DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **22**, Number **4**, June **2017**

doi:10.3934/dcdsb.2017070

pp. **1461–1492**

GLOBAL EXISTENCE FOR A THIN FILM EQUATION WITH SUBCRITICAL MASS

Jian-Guo Liu[2] and Jinhuan Wang[1]

[1]School of Mathematics, Liaoning University
Shenyang 110036, China
[2]Department of Physics and Department of Mathematics
Duke University, Durham, NC 27708, USA

(Communicated by Thomas P. Witelski)

Abstract. In this paper, we study existence of global entropy weak solutions to a critical-case unstable thin film equation in one-dimensional case

$$h_t + \partial_x(h^n\,\partial_{xxx}h) + \partial_x(h^{n+2}\partial_x h) = 0,$$

where $n \geq 1$. There exists a critical mass $M_c = \frac{2\sqrt{6}\pi}{3}$ found by Witelski et al. (2004 Euro. J. of Appl. Math. 15, 223–256) for $n = 1$. We obtain global existence of a non-negative entropy weak solution if initial mass is less than M_c. For $n \geq 4$, entropy weak solutions are positive and unique. For $n = 1$, a finite time blow-up occurs for solutions with initial mass larger than M_c. For the Cauchy problem with $n = 1$ and initial mass less than M_c, we show that at least one of the following long-time behavior holds: the second moment goes to infinity as the time goes to infinity or $h(\cdot, t_k) \rightharpoonup 0$ in $L^1(\mathbb{R})$ for some subsequence $t_k \to \infty$.

1. **Introduction.** This paper deals with the following critical-case long-wave unstable thin film equation

$$h_t + \partial_x\Big(h^n\,\partial_{xxx}h\Big) + \partial_x\Big(h^{n+2}\,\partial_x h\Big) = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{1}$$

where $h(x,t)$ denotes the height of the evolving free-surface and $n \geq 1$ is the exponent of the mobility. We impose the following initial condition

$$h(x,0) = h_0(x),\ x \in \mathbb{R}. \tag{2}$$

The model (1)-(2) can be used to describe pattern formation in physical systems that involve interfaces, c.f. [36]. Here we consider two classes of initial data:

$$h_0 \geq 0,\ \text{supp}\ h_0 \subset (-a,a) \text{ for some } a > 0,\ h_0(x) \in L^1(\mathbb{R}), \tag{3}$$

and

$$h_0 \geq 0,\ h_0(x) = h_0(x + 2L) \text{ for } x \in \mathbb{R},\ h_0(x) \in L^1(-L,L). \tag{4}$$

2010 *Mathematics Subject Classification.* Primary: 35K65, 35K25, 39B62, 76D08.

Key words and phrases. Long-wave instability, free-surface evolution, equilibrium, the Sz. Nagy inequality, long-time behavior.

Corresponding author: Jinhuan Wang, was supported by National Natural Science Foundation of China (Grant No. 11301243) and Program for Liaoning Excellent Talents in University (Grant No. LJQ2015041).

Jian-Guo Liu was partially supported partially supported by KI-Net NSF RNMS grant No. 1107444, NSF DMS grant No. 1514826.

We list below some important properties of the Cauchy problem (1)-(3). First, non-negative solutions $h(x,t)$ to (1)-(2) satisfy conservation of mass, i.e.,

$$\int_{\mathbb{R}} h(x,t)\,dx \equiv \int_{\mathbb{R}} h_0(x)\,dx =: M_0.$$

Second, the equation (1) can be recast in a variational form

$$h_t - \partial_x(h^n\,\partial_x\mu) = 0, \quad \mu = \frac{\delta\mathcal{F}}{\delta h} = -\partial_{xx}h - \frac{h^3}{3}, \tag{5}$$

where μ is the chemical potential. It is given by the variation of the free energy functional:

$$\mathcal{F}(h) = \frac{1}{2}\int_{\mathbb{R}}(\partial_x h)^2\,dx - \frac{1}{12}\int_{\mathbb{R}} h^4\,dx. \tag{6}$$

In thin film equations, the negative chemical potential is referred to as the pressure, $p = -\mu = \partial_{xx}h + \frac{1}{3}h^3$. The variation equation (5) induces the following energy-dissipation relation for $h \geq 0$:

$$\frac{d}{dt}\mathcal{F}(h(\cdot,t)) = -\int_{\mathbb{R}} h^n\Big(\partial_x(\frac{1}{3}h^3 + \partial_{xx}h)\Big)^2\,dx \leq 0. \tag{7}$$

Third, following Bernis and Friedman [5], for any $0 \leq h \leq M$ we define a function

$$G_M(h) = \int_h^M \int_y^M \frac{1}{s^n}\,dsdy, \tag{8}$$

which can be represented exactly by the following form

$$G_M(h) := \begin{cases} \frac{1}{(n-1)(n-2)}(h^{2-n} - M^{2-n}) - \frac{1}{n-1}\left(M^{2-n} - M^{1-n}h\right), & \text{if } n > 2, \\ -\log h + \log M - 1 + \frac{1}{M}h, & \text{if } n = 2, \\ \frac{1}{(n-1)(2-n)}(M^{2-n} - h^{2-n}) - \frac{1}{n-1}\left(M^{2-n} - M^{1-n}h\right), & \text{if } 1 < n < 2, \\ h\log\frac{h}{M} + M - h, & \text{if } n = 1. \end{cases} \tag{9}$$

Define another free energy functional

$$\mathcal{G}_M(h) := \int_{\mathbb{R}} G_M(h(x))\,dx. \tag{10}$$

Noticing that $G''_M(h) = \frac{1}{h^n}$, and from the equation (1), we deduce

$$\frac{d}{dt}\mathcal{G}_M(h(\cdot,t)) = -\int_{\mathbb{R}}(\partial_{xx}h)^2\,dx + \int_{\mathbb{R}} h^2(\partial_x h)^2\,dx. \tag{11}$$

According to (11), one has that $\mathcal{G}_M(h)$ is bounded if it is initially so. This bound can be used to obtain non-negativity of solutions to the model (1)-(2). The details can be found in Subsection 2.2 below.

There is a vast literature on thin film equations, see [10], [3]-[11], [13]-[26], [29]-[31], [33]-[37]. Here we give a short account about results related to (1). For the classical thin film equation

$$h_t + \partial_x\Big(h^n\partial_{xxx}h\Big) = 0, \quad x \in (-L,L),\ t > 0 \tag{12}$$

with appropriate boundary conditions, some fundamental mathematical theories have been developed by Bernis and Friedman (BF) [5] such as regularization, global existence of non-negative weak solutions, and Hölder regularity $h \in C^{\frac{1}{2},\frac{1}{8}}\big([-L,L]\times[0,T]\big)$. In particular, for $n \geq 4$, they proved that there exists a unique strict positive solution $h > 0$ in $(-L,L)\times[0,+\infty)$ under some initial assumptions. The existence of solutions was established for higher dimensions in [11, 15, 16, 20, 21, 22]. Recently, the uniqueness of weak solutions was obtained by John [23], Knüpfer and Masmoudi [24, 25].

One of the most important properties for (12) is finite speed of propagation of the support of solutions. This problem can also be regarded as a free-boundary problem and we refer to Giacomelli, Gnann, Knüpfer and Otto [17, 18, 19, 33], and Mellet [30] for in-depth studies of this problem. The existence of weak solutions for this degenerate parabolic fourth order free boundary problem was proved when $n = 1$ [33]. It was proved in [5] that the support of the solution is expanding. An upper bound on the support expansion was given by Bernis [4],

$$\zeta(t) \leq \zeta(0) + C_0 t^{\alpha}\Big(\int_{\Omega} h_0^{(1+\lambda)}dx\Big)^{\beta}, \tag{13}$$

where $\zeta(t)$ is a curve describing the boundary of the support for solutions, and α and β are positive constants, $\max\Big(-\frac{1}{2}, n-1\Big) < \lambda < 1$ (see also [3]). For the higher dimensional case, the property is derived in [15, 16] for $n \in (0,2)$ and in [22] for $n \in [2,3)$. We refer to [29] and [33] for the approach using gradient flow structure of the thin-film equation.

Notice that the second term in (1) involves the fourth order derivative and it is a stabilizing term. The third term is a destabilizing second derivative term. For short wave solutions, the stabilizing term dominates the destabilizing one so that the linearized equation of (1)-(2) is well-posed. However, for long wave solutions, the destabilizing term may dominate the stabilizing one such that the long wave instability may occur. The competition between stabilizing term and destabilizing term is represented by opposing signs for the corresponding terms in the free energy (6).

To the best of our knowledge, there are no results on existence of weak and strong solutions to (1) with unstable diffusion in multi-dimension before year 2014. In 2014, Taranets and King [35] proved local existence of nonnegative weak and strong solutions in a bounded domain Ω with smooth boundary in $\mathbb{R}^d$ under a more restrictive threshold $M_0 < \bar{M}_d$. In one dimension ($d = 1$), their threshold is $\bar{M}_1 = 1/\sqrt{12} < M_c$. Below we review some results on the long-wave unstable thin film equation in one dimensional case.

A slightly more general version of the long-wave unstable thin film equation is given by

$$h_t + \partial_x\Big(h^n \partial_{xxx} h\Big) + \partial_x\Big(h^m \partial_x h\Big) = 0, \tag{14}$$

and it was studied by Bertozzi, Pugh, and others in a series of papers [6]-[9], [13, 26, 34, 36]. Here $n > 0$ denotes the exponent of the mobility, and $m > 0$ is the power of the destabilizing second order term.

The classification for the critical ($m = n + 2$), super-critical ($m > n + 2$), and sub-critical ($m < n + 2$) cases can be obtained by the mass invariant scaling $h_\lambda = \lambda h(\lambda x, \lambda^{n+4} t)$. These three regimes were first introduced and studied by Bertozzi and Pugh [8] for (14). Details are discussed below.

In the subcritical case $m < n+2$, for relatively thick films (large λ) the stabilizing term (a pre-factor λ^{n+5} for this term in the re-scaling) dominates the destabilizing one (a pre-factor λ^{m+3} for this term in the re-scaling) and the blow-up is precluded. For relatively thin films (small λ), the destabilizing term dominates the stabilizing one and prevents spreading. Global existence and finite speed of propagation of the support of solutions were proved for any large initial data in [8] (also see [13]). On the contrary, for the supercritical case $m > n+2$, the destabilizing term dominates the stabilizing one for relatively thick films so that solutions may blow up in finite time. For example, in [9] blow-up phenomenon is given for $n = 1$. In [13] blow-up phenomenon is obtained for $n \in (0, 2)$. The reference [14] shows blow-up phenomenon and mass concentration for $n \in (0, 3)$. For relatively thin films, the stabilizing term dominates the destabilizing one and solutions have infinite-time spreading.

For $m = n+2$, the fourth order stabilizing term is balanced by the second order destabilizing term and this case is called the critical case. For the critical-case model, there is a critical mass M_c that can be used to distinguish between global existence and finite time blow-up. As discussed below for the special case with $n = 1$, both global existence and blow-up may occur depending on whether the initial mass is less than or larger than the critical mass M_c. Furthermore, let $h(x, t)$ be a solution to (1)-(2), then the mass invariant re-scaling $\lambda h(\lambda x, \lambda^{n+4} t)$ is also a solution to (1)-(2) (see [34]). Using this property, Slepčev and Pugh [34] proved that (14) cannot have self-similar blow-up solutions if $n \geq \frac{3}{2}$. For $0 < n < \frac{3}{2}$, there are compactly supported, symmetric, self-similar solutions that blow up in finite time. They also showed that any self-similar solution must have mass less that M_c. In [36], for the $n = 1$ case of equation (1) Witelski, Bernoff, and Bertozzi studied infinite-time self-similar spreading behavior below the critical mass, and finite time blow-up self-similar solutions beyond the critical mass by numerical simulations and asymptotic analysis.

Another way to understand the critical mass M_c is through the steady solution and the Sz. Nagy inequality as we discuss below. The critical mass sometimes is given by the mass of equilibrium solutions. For equilibrium solutions h_{eq}, the dissipation term on the right side of (7) is zero, and the equilibrium chemical potential is given by

$$\begin{cases} \mu_{eq}(x) = \bar{C}, & x \in \text{ supp } h_{eq}, \\ \mu_{eq}(x) \geq \bar{C}, & \text{otherwise} \end{cases} \tag{15}$$

for some constant $\bar{C}$. In other words, equilibrium solutions are Nash equilibria [12]. Denote an equilibrium profile as h_α with parameter α as the height at the center peak (assumed to be located at $x = 0$), i.e., $h_\alpha(0) = \alpha$, $\partial_x h_\alpha(0) = 0$. From (15), one can solve for the equilibrium profile h_α within its support

$$\partial_{xx} h_\alpha + \frac{1}{3} h_\alpha^3 = -\bar{C}, \quad h_\alpha(0) = \alpha, \quad \partial_x h_\alpha(0) = 0. \tag{16}$$

Together with the decay property at infinity, one knows that a first integral is given by

$$\frac{(\partial_x h_\alpha)^2}{2} + \frac{h_\alpha^4}{12} + \bar{C} h_\alpha \equiv 0. \tag{17}$$

Evaluating the above integral at $x = 0$ and using $h_\alpha(0) = \alpha$, $\partial_x h_\alpha(0) = 0$, one knows that $\bar{C} = -\alpha^3/12$. As shown in (i) below, h_α has a compact support. Outside of its support we have $\mu_{eq} \equiv 0 \geq \bar{C}$. Thus (15) holds. (17) can be recast as

$$(\partial_x h_\alpha)^2 = \frac{1}{6}(\alpha^3 h_\alpha - h_\alpha^4). \tag{18}$$

After some simple calculations, we have the following properties for h_α:

(i) The solution $h_\alpha(x)$ has compact support, i.e., there exists $x_\alpha > 0$ such that $h_\alpha(x) > 0$ for $|x| < x_\alpha$ and $h_\alpha(x) = 0$ as $|x| \geq x_\alpha$, and x_α satisfies

$$\alpha \cdot x_\alpha = \sqrt{6} \int_0^1 \frac{1}{\sqrt{x - x^4}} dx = \frac{\sqrt{6}}{3} \mathcal{B}(1/6, 1/2),$$

where $\mathcal{B}(x, y)$ is the Beta function. From the above, one knows that the support of the solution to (16) can be arbitrarily narrow provided that α is large.

