol{e}_\theta\times(\nabla\times(b\boldsymbol{e}_\theta))\cdot(\nabla\times(c\boldsymbol{e}_\theta)) = \int_{R\times R_+} \frac{a}{r}\left(\partial_x(rb)\partial_r(rc) - \partial_r(rb)\partial_x(rc)\right) drdx$$
$$= \langle \frac{a}{r^2}, J(rb, rc)\rangle.$$

Hence,

$$\langle \boldsymbol{\omega}\times\boldsymbol{u}, \boldsymbol{v}\rangle = \left\langle \frac{\omega}{r^2}, J(r\psi, r\phi)\right\rangle - \left\langle \frac{u}{r^2}, J(ru, r\phi)\right\rangle + \left\langle \frac{v}{r^2}, J(ru, r\psi)\right\rangle. \tag{3.54}$$

Since v and ϕ are independent, it follows from (3.52), (3.53), (3.54), and (3.48) that

$$\langle \partial_t u, v\rangle + \left\langle \frac{v}{r^2}, J(ru, r\psi)\right\rangle + \nu[u, v] = 0, \tag{3.55}$$

$$[\partial_t\psi, \phi] + \left\langle \frac{\omega}{r^2}, J(r\psi, r\phi)\right\rangle - \left\langle \frac{u}{r^2}, J(ru, r\phi)\right\rangle + \nu\,\langle \omega, \mathscr{L}\phi\rangle = 0 \tag{3.56}$$

together with the weak formulation for the relation $\omega = \mathscr{L}\psi$:

$$[\psi, \xi] = \langle \omega, \xi\rangle \quad \text{for all } \xi \in H^1_s(R\times R^+). \tag{3.57}$$

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Hence, (ψ, u, ω) is a weak solution to (3.47). The converse is also true by reversing the calculations above. This proves the equivalence between (i) and (ii).

Since $\mathcal{C}_s^1(R^3, R^3) \cap C_c(R^3, R^3)$ is a subspace of $\{\boldsymbol{v} \in C_0^1(R^3, R^3), \nabla \cdot \boldsymbol{v} = 0\}$, and is dense in $\mathcal{H}_s^1(R \times R^+, R^3)$, (iii) implies (ii).

It remains to show that (ii) implies (iii). Let $\boldsymbol{u}$ be an axisymmetric weak solution of (3.48). From Lemma 12, for any test function $\boldsymbol{v} \in C_0^1(R^3, R^3)$ with $\nabla \cdot \boldsymbol{v} = 0$, we can construct $\boldsymbol{v}^{sym} \in \mathcal{C}_s^1(R^3, R^3) \cap C_0(R^3, R^3)$, such that

$$\int_0^{2\pi} (\boldsymbol{v} - \boldsymbol{v}^{sym})(x, r, \theta) d\theta = 0 \quad \text{for all } (x, r) \in (R \times R^+). \tag{3.58}$$

For any $\boldsymbol{w} \in L_s^2(R^3, R^3)$, one has

$$\int_0^{2\pi} (\boldsymbol{v} - \boldsymbol{v}^{sym}) \cdot \boldsymbol{w}(x, r, \theta) d\theta = 0 \quad \text{for all } (x, r) \in (R \times R^+) \tag{3.59}$$

and

$$\int_0^{2\pi} \nabla \times (\boldsymbol{v} - \boldsymbol{v}^{sym}) \cdot \boldsymbol{w}(x, r, \theta) d\theta = 0 \quad \text{for all } (x, r) \in (R \times R^+). \tag{3.60}$$

Hence,

$$\langle \boldsymbol{v}, \partial_t \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} \rangle + \nu \langle \nabla \times \boldsymbol{v}, \nabla \times \boldsymbol{u} \rangle = \langle \boldsymbol{v}^{sym}, \partial_t \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} \rangle + \nu \langle \nabla \times \boldsymbol{v}^{sym}, \nabla \times \boldsymbol{u} \rangle. \tag{3.61}$$

But now $\boldsymbol{v}^{sym} \in \mathcal{C}_s^1(R^3, R^3) \cap C_0(R^3, R^3) \subset \mathcal{H}_s^1(R \times R^+, R^3)$ is a test function for the axisymmetric weak solution (3.48), so the right-hand side of (3.61) is zero. Therefore, $\boldsymbol{u}$ is a Leray solution. This completes the proof of this theorem. □

COROLLARY 3.