(ii) $\partial_x h_\alpha(x) < 0$ for $x \in (0, x_\alpha)$. From (18), we have $\lim_{x \to x_\alpha^-} \partial_x h_\alpha(x) = 0$ and $\partial_x h_\alpha(x) = 0$ for $x > x_\alpha$. Hence $\partial_x h_\alpha(x)$ is continuous in $\mathbb{R}$ and $\partial_x h_\alpha(x_\alpha) = 0$. Noticing the second derivative $\partial_{xx} h_\alpha$ is discontinuous at x_α (hence μ_{eq} is also discontinuous at x_α), i.e.,

$$\lim_{x \to x_\alpha^-} \partial_{xx} h_\alpha(x) = \frac{1}{12}\alpha^3 < \infty, \quad \lim_{x \to x_\alpha^+} \partial_{xx} h_\alpha(x) = 0.$$

Therefore $\partial_x h_\alpha(x)$ is Lipschitz continuous in $\mathbb{R}$.

(iii) $\|h_\alpha(x)\|_{L^1(\mathbb{R})} = \frac{2\sqrt{6}\pi}{3} =: M_c$, which is a universal constant independent of α and is the critical mass.

(iv) $\mathcal{F}(h_\alpha(x)) = 0$.

In fact, multiplying h_α to (16) and integrating in $(-x_\alpha, x_\alpha)$, we get

$$-\int_{-x_\alpha}^{x_\alpha} (\partial_x h_\alpha)^2 \, dx + \frac{1}{3} \int_{-x_\alpha}^{x_\alpha} h_\alpha^4 \, dx = \frac{\alpha^3}{12} M_c. \tag{19}$$

On the other hand, integrating (18) in $(-x_\alpha, x_\alpha)$, it holds

$$\frac{1}{2} \int_{-x_\alpha}^{x_\alpha} (\partial_x h_\alpha)^2 \, dx + \frac{1}{12} \int_{-x_\alpha}^{x_\alpha} h_\alpha^4 \, dx = \frac{\alpha^3}{12} M_c. \tag{20}$$

Subtracting (19) from (20) gives that

$$3\mathcal{F}(h_\alpha) = \frac{3}{2} \int_{-x_\alpha}^{x_\alpha} (\partial_x h_\alpha)^2 \, dx - \frac{1}{4} \int_{-x_\alpha}^{x_\alpha} h_\alpha^4 \, dx = 0. \tag{21}$$

More properties of steady states for thin film equations can be found in [26, 34].

Witelski et al. in [36] found that the following Sz. Nagy inequality ([32], see also [2, pp. 167]) is closely connected to the critical mass M_c.

Proposition 1. *Let $f \in L^1(\mathbb{R})$ be a non-negative function and $f' \in L^2(\mathbb{R})$. Then f is in $L^4(\mathbb{R})$ and we have the estimate*

$$\frac{1}{12} \int_{\mathbb{R}} f^4 \, dx \leq \left(\frac{M}{M_c} \right)^2 \frac{1}{2} \int_{\mathbb{R}} f'^2 \, dx, \tag{22}$$

where $M = \int_{\mathbb{R}} f \, dx$, $M_c = \frac{2\sqrt{6}\pi}{3}$. The equality holds if $f = c\, h_\alpha(x - x_0)$ for any real numbers $c > 0$, $\alpha > 0$ and x_0. Here h_α is the unique solution to the following free boundary problem

$$h'' + \frac{1}{3} h^3 = \frac{\alpha^3}{12} \quad \text{for } 0 < r < x_\alpha,$$
$$h(0) = \alpha, \; h'(0) = 0, \; h(x_\alpha) = h'(x_\alpha) = 0.$$

If the initial mass M_0 is larger than the critical mass M_c, then there is an initial datum h_0 such that the solution to (1)-(2) blows up in finite time. In fact, taking $h_0(x) = (1+\varepsilon)h_\alpha(x)$, $\varepsilon > 0$, we have

$$\begin{aligned}\mathcal{F}(h_0) &= \frac{(1+\varepsilon)^2}{2}\Big(\int_{\mathbb{R}} \big(\partial_x h_\alpha(x)\big)^2\,dx - \frac{(1+\varepsilon)^2}{6}\int_{\mathbb{R}} h_\alpha^4(x)\,dx\Big) \\ &= -\frac{(1+\varepsilon)^2}{12}(2\varepsilon+\varepsilon^2)\int_{\mathbb{R}} h_\alpha^4(x)\,dx < 0, \end{aligned} \tag{23}$$

where we have used (21) in the second equality. Let $m_2(t) := \int_{\mathbb{R}} |x|^2 h(x,t)dx$ be the second moment. Then for $n = 1$, a simple computation gives that the time derivative of the second moment satisfies

$$\frac{d}{dt}m_2(t) = 6\mathcal{F}(h(\cdot,t)) \le 6\mathcal{F}(h_0) < 0, \tag{24}$$

which implies that there exists a finite time t^* such that $m_2(t^*) = 0$ if the initial second moment is finite. With some computations we obtain (c.f. [12, formula (3.8)])

$$\|h(\cdot,t)\|_{L^4} \ge \left(\frac{M_0}{2}\right)^{\frac{11}{8}} 2^{-\frac{3}{4}}\,(m_2(t))^{-\frac{3}{8}}. \tag{25}$$

Hence by (25) and the fact $m_2(t^*) = 0$, we know that there is a $T_{max} \le t^*$ such that

$$\limsup_{t\to T_{max}} \|h(\cdot,t)\|_{L^4} = \infty.$$

Some blow-up results were also given in papers [9, 36]. Consequently, M_c can be used as the critical mass to distinguish between global existence and finite time blow-up for the thin film equation (1) with $n = 1$. For $n > 1$, (24) does not hold. The question of finite time blow-up is still open.

In this paper we will prove that for $n \ge 1$ and initial data satisfying $M_0 < M_c$, there exists a global non-negative weak solution to the models (1)-(2) with the two classes of initial data (3) and (4). Throughout this paper, we will use C to denote positive constants, which may be different for each calculation.

Before defining weak solutions, we need to review literatures on possible singularities that may appear to the higher derivatives of weak solutions. For the thin film equation without the long-wave unstable term, Giacomelli, Knüpfer, and Otto [18], John [23], and Gnann[19] showed that for $n = 1$ solutions are smooth up to the free boundary. However, for the thin film equation with the long-wave unstable term, the control of solutions at the free boundary is subtle and we do not know if solutions have singularities or not. If there are singularities, however, they will occur in the set $\{h = 0\}$. Hence in the definition of an entropy weak solution, we need to define a set

$$P_T := (-L,L)\times(0,T)\setminus\{(x,t)|h(x,t)=0\}. \tag{26}$$

By Proposition 3 below, we know that P_T is an open set and we can define distribution functions on P_T. Now we give the following definition of an entropy weak solution to (1)-(2) with the periodic initial data (4).

Definition 1. (Entropy weak solution) We say that a non-negative function

$$h \in L^\infty\big(\mathbb{R}_+; L^1 \cap H^1(-L,L)\big), \tag{27}$$

and for any fixed $T > 0$,

$$h \in L^2(0,T;H^2(-L,L)), \quad h^{n/2}\partial_{xxx}h \in L^2(P_T), \tag{28}$$

$$\partial_t h \in L^2(0,T;H^{-1}(-L,L)) \tag{29}$$

is an entropy weak solution to (1)-(2) provided that

(i) For any $2L$-periodic function $\phi \in C^\infty\big(\mathbb{R}\times[0,T]\big)$ the following identity holds

$$\int_0^T\int_{-L}^L \phi h_t\,dxdt = \iint_{P_T} \partial_x\phi\; h^n\,\partial_{xxx}h\,dxdt + \int_0^T\int_{-L}^L \partial_x\phi\; h^{n+2}\partial_x h\,dxdt. \tag{30}$$

(ii) $\mathcal{F}(h(\cdot,t))$ is a non-increasing function in t and satisfies the following energy-dissipation inequality

$$\mathcal{F}(h(\cdot,t)) + \iint_{P_T} h^n\Big|\partial_x(\partial_{xx}h + \frac{h^3}{3})\Big|^2\,dxdt \le \mathcal{F}(h_0) \quad \text{for any } t>0. \tag{31}$$

The definition for an entropy weak solution to the Cauchy problem is similar. We omit details here.

Remark 1. Bertozzi and Pugh [8] introduce the notion of the “BF weak solution” to differentiate this solution from a weak solution in some sense of distributions in their paper. They referred to solutions of (12) with some regularity as BF weak solutions (see [8, Definition 3.1]). Later on for (12) with $0 < n < 3$, it was proved in [3, 4, 6, 7] that BF nonnegative weak solutions satisfy $h(\cdot,t) \in C^1[-L,L]$ for almost all $t > 0$. We shall remark that this kind of solutions was referred to as strong solutions in literature such as [3, 4]. To avoid confusion, we consistently use entropy weak solutions in this paper.

In Section 2 and Section 3, we will prove the following global existence theorem for the periodic problem (1)-(2).

Theorem 1. *Assume initial data h_0 satisfying (4), $M_0 := \int_{-L}^L h_0\,dx < M_c$ and $\mathcal{F}(h_0) < \infty$. Then for $1 \le n < 2$, there is a global non-negative entropy weak solution to (1)-(2) and it satisfies the following uniform estimate*

$$\sup_{0<t<\infty}\{\|h(\cdot,t)\|_{L^4(-L,L)} + \|h(\cdot,t)\|_{H^1(-L,L)}\} \le C(\|h_0\|_{H^1(-L,L)}, \mathcal{F}(h_0)). \tag{32}$$

Theorem 2. *Assume initial data h_0 satisfying (4), $M_0 < M_c$, $\mathcal{F}(h_0) < \infty$ and $G_M(h_0) < \infty$. Then*

1. *for $2 \le n < 4$, there is a global non-negative entropy weak solution to (1)-(2) with the uniform estimate (32);*
2. *for $n \ge 4$, there is a unique global positive entropy weak solution to (1)-(2) with the uniform estimate (32), and additional regularity $h \in L^2(0,T;H^3)$ and $P_T = (-L,L)\times(0,T)$.*

Next, in Section 4, we will consider global existence of entropy weak solutions to the Cauchy problem (1)-(2) with initial data (3). Global existence of the Cauchy problem with compactly supported initial data can be proved by using

(i) periodic extension,
(ii) finite speed of propagation of the support of solutions,
(iii) uniform estimates for H^1-norm of solutions.

The result is given as follows.

Theorem 3. *Assume $n=1$ and initial data h_0 satisfying (3), $M_0 := \int_{\mathbb{R}} h_0\,dx < M_c$ and $\mathcal{F}(h_0) < \infty$. Then there is a global non-negative entropy weak solution to (1)-(2) and it satisfies the following uniform estimate*

$$\sup_{0<t<\infty}\{\|h(\cdot,t)\|_{L^4(\mathbb{R})} + \|h(\cdot,t)\|_{H^1(\mathbb{R})}\} \le C(\|h_0\|_{H^1(\mathbb{R})}, \mathcal{F}(h_0)), \tag{33}$$

Furthermore, we will prove the following long-time behavior in Section 4.

Theorem 4. *Assume $n=1$ and initial data h_0 satisfying (3), $M_0 < M_c$ and $\mathcal{F}(h_0) < \infty$. Let $h(x,t)$ be a global non-negative entropy weak solution of (1)-(2) given by Theorem 3. Then at least one of the following results holds*

$$(a)\ \lim_{t\to\infty} m_2(t) = \infty, \tag{34}$$

$$(b)\ h(\cdot,t_k) \rightharpoonup 0 \text{ in } L^1(\mathbb{R}^d) \text{ for some subsequence } t_k \to \infty. \tag{35}$$

2. Local existence and non-negativity.

In this section, we will prove local existence and non-negativity of entropy weak solutions to the thin film equation (1)-(2) with initial data (4).

2.1. Local existence for a regularized problem.

Define a standard mollifier $J(x) \in C^\infty(\mathbb{R})$ by

$$J(x) := \begin{cases} Ce^{\frac{1}{|x|^2-1}}, & \text{if } |x|<1,\\ 0, & \text{if } |x|\ge 1,\end{cases}$$

where constant $C>0$ selected so that $\int_{\mathbb{R}} J(x)\,dx = 1$. For each $\varepsilon > 0$, set $J_\varepsilon(x) := \frac{1}{\varepsilon}J(\frac{x}{\varepsilon})$.

We consider local existence of solutions for the following regularized problem in a period domain $(-L,L)$

$$\begin{cases} \partial_t h_\varepsilon + \partial_x\Big((h_\varepsilon^2+\varepsilon^2)^{\frac{n}{2}}\ \Big(\partial_{xxx}h_\varepsilon + \partial_x(\frac{h_\varepsilon^3}{3})\Big)\Big) = 0, & x\in(-L,L),\ t>0,\\ h_\varepsilon(x,0) = h_{\varepsilon 0}(x), & x\in(-L,L).\end{cases} \tag{36}$$

Here $h_{\varepsilon 0} := J_\varepsilon * h_0 + c\sqrt{\varepsilon} \in C^\infty(\mathbb{R})$ is a H^1- approximation sequence of initial data $h_0(x)$ and for ε sufficient small, it satisfies

$$h_{\varepsilon 0} \ge h_0, \quad \|h_{\varepsilon 0}\|_{L^1(-L,L)} < M_c, \tag{37}$$

$$\|h_{\varepsilon 0}\|_{H^1(-L,L)} \le \|h_0\|_{H^1(-L,L)} + c\sqrt{\varepsilon}. \tag{38}$$

Here the first inequality used the embedding theorem $H^1(-L,L) \hookrightarrow C^{1/2}[-L,L]$ and c depends only on $\|h_0\|_{H^1(-L,L)}$. We refer to [1] for a different regularized method. Now we give a definition of entropy weak solutions to the regularized problem (36).

Definition 2. (Entropy weak solutions) For any fixed $\varepsilon > 0$ and $T>0$, we say a $2L$-periodic function

$$h \in L^\infty(0,T;H^1(-L,L)), \quad (h^2+\varepsilon^2)^{\frac{n}{4}}\ \partial_{xxx}h \in L^2(0,T;L^2(-L,L)), \tag{39}$$

$$\partial_t h \in L^2(0,T;H^{-1}(-L,L)) \tag{40}$$

is an entropy weak solution of the equation (36) provided that

(i) For any $2L$-periodic function $\phi \in C^\infty\big(\mathbb{R}\times[0,T]\big)$ the following equality holds

$$\int_0^T\int_{-L}^L \phi\partial_t h\,dxdt = \int_0^T\int_{-L}^L \partial_x\phi\ (h^2+\varepsilon^2)^{n/2}\ \partial_x(\partial_{xx}h + \frac{h^3}{3})\,dxdt; \tag{41}$$

(ii) The energy-dissipation equality holds

$$\mathcal{F}(h(\cdot,t)) + \int_0^t \int_{-L}^{L} (h^2+\varepsilon^2)^{n/2} \left|\partial_x(\partial_{xx}h + \frac{h^3}{3})\right|^2 dxdt = \mathcal{F}(h_0), \tag{42}$$

where $t \in [0,T]$.

Proposition 2. *(Local existence of regularized solutions) For any $2L$-periodic initial data $h_0 \in H^1(-L,L)$, there exists a time $T = T(\|h_0\|_{H^1})$ such that if $t \in [0,T]$, the regularized problem (36) has an entropy weak solution h_ε and it satisfies the following uniform in ε estimates*

$$\|h_\varepsilon\|_{L^\infty(0,T;H^1(-L,L))} \le C, \quad \|\partial_t h_\varepsilon\|_{L^2(0,T;H^{-1}(-L,L))} \le C, \tag{43}$$

$$\|(h_\varepsilon^2+\varepsilon^2)^{n/4}\,\partial_{xxx}h_\varepsilon\|_{L^2(0,T;L^2(-L,L))} \le C, \tag{44}$$

and for any $(x_2,t_2),(x_1,t_1) \in [-L,L]\times[0,T]$, the following Hölder continuity holds

$$|h_\varepsilon(x_2,t_2) - h_\varepsilon(x_1,t_1)| \le C(|x_2-x_1|^{\frac{1}{2}} + |t_2-t_1|^{\frac{1}{8}}), \tag{45}$$

where C is independent of ε.

The proof of Proposition 2 is standard, for completeness we provide details in Appendix A.

Remark 2. As noticed by Bernis and Friedman [5], for $n \ge 4$, regularized solutions h_ε are positive, thus we can use directly the Sz. Nagy inequality and the energy functional for h_ε to prove global existence of weak solutions. However, for $1 \le n < 4$, there is no non-negativity of h_ε. Instead, we will show that limit functions h of h_ε are non-negative local weak solutions, and use the same strategy as above to obtain global existence of entropy weak solutions.

2.2. **Local existence of solutions for (1)-(2).** In this subsection, we will prove local existence of solutions to the equation (1)-(2) with initial data (4). In order to prove non-negativity and uniform bound of $\|\partial_{xx}h_\varepsilon\|_{L^2(0,T;L^2(-L,L))}$, analogous to (8), we introduce the following entropy density functions

$$g_{M,\varepsilon}(x) := -\int_x^M \frac{1}{(s^2+\varepsilon^2)^{n/2}}\,ds, \quad G_{M,\varepsilon}(x) := -\int_x^M g_{M,\varepsilon}(s)\,ds, \tag{46}$$

where M is taken so that $M > \max\{1, \|h_\varepsilon\|_{L^\infty(0,T;\ L^\infty(-L,L))}\}$. For $n=1$, $g_\varepsilon(x)$ and $G_\varepsilon(x)$ have the following exact forms for $x \in (-\infty, M]$

$$g_{M,\varepsilon}(x) = \ln\frac{\sqrt{x^2+\varepsilon^2}+x}{\sqrt{M^2+\varepsilon^2}+M} \le 0, \tag{47}$$

$$G_{M,\varepsilon}(x) = x\ln\frac{\sqrt{x^2+\varepsilon^2}+x}{\sqrt{M^2+\varepsilon^2}+M} - \sqrt{x^2+\varepsilon^2} + \sqrt{M^2+\varepsilon^2}. \tag{48}$$

A simple computation gives the following lemma

Lemma 1. *The functions $g_{M,\varepsilon}(x)$ and $G_{M,\varepsilon}(x)$ in (46) satisfy the following properties*

1. *If $x \le M$, then for any $\varepsilon > 0$,*

$$G_{M,\varepsilon}(x) \ge 0, \quad G'_{M,\varepsilon}(x) = g_{M,\varepsilon}(x) \le 0; \tag{49}$$

2. *If $-M \le x < 0$, then*

(a) for $n = 1$, we have

$$G_{M,\varepsilon}(x) \geq |x| \ln \frac{1}{\varepsilon}. \tag{50}$$

(b) for $n > 1$, it holds that

$$G_{M,\varepsilon}(x) \geq C_0 \varepsilon^{1-n} |x|, \tag{51}$$

where $C_0 := \int_0^1 \frac{1}{(s^2+1)^{n/2}} \, ds$ is a positive constant.

Proof. The result (1) is obvious. We only need to prove (2).

For $n = 1$, since $|x| \leq M$, we have

$$G_{M,\varepsilon}(x) \geq x \ln \frac{\sqrt{x^2+\varepsilon^2}+x}{\sqrt{M^2+\varepsilon^2}+M} = x \ln \frac{\varepsilon^2}{(\sqrt{x^2+\varepsilon^2}-x)(\sqrt{M^2+\varepsilon^2}+M)}.$$

Notice that if $x < 0$, then

$$\begin{aligned} G_{M,\varepsilon}(x) &\geq 2x \ln \varepsilon - x \ln(\sqrt{x^2+\varepsilon^2}-x) - x \ln(\sqrt{M^2+\varepsilon^2}+M) \\ &\geq 2|x| \ln \frac{1}{\varepsilon} + |x| \ln(\sqrt{|x|^2+\varepsilon^2}+|x|) \\ &\geq |x| \ln \frac{1}{\varepsilon}. \end{aligned}$$

Hence (50) holds.

For $n > 1$, by (46), we have

$$G_{M,\varepsilon}(x) = \int_x^M \int_s^M \frac{1}{(\tau^2+\varepsilon^2)^{n/2}} \, d\tau ds = \varepsilon^{1-n} \int_x^M \int_{\frac{s}{\varepsilon}}^{\frac{M}{\varepsilon}} \frac{1}{(z^2+1)^{n/2}} \, dz ds.$$

Noticing $-M \leq x < 0$, we deduce

$$G_{M,\varepsilon}(x) \geq \varepsilon^{1-n} \int_x^0 \int_0^1 \frac{1}{(z^2+1)^{n/2}} \, dz ds \geq \varepsilon^{1-n} |x| \int_0^1 \frac{1}{(z^2+1)^{n/2}} \, dz = C_0 \varepsilon^{1-n} |x|,$$

which means that (51) is true. □

The following lemma provides uniform bounds for $\|\partial_{xx} h_\varepsilon\|_{L^2(0,T;L^2(-L,L))}$ and $\int_{-L}^L G_{M,\varepsilon}(h_\varepsilon) \, dx$.

Lemma 2. *Assume h_0 satisfying (4). Let $T = T(\|h_0\|_{H^1})$ be the local time given in Proposition 2. For $n \geq 2$, we further assume that $\int_{-L}^L G_M(h_0) \, dx < \infty$. Then we have*

(i) uniform in ε estimates

$$\|\partial_{xx} h_\varepsilon\|_{L^2(0,T;L^2(-L,L))} \leq C(M_0, \|h_0\|_{H^1(-L,L)}), \tag{52}$$

$$\int_{-L}^L G_{M,\varepsilon}(h_\varepsilon) \, dx \leq C(M_0, \|h_0\|_{H^1(-L,L)}), \tag{53}$$

(ii) mass conservation

$$\int_{-L}^L h_\varepsilon(x,t) \, dx = \int_{-L}^L h_{\varepsilon 0}(x) \, dx, \tag{54}$$

(iii) the L^1-norm goes to zero in the region $\{h_\varepsilon < 0\}$ as $\varepsilon \to 0$, i.e.,

$$\int_{\{h_\varepsilon<0\}} |h_\varepsilon|\,dx \le C\Big(\ln\frac{1}{\varepsilon}\Big)^{-1} \text{ for } n = 1, \tag{55}$$

$$\int_{\{h_\varepsilon<0\}} |h_\varepsilon|\,dx \le C\varepsilon^{n-1} \text{ for } n > 1. \tag{56}$$

Proof. For (i), from the definitions of $G_{M,\varepsilon}(h)$ and $G_M(h)$, it holds that

$$\int_{-L}^{L} G_{M,\varepsilon}(h_{\varepsilon 0})\,dx \le \int_{-L}^{L} G_M(h_{\varepsilon 0})\,dx. \tag{57}$$

For $1 \le n < 2$, using (47) and (48), we can obtain

$$\int_{-L}^{L} G_M(h_{\varepsilon 0})\,dx \le C(M_0, \|h_0\|_{H^1(-L,L)}). \tag{58}$$

For $n \ge 2$, using the facts (37) and (38) and noticing the initial assumption $\int_{-L}^{L} G_M(h_0)\,dx < \infty$, we have for sufficient small ε

$$\int_{-L}^{L} G_M(h_{\varepsilon 0})\,dx \le \int_{-L}^{L} G_M(h_0)\,dx < \infty. \tag{59}$$

Hence (57), (58) and (59) imply that

$$\int_{-L}^{L} G_{M,\varepsilon}(h_{\varepsilon 0}) < \infty \text{ for any } n \ge 1. \tag{60}$$

Multiplying (36) by $g_{M,\varepsilon}(h_\varepsilon)$ after integration on x, we have

$$\begin{aligned} \frac{d}{dt}\int_{-L}^{L} G_{M,\varepsilon}(h_\varepsilon)\,dx &= \int_{-L}^{L} \big(h_\varepsilon^2+\varepsilon^2\big)^{n/2}\ \Big(\partial_{xxx}h_\varepsilon + h_\varepsilon^2\partial_x h_\varepsilon\Big) g'_{M,\varepsilon}(h_\varepsilon)\partial_x h_\varepsilon\,dx \\ &= -\int_{-L}^{L} (\partial_{xx}h_\varepsilon)^2\,dx + \int_{-L}^{L} h_\varepsilon^2(\partial_x h_\varepsilon)^2\,dx, \end{aligned} \tag{61}$$

where we used $g'_{M,\varepsilon}(h_\varepsilon) = \frac{1}{(h_\varepsilon^2+\varepsilon^2)^{n/2}}$. Noticing (43), integrating (61) gives that

$$\int_{-L}^{L} G_{M,\varepsilon}(h_\varepsilon(x,T))\,dx + \int_0^T\int_{-L}^{L} (\partial_{xx}h_\varepsilon)^2\,dx \le \int_{-L}^{L} G_{M,\varepsilon}(h_{\varepsilon 0})\,dx + C, \tag{62}$$

where C is independent of ε. Hence (60) and (62) imply the estimate (53).

Property (ii) is obvious.

For (iii), using (49), (50) and (51) of Lemma 1, we know that

$$\begin{aligned} \int_{-L}^{L} G_{M,\varepsilon}(h_\varepsilon)\,dx &\ge \int_{h_\varepsilon<0} G_{M,\varepsilon}(h_\varepsilon)\,dx \ge \ln\frac{1}{\varepsilon}\int_{h_\varepsilon<0} |h_\varepsilon|\,dx \quad \text{for } n=1, \\ \int_{-L}^{L} G_{M,\varepsilon}(h_\varepsilon)\,dx &\ge C_0\varepsilon^{1-n}\int_{h_\varepsilon<0} |h_\varepsilon|\,dx \quad \text{for } n>1. \end{aligned}$$

Noticing that the left side term in the above inequality is bounded due to Lemma 2, we obtain (55) and (56). □

Now we give local existence of entropy weak solutions to the periodic problem (1)-(2) and its non-negativity. The results on non-negativity of weak solutions were mostly proved by using Hölder continuity in previous papers. In the following proposition, we provide an alternative proof in Sobolev space and do not use Hölder continuity in hope to generalize the results to multi-dimensional thin film equations.

Proposition 3. *Assume that the initial datum h_0 satisfies (4), and $\int_{-L}^{L} G_M(h_0)\,dx < \infty$ when $n \geq 2$. Then the problem (1)-(2) has a non-negative entropy weak solution h in $[0,T]$ and it satisfies $h \in C^{1/2,1/8}([-L,L]\times[0,T])$, where $T = T(\|h_0\|_{H^1(-L,L)})$ is the local time given in Proposition 2.*

Proof. **Step 1**. Strong convergence.