(i) *For any initial data* $u_0 \in L^2(R \times R^+), \psi_0 \in \hat{H}_s^1(R \times R^+)$, *there is a global weak solution* (ψ, u, ω) *to* (3.47), *and* $\boldsymbol{u} = u\boldsymbol{e}_\theta + \nabla \times (\psi \boldsymbol{e}_\theta)$ *is an axisymmetric Leray solution of the Navier–Stokes equation* (1.4).

(ii) *If in addition,*

$$u_0 \in H_s^k(R \times R^+), \quad \psi_0 \in \hat{H}_s^1(R \times R^+), \quad \mathscr{L}\psi_0 \in H_s^{k-1}(R \times R^+), \tag{3.62}$$

with $k \geq 1$, *then there exists a* $T_0 > 0$, *such that the solution satisfies*

$$\begin{aligned} u &\in C^0\left(0, T_0; H_s^k(R \times R^+)\right) \cap L^2\left(0, T_0; H_s^{k+1}(R \times R^+)\right) \\ \omega &\in C^0\left(0, T_0; H_s^{k-1}(R \times R^+)\right) \cap L^2\left(0, T_0; H_s^k(R \times R^+)\right), \end{aligned} \tag{3.63}$$

and it corresponds to the unique strong solution of Navier–Stokes equation (1.4).

(iii) *If* $k \geq 3$ *in* (3.62), *then the solution is classical:*

$$\begin{aligned} u &\in C^0\big(0, T_0; C_s^{k-2}(R \times \overline{R^+})\big) \cap C^1\big(0, T_0; C^{k-3}(R \times \overline{R^+})\big) \\ \psi &\in C^0\big(0, T_0; C_s^{k-1}(R \times \overline{R^+})\big) \cap C^1\big(0, T_0; C^{k-2}(R \times \overline{R^+})\big). \end{aligned} \tag{3.64}$$

Proof. From an initial data $u_0 \in L^2(R \times R^+), \psi_0 \in \hat{H}_s^1(R \times R^+)$, one can construct an axisymmetric vector field $\boldsymbol{u}_0 = u_0 \boldsymbol{e}_\theta + \nabla \times (\psi_0 \boldsymbol{e}_\theta) \in L_s^2(R^3, R^3)$, and then a global weak solution of (3.49) using Leray's method with initial data $\boldsymbol{u}_0$. The weak solution is constructed from a family of approximate solutions obtained via (radially symmetric) mollifiers. See [11, 16] for details. Since the symmetry with respect to the axis of

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rotation is preserved under the action of convolution with the mollifiers, the resulting limit is also axisymmetric. From Theorem 4, it corresponds to a global weak solution (ψ, u, ω) of (3.47). This shows part (i).

If in addition, $u_0 \in H_s^k(R \times R^+)$, $\mathscr{L}\psi_0 \in H_s^{k-1}(R \times R^+)$, $k \geq 1$, then $\boldsymbol{u}_0 \in \mathcal{H}_s^k(R \times R^+, R^3)$. Hence, by classical theory of the Navier–Stokes equation [18], there exists a $T_0 > 0$ depending only on ν and $\|\boldsymbol{u}_0\|_{H^k(R^3,R^3)}$, and a unique solution $(\boldsymbol{u}, \tilde{p})$ in $(0, T_0)$ to (1.4), with regularity

$$\boldsymbol{u} \in H^1\left(0, T_0; H^{k-1}(R^3, R^3)\right) \cap L^2\left(0, T_0; H^{k+1}(R^3, R^3)\right), \tag{3.65}$$

$$\nabla \tilde{p} \in L^2\left(0, T_0; H^{k-1}(R^3, R^3)\right). \tag{3.66}$$

From [6, p. 288], (3.65) implies

$$\boldsymbol{u} \in C^0\left(0, T_0; H^k(R^3, R^3)\right). \tag{3.67}$$

Consequently, any global weak solution of (3.47) coincides with the strong solution (3.65) in $(0, T_0)$, and therefore the strong solution is also axisymmetric. It follows from Lemma 8 that $u \in L^\infty\left(0, T_0; H_s^k(R \times R^+)\right) \cap L^2\left(0, T_0; H_s^{k+1}(R \times R^+)\right)$, $\omega \in L^\infty\left(0, T_0; H_s^{k-1}(R \times R^+)\right) \cap L^2\left(0, T_0; H_s^k(R \times R^+)\right)$. This shows part (ii).