By Proposition 2 and Lemma 2, we know that solutions to (36) satisfy the following uniform estimates

$$\|h_\varepsilon\|_{L^\infty(0,T;H^1(-L,L))} \leq C,\ \|h_\varepsilon\|_{L^2(0,T;H^2(-L,L))} \leq C, \tag{63}$$

$$\|(h_\varepsilon^2+\varepsilon^2)^{\frac{n}{4}}\partial_{xxx}h_\varepsilon\|_{L^2(0,T;L^2(-L,L))} \leq C,\ \|\partial_t h_\varepsilon\|_{L^2(0,T;H^{-1}(-L,L))} \leq C, \tag{64}$$

where the constant C is independent of ε. Consequently, as $\varepsilon \to 0$, there exists a subsequence of h_ε (still denoted by h_ε), and h satisfying (63)-(64) such that

$$h_\varepsilon \overset{*}{\rightharpoonup} h \qquad \text{in } L^\infty(0,T;H^1(-L,L)), \tag{65}$$

$$h_\varepsilon \rightharpoonup h \qquad \text{in } L^2(0,T;H^2(-L,L)), \tag{66}$$

$$\partial_t h_\varepsilon \rightharpoonup \partial_t h \qquad \text{in } L^2(0,T;H^{-1}(-L,L)). \tag{67}$$

Using the Lions-Aubin lemma [27], there exists a subsequence of h_ε (still denoted by h_ε) such that

$$h_\varepsilon \to h \text{ in } L^2(0,T;H^1(-L,L)) \text{ as } \varepsilon \to 0. \tag{68}$$

Hence

$$h_\varepsilon \to h \quad a.e. \quad \text{as } \varepsilon \to 0. \tag{69}$$

From (45), we have for any $x_1, x_2 \in [-L,L]$ and $t_1, t_2 \in [0,T]$, the following estimate holds

$$|h(x_2,t_2) - h(x_1,t_1)| \leq C(|x_2-x_1|^{\frac12} + |t_1-t_2|^{\frac18}), \quad \text{for } t \in (0,T), \tag{70}$$

that is $h \in C^{1/2,1/8}([-L,L]\times[0,T])$.

Step 2. Non-negativity of h almost everywhere.

In this step, we use the contradiction method to prove that the limit function h is non-negative in $(-L,L)\times[0,T]$ almost everywhere. If not, we have $|\{(x,t)|h(x,t) < 0\}| \neq 0$. Noticing that $\{(x,t)|h(x,t)<0\} = \bigcup_{m=1}^\infty \{(x,t)|h(x,t) < -\frac{1}{m}\}$, then there exist m_0 and

$$A := \Big\{(x,t)|h(x,t) < -\frac{1}{m_0}\Big\} \tag{71}$$

such that $|A| := \alpha_0 > 0$. Let

$$C_{\alpha_0}(t) = \{x|(x,t)\in A\}, \quad D_{\alpha_0} = \{t\big||C_{\alpha_0}(t)| > 0\}.$$

We know that $|D_{\alpha_0}| > 0$, and $\bigcup_{t\in D_{\alpha_0}}\{t\}\times C_{\alpha_0}(t) \subset A$.

Due to (68), there exists a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ of ε ($\varepsilon_k > 0$, and $\varepsilon_k \to 0$ as $k \to +\infty$) such that for any $(x,t) \in \bigcup_{t\in D_{\alpha_0}}\{t\}\times C_{\alpha_0}(t)$, we have

$$\int_{D_{\alpha_0}} \|h_{\varepsilon_k}(\cdot,t) - h(\cdot,t)\|^2_{L^\infty(C_{\alpha_0}(t))}\,dt < \big(\frac{1}{2m_0}\big)^2\frac{|D_{\alpha_0}|}{2^{k+1}}. \tag{72}$$

We define

$$B = \{t \in D_{\alpha_0},\ \text{for all } k, x \in C_{\alpha_0}(t), h_{\varepsilon_k}(x,t) \leq -\frac{1}{2m_0}\},$$

and

$$A_k = \{t \in D_{\alpha_0},\ \text{there is } x \in C_{\alpha_0}(t) \text{ such that } h_{\varepsilon_k}(x,t) > -\frac{1}{2m_0}\}.$$

From the definition of A_k and (71), we know that

$$\begin{aligned}(\frac{1}{2m_0})^2\frac{|D_{\alpha_0}|}{2^{k+1}} &\geq \int_{D_{\alpha_0}} \|h_{\varepsilon_k}(\cdot,t)-h(\cdot,t)\|^2_{L^\infty(C_{\alpha_0}(t))}\,dt\\ &\geq \int_{A_k} \|h_{\varepsilon_k}(\cdot,t)-h(\cdot,t)\|^2_{L^\infty(C_{\alpha_0}(t))}\,dt\\ &\geq (\frac{1}{2m_0})^2|A_k|,\end{aligned}$$

which means $|A_k| \leq |D_{\alpha_0}|/2^{k+1}$. Hence we have

$$|\cup_{k=1}^\infty A_k| \leq \sum_{k=1}^\infty |D_{\alpha_0}|/2^{k+1} = |D_{\alpha_0}|/2.$$

Noticing that for any k, $B\cap A_k = \emptyset$, from the definition of these two subsets, we know $B\cap[\cup_{k=1}^\infty A_k] = \emptyset$. Thus we obtain

$$|B| = |D_{\alpha_0}| - |\cup_{k=1}^\infty A_k| \geq |D_{\alpha_0}|/2 > 0.$$

By non-negativity of $G_{M,\varepsilon}(x)$ for any $|x|\leq M$, we have

$$\int_0^T\int_{-L}^L G_{M,\varepsilon}(h_\varepsilon(x,t))\,dxdt \geq \int_B\int_{C_{\alpha_0}(t)} G_{M,\varepsilon}(h_\varepsilon(x,t))\,dxdt. \tag{73}$$

On the other hand, for $n=1$, by the formula (50) in Lemma 1, we know for any $-M\leq h_\varepsilon \leq -\frac{1}{2m_0}$,

$$G_{M,\varepsilon}(h_\varepsilon) \quad\geq\quad \frac{1}{2m_0}\ln\frac{1}{\varepsilon} \to +\infty,\ \text{as } \varepsilon\to 0^+. \tag{74}$$

Thus by (73), we deduce

$$\limsup_{\varepsilon\to 0^+}\int_0^T\int_{-L}^L G_{M,\varepsilon}(x)\,dxdt \geq \lim_{\varepsilon\to 0^+}\frac{1}{2m_0}|C_{\alpha_0}(t)|\cdot|B|\ln\frac{1}{\varepsilon} = +\infty, \tag{75}$$

which is a contradiction with the uniform estimate

$$\int_0^T\int_{-L}^L G_{M,\varepsilon}(h_\varepsilon)\,dxdt \leq C(M_0,\|h_0\|_{H^1(-L,L)}).$$

For $n>1$, all the arguments above are exactly same except $\ln\frac{1}{\varepsilon}$ in (75) and (74) is replaced by ε^{1-n} in view of using (51) in Lemma 1.

Step 3. h is a local solution of (1)-(2).

Now we show that the non-negative limit function h in (65) is a local entropy weak solution to (1)-(2). Passing to the limit for h_ε in (41) as $\varepsilon\to 0$, we can obtain for any $2L$-periodic function $\phi\in C^\infty(\mathbb{R}\times[0,T])$

$$\int_0^T\int_{-L}^L \phi\ \partial_t h\,dxdt = \iint_{P_T}\partial_x\phi\ h^n\ \partial_{xxx}h\,dxdt + \int_0^T\int_{-L}^L\partial_x\phi\ h^{n+2}\partial_x h\,dxdt. \tag{76}$$

The details of the proof for (76) are given below. The convergence of the left side term and the second term on the right side in (41) can be directly obtained from the convergence of (66), (67) and (68), i.e.,

$$\int_0^T \int_{-L}^L \phi \, \partial_t h_\varepsilon \, dxdt \to \int_0^T \int_{-L}^L \phi \, \partial_t h \, dxdt, \tag{77}$$

$$\int_0^T \int_{-L}^L \partial_x \phi \, \left(h_\varepsilon^2 + \varepsilon^2\right)^{n/2} h_\varepsilon^2 \, \partial_x h_\varepsilon \, dxdt \to \int_0^T \int_{-L}^L \partial_x \phi \, h^{n+2} \partial_x h \, dxdt. \tag{78}$$

The limit for the first term on the right side in (41) is given by the following claim.

Claim. *We have as* $\varepsilon \to 0$

$$\int_0^T \int_{-L}^L \partial_x \phi \, \left(h_\varepsilon^2 + \varepsilon^2\right)^{n/2} \, \partial_{xxx} h_\varepsilon \, dxdt \to \iint_{P_T} \partial_x \phi \, h^n \, \partial_{xxx} h \, dxdt, \tag{79}$$

where P_T *is defined in (26).*

Proof of claim. For any fixed $\delta > 0$, using the strong convergence (68) in $L^2(0,T;H^1(-L,L))$, similar to (72), we know that there exists a subsequence $\{\varepsilon_k\}_{k=1}^\infty$ of ε ($\varepsilon_k > 0$, and $\varepsilon_k \to 0$ as $k \to +\infty$) such that

$$\int_0^T \|h_{\varepsilon_k}(\cdot,t) - h(\cdot,t)\|^2_{L^\infty(-L,L)} \, dt < \frac{\delta^3}{2^{2k}}. \tag{80}$$

Notice that

$$\begin{aligned}
& \int_0^T \int_{-L}^L \partial_x \phi \, \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \, \partial_{xxx} h_{\varepsilon_k} \, dxdt \\
= \; & \iint_{\{h>\delta\}} \partial_x \phi \, \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \, \partial_{xxx} h_{\varepsilon_k} \, dxdt \\
& + \iint_{\{h\le\delta\}} \partial_x \phi \, \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \, \partial_{xxx} h_{\varepsilon_k} \, dxdt \\
=: \; & I_1 + I_2.
\end{aligned} \tag{81}$$

For I_2, denote

$$\begin{aligned}
& C_\delta(t) := \{x \in (-L,L) \mid 0 \le h(x,t) \le \delta\}, \\
& B := \{t \in [0,T] \mid \text{ for all } k \in \mathbb{N}, x \in C_\delta(t), |h_{\varepsilon_k}(x,t)| \le 2\delta\}, \\
& A_k := \{t \in [0,T] \mid \text{ there is } x \in C_\delta(t) \text{ such that } |h_{\varepsilon_k}(x,t)| > 2\delta\}.
\end{aligned} \tag{82}$$

From the above definitions, one has $B \cup (\cup_{k=1}^\infty A_k) = [0,T]$. Using the definition of A_k, we know that

$$\begin{aligned}
\int_0^T \|h_{\varepsilon_k}(\cdot,t) - h(\cdot,t)\|^2_{L^\infty(C_\delta(t))} \, dt \;\; & \ge \;\; \int_{A_k} \|h_{\varepsilon_k}(\cdot,t) - h(\cdot,t)\|^2_{L^\infty(C_\delta(t))} \, dt \\
& \ge \;\; \delta^2 |A_k|.
\end{aligned} \tag{83}$$

Thus (80) and (83) imply $|A_k| \le \delta/2^{2k}$. We decompose I_2 as follows

$$\begin{aligned}
I_2 \;\; = \;\; & \int_{\cup_{k=1}^\infty A_k} \int_{C_\delta(t)} \partial_x \phi \, \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \, \partial_{xxx} h_{\varepsilon_k} \, dxdt \\
& + \int_B \int_{C_\delta(t)} \partial_x \phi \, \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \, \partial_{xxx} h_{\varepsilon_k} \, dxdt \\
=: \;\; & J_1 + J_2.
\end{aligned}$$

Using the Hölder inequality and the estimate (64), we have

$$\begin{aligned} J_2 &= \int_B \int_{C_\delta(t)} \partial_x \phi \ \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \ \partial_{xxx} h_{\varepsilon_k} \, dxdt \\ &\leq \Big(\int_B \int_{C_\delta(t)} (\partial_x \phi)^2 \ \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{\frac{n}{2}} \ dxdt \Big)^{\frac{1}{2}} \\ &\quad \cdot \Big(\int_B \int_{C_\delta(t)} \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{\frac{n}{2}} \ (\partial_{xxx} h_{\varepsilon_k})^2 \, dxdt \Big)^{\frac{1}{2}} \\ &\leq C(\delta + \varepsilon_k)^{1/2}, \end{aligned} \tag{84}$$

and

$$\begin{aligned} J_1 &\leq \sum_{k=1}^{\infty} \int_{A_k} \int_{C_\delta(t)} \Big| \partial_x \phi \ \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \ \partial_{xxx} h_{\varepsilon_k} \Big| \, dxdt \\ &\leq C \sum_{k=1}^{\infty} \Big(\int_{A_k} \int_{C_\delta(t)} (\partial_x \phi)^2 \ \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \ dxdt \Big)^{1/2}. \end{aligned}$$

Since $\|h_{\varepsilon_k}\|_{L^\infty((-L,L)\times(0,T))} < C$, we obtain

$$J_1 \leq C \sum_{k=1}^{\infty} |A_k|^{1/2} \leq C \delta^{1/2} \sum_{k=1}^{\infty} \frac{1}{2^k} = C \delta^{1/2}. \tag{85}$$

Hence (85) and (84) imply

$$I_2 = J_1 + J_2 \leq C(\delta + \varepsilon_k)^{1/2}. \tag{86}$$

For I_1, similar to the above process, we denote

$$\begin{aligned} & C_\delta(t) := \{x \in (-L, L) \mid h(x,t) > \delta\}, \\ & B := \{t \in (0,T] \mid \ \text{for all } k \in \mathbb{N}, x \in C_\delta(t), h_{\varepsilon_k}(x,t) > \frac{\delta}{2}\}, \\ & A_k := \{t \in (0,T] \mid \ \text{there is } x \in C_\delta(t) \text{ such that } h_{\varepsilon_k}(x,t) \leq \frac{\delta}{2}\}, \\ & D_\delta := \bigcup_{t \in B} C_\delta(t) \times \{t\}, \quad E_\delta := \bigcup_{t \in (0,T]} C_\delta(t) \times \{t\}. \end{aligned} \tag{87}$$

From the above definitions, one has that $B \cup (\cup_{k=1}^{\infty} A_k) = (0,T]$, $D_\delta \subseteq E_\delta \subseteq P_T$ and the subset D_δ increases as $\delta \to 0^+$. By the definition of A_k, we know that

$$\begin{aligned} \int_0^T \|h_{\varepsilon_k}(\cdot,t) - h(\cdot,t)\|_{L^\infty(C_\delta(t))}^2 \, dt &\geq \int_{A_k} \|h_{\varepsilon_k}(\cdot,t) - h(\cdot,t)\|_{L^\infty(C_\delta(t))}^2 \, dt \\ &\geq (\delta/2)^2 |A_k|. \end{aligned} \tag{88}$$

So, (80) and (88) give $|A_k| \leq \frac{4\delta}{2^{2k}}$, which implies

$$\sum_{k=1}^{\infty} |A_k| \leq C\delta.$$

Hence we have

$$|E_\delta \setminus D_\delta| \leq \sum_k 2L|A_k| \leq C\delta. \tag{89}$$

Therefore for any $\eta > 0$, take $\delta_j = \frac{\eta}{2^j}$, we have the following relations

$$P_T = \bigcup_{j=1}^{\infty} E_{\delta_j}, \tag{90}$$

$$P_T \setminus \bigcup_{j=1}^{\infty} D_{\delta_j} \subset \bigcup_{j=1}^{\infty} (E_{\delta_j} \setminus D_{\delta_j}), \tag{91}$$

which give

$$|P_T \setminus \bigcup_{\delta>0} D_\delta| = |P_T \setminus \bigcup_{j=1}^{\infty} D_{\delta_j}| \le C\eta \sum_{j=1}^{\infty} \frac{1}{2^j} = C\eta.$$

Thus taking $\eta \to 0$, we have

$$|P_T \setminus \bigcup_{\delta>0} D_\delta| = 0. \tag{92}$$

We decompose I_1 as follows

$$\begin{aligned} I_1 &= \left(\int_{\cup_{k=1}^{\infty} A_k} \int_{C_\delta(t)} + \int_B \int_{C_\delta(t)} \right) \partial_x \phi \ \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \ \partial_{xxx} h_{\varepsilon_k} \, dxdt \\ &=: \ L_1 + L_2. \end{aligned}$$

Similar to J_1, we get

$$L_1 \le C\delta^{1/2}. \tag{93}$$

To estimate L_2, using the fact $h_{\varepsilon_k} > \frac{\delta}{2}$ in D_δ, and (64), we have

$$\iint_{[0,T]\times(-L,L)} \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \ (\partial_{xxx} h_{\varepsilon_k})^2 \, dxdt \le C.$$

Thus we have

$$\iint_{D_\delta} (\partial_{xxx} h_{\varepsilon_k})^2 \, dxdt \le \frac{C}{\left(\left(\frac{\delta}{2}\right)^2 + \varepsilon_k^2\right)^{n/2}} \le \frac{C}{\delta^n},$$

which implies that there exists a subsequence of h_{ε_k} (still denote h_{ε_k}) such that

$$\partial_{xxx} h_{\varepsilon_k} \rightharpoonup \partial_{xxx} h, \ \text{ as } k \to \infty \text{ in } L^2(D_\delta).$$

Together with (68), we have for $k \to \infty$,

$$L_2 = \int_B \int_{C_\delta(t)} \partial_x \phi \ \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \ \partial_{xxx} h_{\varepsilon_k} \, dxdt \to \int_B \int_{C_\delta(t)} \partial_x \phi \ h^n \ \partial_{xxx} h \, dxdt. \tag{94}$$

Hence from (81), (86), (93) and (94), we know that there exists $K > 0$ such that as $k > K$,

$$\Big| \int_0^T \int_{-L}^{L} \partial_x \phi \ \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{\frac{n}{2}} \ \partial_{xxx} h_{\varepsilon_k} \, dxdt - \int_B \int_{C_\delta(t)} \partial_x \phi \ h^n \ \partial_{xxx} h \, dxdt \Big| \le C\sqrt{\delta}, \tag{95}$$

which implies that

$$\lim_{\delta \to 0^+} \lim_{k \to \infty} \Big(\int_0^T \int_{-L}^{L} \partial_x \phi \ \left(h_{\varepsilon_k}^2 + \varepsilon_k^2\right)^{n/2} \ \partial_{xxx} h_{\varepsilon_k} \, dxdt - \iint_{D_\delta} \partial_x \phi \ h^n \ \partial_{xxx} h \, dxdt \Big) = 0.$$

Noticing (92), we get

$$\iint_{P_T} \partial_x \phi \ h^n \ \partial_{xxx} h \, dxdt = \lim_{\delta \to 0^+} \iint_{D_\delta} \partial_x \phi \ h^n \ \partial_{xxx} h \, dxdt.$$

Therefore the following limit holds

$$\lim_{k\to\infty}\int_0^T\int_{-L}^L \partial_x\phi\ \left(h_{\varepsilon_k}^2+\varepsilon_k^2\right)^{n/2}\ \partial_{xxx}h_{\varepsilon_k}\,dxdt=\iint_{P_T}\partial_x\phi\ h^n\ \partial_{xxx}h\,dxdt,$$

i.e., we obtain (79). This completes the proof of the claim. □

Therefore (77), (78) and (79) imply (76), i.e., h satisfies the weak form (30) in Definition 1.

Next we will prove that h is an entropy weak solution. The solution h_ε of the regularized problem (36) satisfies the energy-dissipation equality (42). Here we recall it,

$$\mathcal{F}(h_\varepsilon(\cdot,t))+\int_0^t\int_{-L}^L\left(h_\varepsilon^2+\varepsilon^2\right)^{n/2}\ \left(\partial_{xxx}h_\varepsilon+h_\varepsilon^2\partial_x h_\varepsilon\right)^2dxd\tau=\mathcal{F}(h_{\varepsilon 0}).$$

By the convergent relations (65)–(68) and (79), we know that if $\varepsilon\to 0$,

$$\int_{-L}^L(\partial_x h)^2\,dx\le\liminf_{\varepsilon\to 0}\int_{-L}^L(\partial_x h_\varepsilon)^2\,dx\quad a.e.\ t\in(0,T];$$
$$\int_{-L}^L h_\varepsilon^4\,dx\to\int_{-L}^L h^4\,dx\quad a.e.\ t\in(0,T].$$

Hence we get

$$\mathcal{F}(h(\cdot,t))\le\liminf_{\varepsilon\to 0}\mathcal{F}(h_\varepsilon(\cdot,t))\quad a.e.\ t\in(0,T].$$

From (78) and (79), we have

$$(h_\varepsilon^2+\varepsilon^2)^{n/2}\left(\partial_{xxx}h_\varepsilon+h_\varepsilon^2\partial_x h_\varepsilon\right)\rightharpoonup h^n\left(\partial_{xxx}h+h^2\partial_x h\right)\ \text{in}\ L^2(P_T),$$

which means

$$\iint_{P_T}h^n\left(\partial_{xxx}h+h^2\partial_x h\right)^2dxd\tau\le\liminf_{\varepsilon\to 0}\iint_{P_T}\left(h_\varepsilon^2+\varepsilon^2\right)^{n/2}\left(\partial_{xxx}h_\varepsilon+h_\varepsilon^2\partial_x h_\varepsilon\right)^2dxd\tau.$$

Together with

$$\mathcal{F}(h_{\varepsilon 0})\to\mathcal{F}(h_0),$$

we have that the non-negative function h satisfies the following energy-dissipation inequality

$$\mathcal{F}(h(\cdot,t))+\iint_{P_t}h^n\left(\partial_{xxx}h+h^2\partial_x h\right)^2dxdt\le\mathcal{F}(h_0),\ \text{a.e.}\ t\in[0,T]. \tag{96}$$

Since $\mathcal{F}(h_\varepsilon(\cdot,t))$ is decreasing in $(0,T)$ from (42), then Helly's selection theorem implies that $\mathcal{F}(h(\cdot,t))$ is also a decreasing function in $(0,T)$. Hence h is an entropy weak solution in [0,T] as that in Definition 1. □

3. **Global existence.** In this section, we will prove global existence of entropy weak solutions to the problem (1)-(2) under the sharp initial condition $M_0<M_c$ in the periodic domain. Namely, we prove that local weak solutions given by Proposition 3 are indeed global solutions.

Firstly, we use the energy-dissipation inequality (96) and the Sz. Nagy inequality (22) to prove the following lemma.

Lemma 3. *Assume that initial datum h_0 satisfies $M_0<M_c$, $\mathcal{F}(h_0)<\infty$. Let h, $T=T(\|h_0\|_{H^1})$ be a local entropy weak solution in $(0,T)$ given by Proposition 3. Then*

$$\|h\|_{L^\infty(0,T;H^1(-L,L))}\le C(M_0,\mathcal{F}(h_0)). \tag{97}$$

Proof. From (96) and non-negativity of h, we have

$$\mathcal{F}(h(\cdot,t)) \le \mathcal{F}(h_0), \quad t\in[0,T]. \tag{98}$$

To apply the Sz. Nagy inequality (22) in the periodic setting, we use a trick from [36] below. Suppose that h achieves its minimum h_{min} at $x^*(t)$. Hence $h_{min}\ge 0$. Denote

$$\bar h(x) = \begin{cases} h(x)-h_{min}\ge 0, & \text{if } x\in[x^*,x^*+2L],\\ 0, & \text{otherwise.}\end{cases}$$

Using the Sz. Nagy inequality (22) for $\bar h(x)$, we have

$$\int_{\mathbb{R}} \bar h^4(x)\,dx \le \frac{9}{4\pi^2}\Big(\int_{\mathbb{R}} \bar h(x)\,dx\Big)^2 \int_{\mathbb{R}} (\partial_x \bar h(x))^2\,dx. \tag{99}$$

By the definition of $\bar h(x)$, we obtain

$$\int_{x^*}^{x^*+2L} \bar h^4(x)\,dx \le \frac{9}{4\pi^2}\Big(\int_{x^*}^{x^*+2L} \bar h(x)\,dx\Big)^2 \int_{x^*}^{x^*+2L} (\partial_x \bar h(x))^2\,dx. \tag{100}$$

A simple computation gives

$$\begin{aligned}\int_{-L}^{L} h(x)^4\,dx &= \int_{x^*}^{x^*+2L} h(x)^4\,dx = \int_{x^*}^{x^*+2L} (\bar h(x)+h_{min})^4\,dx\\ &= \int_{x^*}^{x^*+2L} \bar h(x)^4\,dx + 4h_{min}\int_{x^*}^{x^*+2L} \bar h(x)^3\,dx\\ &\quad +6h_{min}^2\int_{x^*}^{x^*+2L} \bar h(x)^2\,dx + 4h_{min}^3\int_{x^*}^{x^*+2L} \bar h(x)\,dx + 2Lh_{min}^4.\end{aligned}$$

Using Young's inequality, we get

$$\int_{-L}^{L} h(x)^4\,dx \le (1+3\nu)\int_{x^*}^{x^*+2L} \bar h(x)^4\,dx + C(\nu)h_{min}^4 \tag{101}$$

with some $\nu>0$. Hence from (100) and (101), we know that

$$\int_{-L}^{L} h(x)^4\,dx \le \frac{6(1+3\nu)}{M_c^2}\Big(\int_{x^*}^{x^*+2L} \bar h(x)\,dx\Big)^2 \int_{x^*}^{x^*+2L} (\partial_x \bar h(x))^2\,dx + C(\nu)h_{min}^4. \tag{102}$$

Noticing

$$\int_{-L}^{L} (\partial_x h(x))^2\,dx = \int_{x^*}^{x^*+2L} (\partial_x \bar h(x))^2\,dx,$$

we obtain

$$\int_{-L}^{L} h(x)^4\,dx \le (1+3\nu)\frac{6}{M_c^2}\Big(M_0-2Lh_{min}\Big)^2 \int_{-L}^{L} (\partial_x h(x))^2\,dx + C(\nu)h_{min}^4. \tag{103}$$

From conservation of mass, we have

$$M_0 = \int_{-L}^{L} h\,dx \ge \int_{-L}^{L} h_{min}\,dx = 2Lh_{min},$$

which means

$$0\le h_{min}\le \frac{M_0}{2L}. \tag{104}$$

Thus we deduce

$$\begin{aligned}
&\mathcal{F}(h(\cdot,t))\\
=\ & \frac{1}{2}\int_{-L}^{L}(\partial_x h(x,t))^2\,dx-\frac{1}{12}\int_{-L}^{L}h(x,t)^4\,dx\\
\geq\ & \frac{1}{2}\Big(1-(1+3\nu)M_c^{-2}\big(M_0-2Lh_{min}\big)^2\Big)\int_{-L}^{L}(\partial_x h(x,t))^2\,dx-C(\nu)h_{min}^4\\
\geq\ & \frac{1}{2}\Big(1-(1+3\nu)\Big(\frac{M_0}{M_c}\Big)^2\Big)\int_{-L}^{L}(\partial_x h(x,t))^2\,dx-C(\nu)h_{min}^4.
\end{aligned}$$

Taking $\nu=\frac{M_c^2-M_0^2}{6M_0^2}$, we get $1-(1+3\nu)\Big(\frac{M_0}{M_c}\Big)^2>0$. Using (98) and (104), we have

$$\int_{-L}^{L}(\partial_x h(x,t))^2\,dx\leq C(M_0,\mathcal{F}(h_0)).$$

The formula (103) implies

$$\int_{-L}^{L}h(x)^4\,dx\leq C(M_0,\mathcal{F}(h_0)).$$

Hence (97) holds. This completes the proof of the lemma. □

Now we will use the H^1-estimate (97) and a bootstrap iterative technique to prove global existence of non-negative entropy weak solutions to the periodic problem (1)-(2).