Since $H^2(R^3, R^3) \subset C^0(R^3, R^3)$, it follows from (3.67) that, when $k \geq 3$,

$$\boldsymbol{u} \in C^0\left(0, T_0; C_s^{k-1}(R^3, R^3)\right). \tag{3.68}$$

Since $\partial_t \boldsymbol{u}$ is the Leray projection of $\nu \nabla^2 \boldsymbol{u} - (\nabla \times \boldsymbol{u}) \times \boldsymbol{u}$, it follows that

$$\partial_t \boldsymbol{u} \in C^0\left(0, T_0; C_s^{k-3}(R^3, R^3)\right). \tag{3.69}$$

This gives (3.64) and proves (iii). ∎

From well-known regularity results of the 3D Euler equation, the counterpart of Corollary 3 for the Euler equation can be obtained using a similar argument. We state it without proof.

COROLLARY 4. *For any initial data* $u_0 \in H_s^k(R \times R^+), \psi_0 \in \hat{H}_s^1(R \times R^+), \mathscr{L}\psi_0 \in H_s^{k-1}(R \times R^+)$*,* $k \geq 3$*, there exists a unique local-in-time classical solution* (ψ, u, ω) *to the Euler equation* (3.1) *with*

$$\begin{aligned} u \in C^0\big(0, T_0; C_s^{k-2}(R \times \overline{R^+}) \cap H_s^k(R \times R^+)\big) \\ \cap C^1\big(0, T_0; C^{k-3}(R \times \overline{R^+}) \cap H_s^{k-1}(R \times R^+)\big) \\ \psi \in C^0\big(0, T_0; C_s^{k-1}(R \times \overline{R^+}) \cap H_s^{k+1}(R \times R^+)\big) \\ \cap C^1\big(0, T_0; C^{k-2}(R \times \overline{R^+}) \cap H_s^k(R \times R^+)\big). \end{aligned} \tag{3.70}$$

As remarked earlier, the weak formulation (3.47) is not standard, and it only imposes the boundary condition $\omega = 0$ in an implicitly way. In fact, if the solution is regular enough, then one recovers this boundary condition and the usual weak formulation follows. This becomes more clear as we recast part (ii) of Corollary 3 as follows.

COROLLARY 5. *Let* (ψ, u, ω) *be a weak solution of* (3.47) *and* $\boldsymbol{u} = \nabla \times (\psi \boldsymbol{e}_\theta) + u\boldsymbol{e}_\theta)$*. If*

$$\boldsymbol{u} \in L_{loc}^\infty\left((0, T); \mathcal{H}_s^1(R \times R^+, R^3)\right),$$

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then

$$\boldsymbol{u} \in L^2_{loc}\left((0,T);\mathcal{H}^2_s(R\times R^+,R^3)\right).$$

In particular, the trace of $\omega = \mathscr{L}\psi$ *on* $r = 0^+$ *vanishes almost everywhere on* $(0,T)$.

Remark 1. The standard variational formulation for (1.5) is as follows.

Find $u \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1_s)$, $\psi \in L^\infty(0,T;H^1_s)$, and $\omega \in L^2(0,T;H^1_s)$, such that

$$\begin{aligned}
&\langle \partial_t u, v\rangle + \langle \tfrac{v}{r^2}, J(ru, r\psi)\rangle + \nu[u,v] = 0\\
&[\partial_t\psi,\phi] + \langle \tfrac{\omega}{r^2}, J(r\psi, r\phi)\rangle - \langle \tfrac{u}{r^2}, J(ru, r\phi)\rangle - \nu[\omega,\phi] = 0\\
&\langle \omega,\xi\rangle = [\psi,\xi]
\end{aligned} \tag{3.71}$$

for all $v \in H^1_s(R\times\overline{R^+})$, $\phi \in H^1_s(R\times\overline{R^+})$, and $\xi \in H^1_s(R\times\overline{R^+})$.

The main difference between (3.47) and (3.71) is the viscous term of the vorticity equation. The formulation (3.71) is natural for a standard C^0 finite element setting. The regularity requirement for (3.71) lies between weak solution (3.47) and the strong solution (3.65). The well posedness of (3.71), including uniqueness and local existence of solution for initial data $u_0 \in L^2_s(R\times R^+)$, $\omega_0 \in L^2_s(R\times R^+)$, is still unclear.

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