Proof of Theorem 1. By (97) in Lemma 3, for any non-negative initial data satisfying that $M_0<M_c$ and initial free energy is finite, we have

$$\|h(\cdot,t)\|_{H^1(-L,L)}\leq C(M_0,\mathcal{F}(h_0))\qquad\text{for any } t\in[0,T],$$

where $T=T(\|h_0\|_{H^1})$ is given by Proposition 2.

For any initial data $\tilde{h}_0\geq 0$ satisfying that (i) $\int_{-L}^{L}\tilde{h}_0\,dx=M_0$, (ii) $\mathcal{F}(\tilde{h}_0)\leq\mathcal{F}(h_0)$ and (iii) $\|\tilde{h}(\cdot,t)\|_{H^1(-L,L)}\leq C(M_0,\mathcal{F}(h_0))$, by Proposition 2, we know that there is a fixed T_1 depending only on M_0 and $\mathcal{F}(h_0)$ such that a entropy weak solution exists in $[0,T_1]$.

At $t=T_1$, $h(x,t)$ still satisfies above (i), (ii) and (iii). Taking T_1 as a new initial time and $h(\cdot,T_1)$ as a new initial datum, we can obtain a non-negative entropy weak solution, which exists in $t\in[T_1,2T_1]$. We can continue this process and obtain a global solution in $\mathbb{R}_+$, and it satisfies

$$\|h\|_{L^\infty(\mathbb{R}_+;H^1(-L,L))}\leq C(M_0,\mathcal{F}(h_0)).$$

Therefore again using the Sz. Nagy inequality (22), we have (32). Furthermore, estimates (27)–(31) hold for any $T>0$. This completes the proof of Theorem 1. □

Proof of Theorem 2. For $n\geq 2$, we further assume $\int_{-L}^{L}G_M(h_0)\,dx<\infty$. By Lemma 2, we have that (52) and (53) still hold. Hence we can prove that there is a global non-negative entropy weak solution, which satisfies Hölder continuity (70).

Now we only need to prove positivity of weak solutions for $n \geq 4$ in part (ii) of Theorem 2.

Step 1. Positivity of h_ε for ε sufficiently small.

From Proposition 2 and Lemma 2, we know that solutions h_ε of the regularized problem (36) satisfy the space Hölder continuity (45) and for any fixed $T > 0$

$$\int_{-L}^{L} G_{M,\varepsilon}(h_\varepsilon)\,dx \leq C(T, M_0, \|h_0\|_{H^1(-L,L)}).$$

Now we use a contradiction method to prove $h_\varepsilon > 0$. If not, there is a point $(x_*, t_*) \in [-L, L] \times [0, T])$ such that $h_\varepsilon(x_*, t_*) = 0$. Then by (45) we have

$$|h_\varepsilon(x, t_*)| \leq C|x - x_*|^{1/2}, \text{ for any } x \in [-L, L].$$

Thus there is $\delta > 0$ such that if $|x - x_*| < \delta$, we have $C|x - x_*|^{1/2} < M$. Hence we deduce that

$$\begin{aligned} G_{M,\varepsilon}(h_\varepsilon) &= \int_{h_\varepsilon}^{M} \int_{y}^{M} \frac{1}{(s^2+\varepsilon^2)^{n/2}}\,dsdy \\ &\geq \int_{C|x-x_*|^{1/2}}^{M} \int_{y}^{M} \frac{1}{(s^2+\varepsilon^2)^{n/2}}\,dsdy \\ &\geq \int_{C|x-x_*|^{1/2}}^{M} \int_{y}^{M} \frac{1}{(s+\varepsilon)^{n}}\,dsdy \\ &= \int_{C|x-x_*|^{1/2}+\varepsilon}^{M+\varepsilon} \int_{y}^{M+\varepsilon} \frac{1}{s^n}\,dsdy \geq G_M(C|x-x_*|^{1/2}+\varepsilon). \end{aligned} \tag{105}$$

By (8), we know that for $n \geq 4$ and $0 \leq h \leq M$, it holds that

$$G_M(h) \geq \frac{1}{(n-1)(n-2)} h^{2-n} - \frac{1}{n-2} M^{2-n}. \tag{106}$$

Hence from (105) and (106) we have

$$G_{M,\varepsilon}(h_\varepsilon) \geq \frac{1}{(n-1)(n-2)} (C|x-x_*|^{1/2}+\varepsilon)^{2-n} - \frac{1}{n-2} M^{2-n}.$$

A simple computation gives that

$$\begin{aligned} \int_{-L}^{L} G_{M,\varepsilon}(h_\varepsilon)\,dx &\geq \int_{x_*}^{x_*+\delta} G_{M,\varepsilon}(h_\varepsilon)\,dx \\ &\geq C\int_{x_*}^{x_*+\delta} (C|x-x_*|^{1/2}+\varepsilon)^{2-n}\,dx - C, \end{aligned} \tag{107}$$

and

$$\int_{x_*}^{x_*+\delta} (C|x-x_*|^{1/2}+\varepsilon)^{2-n}\,dx \geq C \begin{cases} \dfrac{1}{\varepsilon^{n-4}}, & \text{if } n > 4, \\ \log\dfrac{1}{\varepsilon}, & \text{if } n = 4, \end{cases} \tag{108}$$

where the constant C is independent of ε. Hence (107) and (108) imply

$$C_T \geq \int_{-L}^{L} G_{M,\varepsilon}(h_\varepsilon)\,dx \geq C \begin{cases} \dfrac{1}{\varepsilon^{n-4}}, & \text{if } n > 4, \\ \log\dfrac{1}{\varepsilon}, & \text{if } n = 4, \end{cases} \tag{109}$$

which is a contradiction for sufficiently small ε.

Step 2. Positivity of limit functions h.

From Step 1, for any fixed T and sufficient small ε, there exists a minimum point $(x_*, t_*) \in [-L, L] \times [0, T]$, such that for any $(x, t) \in [-L, L] \times [0, T]$

$$0 < h_{\varepsilon,min} := h_\varepsilon(x_*, t_*) \le h_\varepsilon(x, t) \le M,$$

and for any $x \in [-L, L]$, it holds

$$0 < h_{\varepsilon,min} \le h_\varepsilon(x, t_*) \le h_{\varepsilon,min} + C_T|x - x_*|^{\frac{1}{2}}.$$

Noticing

$$G_{M,\varepsilon}(h_\varepsilon) \ge G_M(h_\varepsilon + \varepsilon),$$

we have

$$C_T \ge \int_{-L}^{L} G_{M,\varepsilon}(h_\varepsilon)\,dx \ge \int_{-L}^{L} G_M(h_\varepsilon + \varepsilon)\,dx \ge C_T \begin{cases} \dfrac{1}{(h_{\varepsilon,min} + \varepsilon)^{n-4}}, & \text{if } n > 4, \\ \log \dfrac{1}{h_{\varepsilon,min} + \varepsilon}, & \text{if } n = 4. \end{cases}$$

Hence $h_{\varepsilon,min} + \varepsilon \ge C_T$. Taking $0 < \varepsilon < \frac{C_T}{2}$, we have $h_\varepsilon \ge h_{\varepsilon,min} \ge \frac{C_T}{2}$. Hence limit functions h satisfy $h \ge \frac{C_T}{2}$. In other words, the thin film equation is non-degenerate and its solutions have further regularity $h \in L^2(0, T; H^3(-L, L))$ and $P_T = (-L, L) \times (0, T)$.

Step 3. Uniqueness for $n \ge 4$.

Let h_1 and h_2 be two solutions to (1)-(2) with the same initial datum. Denote $h = h_1 - h_2$. Using the fact $h_1, h_2 \ge c_0 > 0$ for $n \ge 4$ and some simple estimates, we have

$$\frac{d}{dt}\|h\|^2_{L^2(-L,L)} \le C\left(1 + \|(h_1)_{xxx}\|^2_{L^2(-L,L)}\right)\|h\|^2_{L^2(-L,L)}.$$

Hence the facts $\|h_1\|_{L^2(0,T;H^3(-L,L))} \le C$, $h(0) = 0$ and Grönwall's inequality imply that $h \equiv 0$. This completes the proof of Theorem 2. □

4. **Global existence and long-time behavior for the Cauchy problem.** In this section, we first prove global existence for the Cauchy problem (1)-(2) with $n = 1$ as stated in Theorem 3.

Proof of Theorem 3. **Step 1**. Local existence of weak solutions.

Since the initial data h_0 has compact support, $h_0 \in H^1(\mathbb{R})$, and supp $h_0 \in [-a, a]$, we take a $2(a + A_0)-$ periodic extension function $h_{0,period}$ defined in $\mathbb{R}$ such that $h_{0,period}$ satisfies the assumptions of Theorem 1, where A_0 is a constant and is given by (121) below. Hence from Theorem 1, we know that there exists a global non-negative weak solution h_{period} to the periodic problem (1)-(4) with initial data $h_{0,period}$ and h_{period} satisfies (27)-(29).

Now we apply results on finite speed of propagation for support of solutions in [4, 8, 13] to h_{period} within the periodic domain $(-(a + A_0), (a + A_0))$. Let $\zeta(t)$ denote the right boundary of support of solutions h_{period} within the periodic domain $(-(a + A_0), (a + A_0))$. Following the results in [8, Lemma 3.7 and Lemma 3.8] or [13] and using the method provided by Bernis in [4, Theorem 5.1], we have that there is a positive constant T_* such that if $0 < t < T_*$

$$\zeta(t) \le a + C_0 t^k \Big(\int_{-L}^{L} h_0^{(1+\lambda)} dx\Big)^\beta, \quad 0 < \lambda < 1, \tag{110}$$

where $\beta = \frac{1}{4\lambda+5}$ is a decreasing function of λ, and $k = \frac{\lambda+1}{4\lambda+5}$ is a increasing function of λ. As explained in [8], the unstable term in (1) is attractive and it shrinks the support of solutions. We omit details here.

Using (110) and taking $T_1 \in [0, T_*]$ such that

$$C_0 T_1^k \Big(\int_\Omega h_0^{(1+\lambda)} dx \Big)^\beta < A_0, \tag{111}$$

one has that for $0 \le t \le T_1$,

$$\text{supp } h_{period}(\cdot, t) \cap [-(a + A_0), a + A_0] \subset (-(a + A_0), (a + A_0)). \tag{112}$$

In other words, the support of solutions h_{period} to the local equation (1) in every periodic domain stays within the interior of the periodic domain for any $t \le T_1$. Thus we use h_{period} within one period to construct h as below, for any $t \in [0, T_1]$,

$$h(x,t) = \begin{cases} h_{period}(x,t), & \text{if } x \in (-(a + A_0), (a + A_0)), \\ 0, & \text{otherwise .} \end{cases} \tag{113}$$

Clearly, $h(x, t)$ is a local non-negative entropy weak solution for the Cauchy problem (1)-(2) with initial data h_0, and satisfies the following regularities

$$h \in L^\infty\big(0, T_1; L^1 \cap H^1(\mathbb{R})\big) \cap L^2\big(0, T_1; H^2(\mathbb{R})\big), \tag{114}$$

$$\partial_t h \in L^2\big(0, T_1; H^{-1}(\mathbb{R})\big), \tag{115}$$

and the energy-dissipation inequality (96).

Step 2. Estimate on $\|h\|_{L^\infty(0,T_1;H^1(\mathbb{R}))}$.

Using the Sz. Nagy inequality (22), we deduce

$$\begin{aligned} \mathcal{F}(h(\cdot,t)) &= \frac{1}{2}\int_{\mathbb{R}} (\partial_x h(x,t))^2\, dx - \frac{1}{12}\int_{\mathbb{R}} h(x,t)^4\, dx \\ &\ge \frac{1}{2}\Big(1 - M_c^{-2}\big(\int_{\mathbb{R}} h(x,t)\, dx\big)^2\Big) \int_{\mathbb{R}} (\partial_x h(x,t))^2\, dx. \\ &= \frac{1}{2}\Big(1 - \frac{M_0^2}{M_c^2}\Big) \int_{\mathbb{R}} (\partial_x h(x,t))^2\, dx. \end{aligned} \tag{116}$$

From $M_0 < M_c$ and the energy-dissipation inequality (96), one has

$$\int_{\mathbb{R}} \big(\partial_x h(x,t)\big)^2\, dx \le C\big(M_0, \mathcal{F}(h_0)\big), \quad 0 \le t \le T_1. \tag{117}$$

Again, using (22), we know

$$\int_{\mathbb{R}} h(x,t)^4\, dx \le C(M_0, \mathcal{F}(h_0)), \quad 0 \le t \le T_1. \tag{118}$$

Hence

$$\|h\|_{L^\infty(0,T_1;H^1(\mathbb{R}))} \le C(M_0, \mathcal{F}(h_0)). \tag{119}$$

Using (118), we can get for any $t \in [0, T_1]$

$$\|h\|_{L^{\lambda+1}(\mathbb{R})} \le \|h\|_{L^1(\mathbb{R})}^{1-\theta} \|h\|_{L^4(\mathbb{R})}^{\theta} \le \widetilde{C}(M_0, \mathcal{F}(h_0)), \quad \theta = \frac{3-\lambda}{3(\lambda+1)}. \tag{120}$$

Now, we take

$$A_0 := 2C_0 T_1^k \Big(\widetilde{C}(M_0, \mathcal{F}(h_0))\Big)^\beta \tag{121}$$

satisfying (111), hence (112) holds for any $t \in [0, T_1]$.

Step 3. Global existence under the sharp condition $M_0 < M_c$.

In the above two steps, we have shown that

1. $\int_{\mathbb{R}} h(x, T_1)\, dx = M_0$,
2. $\mathcal{F}(h(\cdot, T_1)) \leq \mathcal{F}(h_0)$,
3. $\text{supp } h(\cdot, T_1) \subset (-(a + A_0), (a + A_0))$.

Taking T_1 as a new starting time and $h(x, T_1) \in H^1$ as a new initial data and repeating the arguments used in Step 1 and Step 2, we can obtain a non-negative entropy weak solution h to the Cauchy problem (1)-(2) and it satisfies $\text{supp } h(\cdot, t) \subset (-(a + 2A_0), a + 2A_0)$ in $t \in [T_1, 2T_1]$. At $t = 2T_1$, (i), (ii) and (iii) are also true. Hence we can continue this process and obtain a global solution for the Cauchy problem in $\mathbb{R}_+$ and it satisfies for any fixed $T > 0$

$$h \in L^\infty\big(\mathbb{R}_+; L^1 \cap H^1(\mathbb{R})\big) \cap L^2\big(0, T; H^2(\mathbb{R})\big), \tag{122}$$

$$\partial_t h \in L^2\big(0, T; H^{-1}(\mathbb{R})\big). \tag{123}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. Let $h(x, t)$ be a global non-negative entropy weak solution of (1)-(2) given by Theorem 3 with initial data h_0 satisfying (3), $M_0 < M_c$ and $\mathcal{F}(h_0) < \infty$. Since $M_0 < M_c$, the Sz. Nagy inequality (22) implies that $\mathcal{F}(h(\cdot, t)) > 0$ for any $t \geq 0$. Noticing that the free energy is decreasing in time t, we know that there is a $\mathcal{F}_\infty$ such that

$$\lim_{t \to \infty} \mathcal{F}(h(\cdot, t)) = \mathcal{F}_\infty \geq 0.$$

On the other hand, a simple computation gives

$$\frac{d}{dt} m_2(t) = 6\mathcal{F}(h(\cdot, t)) \geq 6\mathcal{F}_\infty \geq 0, \tag{124}$$

which says that the second moment is increasing in t.

Now we prove that at least one of (34) and (35) holds. Suppose that

$$m_2(t) \not\to +\infty \text{ as } t \to \infty. \tag{125}$$

By (124), we have that there exists a constant $\tilde{C} > 0$ such that $m_2(t) \leq \tilde{C}$ for any $t \in (0, \infty)$. In this case, we claim that there is a sequence t_k and h_∞ such that as $t_k \to \infty$

$$h(t_k) \to h_\infty \text{ strongly in } L^4(\mathbb{R}). \tag{126}$$

In fact, since $h \in L^\infty(\mathbb{R}_+, L^1 \cap H^1(\mathbb{R}))$ and the second moment is finite, we have

1. $\forall\, \varepsilon > 0$, there exists a $R_\varepsilon > 0$ such that

$$\begin{aligned} \int_{|x| > R_\varepsilon} h^4\, dx &\leq \frac{1}{R_\varepsilon^2} \|h\|_{L^\infty}^3 \int_{|x| > R_\varepsilon} |x|^2 h\, dx \\ &\leq \frac{m_2(t)}{R_\varepsilon^2} \|h\|_{L^\infty}^3. \end{aligned}$$

   Hence taking

$$R_\varepsilon \geq \Big(\frac{2\|h\|_{L^\infty}^3 \tilde{C}}{\varepsilon}\Big)^{1/2} \geq \Big(\frac{2\|h\|_{L^\infty}^3 m_2(t)}{\varepsilon}\Big)^{1/2}, \tag{127}$$

   we obtain

$$\int_{|x| > R_\varepsilon} h^4\, dx < \varepsilon. \tag{128}$$

Then there is a subsequence t_k (without relabel) and $h_{1,\infty}$ such that

$$h(\cdot, t_k) \rightharpoonup h_{1,\infty} \quad \text{in } L^4(|x| \geq R_\varepsilon)$$

and

$$\int_{|x|>R_\varepsilon} h_{1,\infty}^4 \, dx \leq \liminf_{k\to\infty} \int_{|x|>R_\varepsilon} h(x,t_k)^4 \, dx \leq \varepsilon. \tag{129}$$

2. For fixed R_ε satisfying (127), we know that $h(x,t_k) \in L^\infty(\mathbb{R}_+; H^1(B(0,R_\varepsilon)))$. Thus by the Sobolev embedding theorem, one obtains that there is a strong convergent subsequence, still denoted by $h(x,t_k)$, and $h_{2,\infty}$ such that

$$h(\cdot,t_k) \to h_{2,\infty} \text{ strongly in } L^4(B(0,R_\varepsilon)). \tag{130}$$

Let h_∞ be the combination of $h_{1,\infty}$ and $h_{2,\infty}$ defined in $\mathbb{R}$. Hence, from (128), (129) and (130), we have that there is a K such that if $k \geq K$, then

$$\begin{aligned}
&\int_{\mathbb{R}} |h(x,t_k) - h_\infty|^4 \, dx \\
= \;& \int_{|x|>R_\varepsilon} |h(x,t_k) - h_\infty|^4 \, dx + \int_{|x|\leq R_\varepsilon} |h(x,t_k) - h_\infty|^4 \, dx \\
\leq \;& C\int_{|x|>R_\varepsilon} \left(|h(x,t_k)|^4 + |h_\infty|^4\right) \, dx + \int_{|x|\leq R_\varepsilon} |h(x,t_k) - h_\infty|^4 \, dx \\
< \;& C\varepsilon,
\end{aligned}$$

which proves our claim (126). Thus we have

$$\lim_{k\to\infty} \int_{\mathbb{R}} h^4(x,t_k) \, dx = \int_{\mathbb{R}} h_\infty^4 \, dx. \tag{131}$$

By uniform in time estimate (33), we know that there is a subsequence of t_k (still denoted by t_k) such that as $t_k \to \infty$

$$\partial_x h(\cdot,t_k) \rightharpoonup \partial_x h_\infty \text{ in } L^2(\mathbb{R}),$$

and hence

$$\int_{\mathbb{R}} (\partial_x h_\infty)^2 \, dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}} h_x^2(x,t_k) \, dx. \tag{132}$$

Therefore (131) and (132) give

$$\begin{aligned}
\mathcal{F}(h_\infty) &= \frac{1}{2}\int_{\mathbb{R}} (\partial_x h_\infty)^2 \, dx - \frac{1}{12}\int_{\mathbb{R}} h_\infty^4 \, dx \\
&\leq \liminf_{k\to\infty} \frac{1}{2}\int_{\mathbb{R}} h_x^2(x,t_k) \, dx - \lim_{k\to\infty} \frac{1}{12}\int_{\mathbb{R}} h^4(x,t_k) \, dx \\
&= \liminf_{k\to\infty} \mathcal{F}(h(\cdot,t_k)) = \mathcal{F}_\infty.
\end{aligned} \tag{133}$$

Finally, noticing that $\|h(\cdot,t_k)\|_{L^1(\mathbb{R})} = M_0$, $\|h(\cdot,t_k)\|_{L^4(\mathbb{R})} \leq C$ and the second moment is finite, from the Dunford-Pettis theorem we can get

$$h(\cdot,t_k) \rightharpoonup h_\infty \text{ in } L^1(\mathbb{R}). \tag{134}$$

Hence Fatou's lemma implies

$$\int_{\mathbb{R}} h_\infty \, dx \leq \liminf_{k\to\infty} \int_{\mathbb{R}} h(x,t_k) \, dx = M_0 < M_c. \tag{135}$$

We have two cases: (i) $h_\infty = 0$, (ii) $h_\infty \neq 0$. In the case (i), the formula (134) implies that there exists a subsequence t_k such that $h(\cdot,t_k) \rightharpoonup 0$ as $t_k \to \infty$, i.e.,

(35) holds. In the case (ii), by the inequality (22), we know $\mathcal{F}(h_\infty) > 0$. Hence (133) gives $\mathcal{F}_\infty > 0$. Note that

$$m_2(t) \geq m_2(0) + 2(d+2)\mathcal{F}_\infty t \to +\infty \text{ as } t \to \infty,$$

which contradicts with (125). Hence (34) holds. This finishes the proof of Theorem 4. □

Appendix A. **The proof of Proposition 2.** Denote $\bar{h}_{\delta\varepsilon} := J_\delta * h_{\delta\varepsilon} + c\sqrt{\varepsilon}$. J_δ is defined in Subsection 2.1. We introduce the following further regularized problem by using the modified method in $[-L, L] \times [0, \infty)$

$$\begin{cases} (h_{\delta\varepsilon})_t + \partial_x\Big(J_\delta * \Big[\left(\bar{h}_{\delta\varepsilon}^2 + \varepsilon^2\right)^{\frac{n}{2}} \left(\partial_{xxx}\bar{h}_{\delta\varepsilon} + \partial_x(\frac{\bar{h}_{\delta\varepsilon}^3}{3})\right)\Big]\Big) = 0, \\ h_{\delta\varepsilon}(x,0) = h_{\varepsilon 0}(x) \end{cases} \tag{136}$$

with the $2L$-periodic boundary condition. Here we notice

$$\|h_{\delta\varepsilon}(\cdot,0)\|_{H^1(-L,L)} = \|h_{\varepsilon 0}\|_{H^1(-L,L)} \leq \|h_0\|_{H^1(-L,L)} + c\sqrt{\varepsilon}.$$

Step 1. An estimate on H^1-norm of solutions $h_{\delta\varepsilon}$.

Taking $h_{\delta\varepsilon} - \partial_{xx}h_{\delta\varepsilon}$ as a test function in the equation (136), and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\frac{d}{dt}\|h_{\delta\varepsilon}\|^2_{H^1(-L,L)} \\ = \;& 2\int_{-L}^{L} \partial_x(\bar{h}_{\delta\varepsilon} - \partial_{xx}\bar{h}_{\delta\varepsilon})\left(\bar{h}_{\delta\varepsilon}^2 + \varepsilon^2\right)^{n/2} \left(\partial_{xxx}\bar{h}_{\delta\varepsilon} + \bar{h}_{\delta\varepsilon}^2\partial_x\bar{h}_{\delta\varepsilon}\right) dx \\ \leq \;& -\int_{-L}^{L} \left(\bar{h}_{\delta\varepsilon}^2 + \varepsilon^2\right)^{n/2} (\partial_{xxx}\bar{h}_{\delta\varepsilon})^2\, dx + C\Big(\int_{-L}^{L} \left(\bar{h}_{\delta\varepsilon}^2 + \varepsilon^2\right)^{n/2} \bar{h}_{\delta\varepsilon}^4(\partial_x\bar{h}_{\delta\varepsilon})^2\, dx \\ &+ \int_{-L}^{L} \left(\bar{h}_{\delta\varepsilon}^2 + \varepsilon^2\right)^{n/2} (\partial_x\bar{h}_{\delta\varepsilon})^2\, dx + \int_{-L}^{L} \left(\bar{h}_{\delta\varepsilon}^2 + \varepsilon^2\right)^{n/2} \bar{h}_{\delta\varepsilon}^2(\partial_x\bar{h}_{\delta\varepsilon})^2\, dx\Big). \end{aligned}$$

Notice that if $h_{\delta\varepsilon} \in H^1(-L,L)$, then by the embedding theorem $H^1(-L,L) \hookrightarrow C^{1/2}[-L,L]$, we have

$$\|h_{\delta\varepsilon}\|_{L^\infty(-L,L)} \leq C\|h_{\delta\varepsilon}\|_{H^1(-L,L)}.$$

So, we have

$$\frac{d}{dt}\|h_{\delta\varepsilon}\|^2_{H^1(-L,L)} + \int_{-L}^{L} \left(\bar{h}_{\delta\varepsilon}^2 + \varepsilon^2\right)^{n/2} (\partial_{xxx}\bar{h}_{\delta\varepsilon})^2\, dx \leq C(\|h_{\delta\varepsilon}\|^2_{H^1(-L,L)} + 1)^{(n+6)/2},$$

where C is a constant independent of T, δ, ε and $h_{\delta\varepsilon}$. Solving the above ordinary differential inequality, we obtain

$$\|h_{\delta\varepsilon}\|^2_{H^1(-L,L)} + 1 \leq \frac{1}{\Big((\|h_{\varepsilon 0}\|^2_{H^1(-L,L)} + 1)^{-\frac{n+4}{2}} - \frac{n+4}{2}Ct\Big)^{\frac{2}{n+4}}}. \tag{137}$$

This implies that there exists a $T = T(\|h_0\|_{H^1(-L,L)})$ independent of δ and ε such that the following estimates hold

$$\|h_{\delta\varepsilon}\|_{L^\infty(0,T;H^1(-L,L))} \leq C, \tag{138}$$

$$\varepsilon^n \int_0^T \int_{-L}^{L} (\partial_{xxx}\bar{h}_{\delta\varepsilon})^2\, dxdt \leq C, \tag{139}$$

$$\|\partial_t h_{\delta\varepsilon}\|_{L^2(0,T;H^{-1}(-L,L))} \leq C, \tag{140}$$

where constants C are independent of δ and ε. A direct consequence of (138) and (140) is that

$$\|h_{\delta\varepsilon}\|_{L^\infty((-L,L)\times(0,T))} \le C, \tag{141}$$
$$\|\bar{h}_{\delta\varepsilon}\|_{L^\infty(0,T;H^1(-L,L))} \le C, \quad \|\partial_t \bar{h}_{\delta\varepsilon}\|_{L^2(0,T;H^{-1}(-L,L))} \le C, \tag{142}$$

where constants C are also independent of δ and ε.

Step 2. Existence of $h_{\delta\varepsilon}$ in $[0,T]$ is given by an ODE theory in Banach space $L^\infty(-L,L)$, where $T = T(\|h_0\|_{H^1(-L,L)})$ is stated in Step 1.

Define $F_\delta : L^\infty(-L,L) \to L^\infty(-L,L)$ as

$$F_\delta(h_{\delta\varepsilon}) := -\partial_x \left(J_\delta * \left[\left((J_\delta * h_{\delta\varepsilon})^2 + \varepsilon^2 \right)^{\frac{n}{2}} \left(\partial_{xxx}(J_\delta * h_{\delta\varepsilon}) + \partial_x \left(\frac{(J_\delta * h_{\delta\varepsilon})^3}{3} \right) \right) \right] \right).$$

We easily prove that $F_\delta(h)$ is locally Lipschitz continuous in $L^\infty(-L,L)$, i.e., for any $h_1, h_2 \in L^\infty(-L,L)$, it holds

$$\|F_\delta(h_1) - F_\delta(h_2)\|_{L^\infty(-L,L)} \le C(\|h_1\|_{L^\infty(-L,L)}, \|h_2\|_{L^\infty(-L,L)}, \delta, \varepsilon)\|h_1 - h_2\|_{L^\infty(-L,L)}.$$

Hence the equation (136) can be written as the following ODE in $L^\infty(-L,L)$,

$$\partial_t h_{\delta\varepsilon} = F_\delta(h_{\delta\varepsilon}), \quad h_{\delta\varepsilon}(0) = h_{\varepsilon 0}. \tag{143}$$

By Theorem 3.1 of [28, pp.100], we know that there is $T_\delta > 0$ such that (143) has a unique local solution $h_{\delta\varepsilon} \in C^1(0,T_\delta;\ L^\infty(-L,L))$. Then the extension theorem [28, Theorem 3.3] tells us that the solution $h_{\delta\varepsilon}$ exists in $[0,T]$ due to the uniform L^∞ estimate (141). Hence all estimates (138) and (140) are actually valid for the solution $h_{\delta\varepsilon}$ of the equation (136).

Step 3. Existence of an entropy weak solution to (36).

From estimates (138) and (140), we know that for any fixed $\varepsilon > 0$, there exists a subsequence (still denoted by $h_{\delta\varepsilon}$) such that as $\delta \to 0$ the following convergent relations hold

$$\begin{aligned} &h_{\delta\varepsilon} \overset{*}{\rightharpoonup} h_\varepsilon \qquad in\ L^\infty(0,T;H^1(-L,L)),\\ &\partial_t h_{\delta\varepsilon} \rightharpoonup \partial_t h_\varepsilon \qquad in\ L^2(0,T;H^{-1}(-L,L)). \end{aligned}$$

Hence by using properties of mollifiers and (142) , we have that as $\delta \to 0$, $\bar{h}_{\delta\varepsilon} = J_\delta * h_{\delta\varepsilon}$ satisfies

$$\begin{aligned} &\bar{h}_{\delta\varepsilon} \overset{*}{\rightharpoonup} h_\varepsilon \qquad in\ L^\infty(0,T;H^1(-L,L)),\\ &\partial_t \bar{h}_{\delta\varepsilon} \rightharpoonup \partial_t h_\varepsilon \qquad in\ L^2(0,T;H^{-1}(-L,L)). \end{aligned} \tag{144}$$

From the uniform bounds (139) and (144), one has that there exists a subsequence of $\bar{h}_{\delta\varepsilon}$ (still denoted by $\bar{h}_{\delta\varepsilon}$) such that as $\delta \to 0$

$$\bar{h}_{\delta\varepsilon} \rightharpoonup h_\varepsilon \qquad in\ L^2(0,T;H^3(-L,L)). \tag{145}$$

Therefore by the Sobolev embedding theorems, we know

$$H^3(-L,L) \hookrightarrow\hookrightarrow H^2(-L,L) \subset H^{-1}(-L,L).$$

The Lions-Aubin lemma [27] with (139) and (142) implies that there is a subsequence of $\bar{h}_{\delta\varepsilon}$ (still denoted by $\bar{h}_{\delta\varepsilon}$) such that as $\delta \to 0$

$$\bar{h}_{\delta\varepsilon} \to h_\varepsilon \qquad in\ L^2(0,T;H^2(-L,L)). \tag{146}$$

Hence for any fixed $\varepsilon > 0$, and any $2L$- periodic function $\phi \in C^\infty\big(\mathbb{R}\times[0,T]\big)$, we have that the following weak form holds

$$\begin{aligned}\int_0^T\int_{-L}^L \phi\,\partial_t h_\varepsilon\,dxdt &= \int_0^T\int_{-L}^L \partial_x\phi\ \left(h_\varepsilon^2+\varepsilon^2\right)^{n/2}\ \partial_{xxx}h_\varepsilon\,dxdt\\ &\quad+\int_0^T\int_{-L}^L \partial_x\phi\ \left(h_\varepsilon^2+\varepsilon^2\right)^{n/2}\ h_\varepsilon^2\partial_x h_\varepsilon\,dxdt. \end{aligned}\tag{147}$$

Moreover, there is a constant C independent of ε such that h_ε satisfies (43)-(44). Hence for any fixed $\varepsilon > 0$, $h_\varepsilon \in L^\infty(0,T;H^1(-L,L)) \cap L^2(0,T;H^3(-L,L))$. Let $p_\varepsilon = \partial_{xx}h_\varepsilon + \frac{h_\varepsilon^3}{3} \in L^2(0,T;H^1(-L,L))$. Taking $\phi_i \in C^\infty([-L,L]\times[0,T])$ such that $\phi_i \to p_\varepsilon$ as $i\to\infty$ in $L^2(0,T;H^1(-L,L))$, i.e.,

$$\int_0^T \|\phi_i - p_\varepsilon\|_{H^1}^2\,dt \to 0 \qquad \text{as } i\to\infty. \tag{148}$$

By the weak form (147), we know

$$\int_0^t\int_{-L}^L \phi_i\partial_t h_\varepsilon\,dxdt = \int_0^t\int_{-L}^L \partial_x\phi_i\ \left(h_\varepsilon^2+\varepsilon^2\right)^{n/2}\ \partial_x\Big(\partial_{xx}h_\varepsilon+\frac{h_\varepsilon^3}{3}\Big)\,dxdt.$$

Using the estimates (43), (44) and (148), we can prove in the limit $i\to\infty$

$$\int_0^t\int_{-L}^L p_\varepsilon\partial_t h_\varepsilon\,dxdt = \int_0^t\int_{-L}^L \partial_x p_\varepsilon\ \left(h_\varepsilon^2+\varepsilon^2\right)^{n/2}\ \partial_x\Big(\partial_{xx}h_\varepsilon+\frac{h_\varepsilon^3}{3}\Big)\,dxdt.$$

Thus a simple computation leads the following energy-dissipation equality, for any $t\in[0,T]$,

$$\mathcal{F}(h_\varepsilon) + \int_0^t\int_{-L}^L \left(h_\varepsilon^2+\varepsilon^2\right)^{n/2}\ \Big|\partial_x\Big(\partial_{xx}h_\varepsilon+\frac{h_\varepsilon^3}{3}\Big)\Big|^2\,dxdt = \mathcal{F}(h_{\varepsilon 0}). \tag{149}$$

From the above arguments, we know that there is a local entropy weak solution h_ε to (36).

Step 4. h_ε is space-time Hölder continuous uniformly in ε. Indeed, for any $x_1, x_2 \in (-L,L)$ and $t_1,t_2\in(0,T)$, the estimate (45) holds. The property was proved in the paper [5]. For completeness, we provide the proof in Appendix B. □

Appendix B. **Hölder continuity.** In this Appendix, we consider the uniform in ε Hölder continuity (70) of weak solutions h_ε to the regularized problem (36). Before proving the main result, we first show the following two lemmas.

Lemma 4. *Suppose $h(x,t) \in L^\infty(0,T;H^1(-L,L))$, then for almost everywhere $x_1,x_2\in[-L,L]$, $t\in[0,T]$, it holds that*

$$|h(x_1,t)-h(x_2,t)| \le \|h\|_{L^\infty(0,T;H^1(-L,L))}|x_1-x_2|^{1/2}. \tag{150}$$

Proof. This a direct consequence of the embedding theorem $H^1(-L,L)\hookrightarrow C^{1/2}[-L,L]$ and $h(x,t)\in L^\infty(0,T;H^1(-L,L))$. □

For any $t_1,t_2\in[0,T]$, $t_1<t_2$, we construct a cut-off function $b_\delta(t) = \int_{-\infty}^t b'_\delta(t)\,dt$, $b'_\delta(t)$ satisfying

$$b'_\delta(t) = \begin{cases} \frac{1}{\delta}, & |t-t_2|<\delta,\\ -\frac{1}{\delta}, & |t-t_1|<\delta,\\ 0, & \text{otherwise,}\end{cases} \tag{151}$$

where the constant δ satisfies $0 < \delta < \frac{|t_2-t_1|}{2}$. Using the definition of $b_\delta(t)$, we know that $b_\delta(t)$ is Lipschitz continuous, and satisfies $|b_\delta(t)| \le 2$.

For any $x_0 \in (-L, L)$, constructing an auxiliary function

$$a(x) = a_0\left(\frac{K(x-x_0)}{|t_2-t_1|^\alpha}\right),$$

where $0 < \alpha < 1$ and $K > 0$ are two constants to be determined later, and $a_0(x) \in C_0^\infty(\mathbb{R})$ is defined by

$$a_0(x) = \begin{cases} 1, & -\frac{1}{2} \le x \le \frac{1}{2}, \\ 0, & |x| \ge 1 \end{cases} \tag{152}$$

satisfying $|a_0'(x)| \le C$. From (152), we know

$$a(x) = \begin{cases} 1, & |x-x_0| \le \frac{1}{2K}|t_2-t_1|^\alpha, \\ 0, & |x-x_0| \ge \frac{1}{K}|t_2-t_1|^\alpha. \end{cases} \tag{153}$$

Taking $K > \frac{T^\alpha}{L}$ such that

$$\left(-\frac{1}{K}|t_2-t_1|^\alpha, \frac{1}{K}|t_2-t_1|^\alpha\right) \subset (-L, L),$$

hence $a(-L) = a(L) = 0$.

Lemma 5. *Let $h(x,t) \in L^\infty(0,T;H^1(-L,L))$, and $a(x)$ and $b_\delta(t)$ be defined above. Then for almost everywhere $x_0 \in [-L,L], t_1, t_2 \in [0,T]$, $t_1 < t_2$, it holds*

$$| h(x_0,t_2) - h(x_0,t_1) | \le C\left(\langle h(x,t), a(x)b_\delta'(t)\rangle |t_2-t_1|^{-\alpha} + |t_2-t_1|^{\frac{\alpha}{2}}\right), \tag{154}$$

where C depends only on T, L and $\|h\|_{L^\infty(0,T;H^1(-L,L))}$.

Proof. Without loss of generality, we suppose $h(x_0,t_2) > h(x_0,t_1)$. Computing the inner product of h and $a(x)b_\delta'(t)$, we obtain

$$\begin{aligned} \langle h(x,t), a(x)b_\delta'(t)\rangle &= \int_{x_0-\frac{1}{K}(\Delta t)^\alpha}^{x_0+\frac{1}{K}(\Delta t)^\alpha} \int_0^T h(x,t)a(x)b_\delta'(t)\,dxdt \\ &= \int_{x_0-\frac{1}{K}(\Delta t)^\alpha}^{x_0+\frac{1}{K}(\Delta t)^\alpha} \left(\int_{t_2-\delta}^{t_2+\delta} \frac{1}{\delta}\ h(x,t)a(x) - \int_{t_1-\delta}^{t_1+\delta} \frac{1}{\delta}\ h(x,t)a(x)\right) dxdt \\ &= \frac{1}{\delta}\int_{x_0-\frac{1}{K}(\Delta t)^\alpha}^{x_0+\frac{1}{K}(\Delta t)^\alpha} \left(\int_{-\delta}^{\delta} h(x,t_2+\tau)a(x)\,d\tau - h(x,t_1+\tau)a(x)d\tau\right) dx \\ &= \frac{1}{\delta}\int_{-\delta}^{\delta}\int_{x_0-\frac{1}{K}(\Delta t)^\alpha}^{x_0+\frac{1}{K}(\Delta t)^\alpha} a(x)\left(h(x,t_2+\tau) - h(x,t_1+\tau)\right) dxd\tau \\ &= \frac{1}{\delta}\int_{-\delta}^{\delta}\int_{-\frac{1}{K}(\Delta t)^\alpha}^{\frac{1}{K}(\Delta t)^\alpha} a(x_0+y)\left(h(x_0+y,t_2+\tau) - h(x_0+y,t_1+\tau)\right) dyd\tau \end{aligned} \tag{155}$$

By Hölder continuity of h respect to the space variable, we have

$$|h(x_0+y,t) - h(x_0,t)| \le C|y|^{\frac{1}{2}}.$$

Noticing that $a(x)$ is a nonnegative function, from (155) we have

$$\begin{aligned}
&\langle h(x,t), a(x)b'_\delta(t)\rangle \qquad (156)\\
\geq\ & \frac{1}{\delta}\int_{-\delta}^{\delta}\int_{-\frac{1}{K}(\Delta t)^\alpha}^{\frac{1}{K}(\Delta t)^\alpha} a(x_0+y)\left(h(x_0,t_2+\tau)-h(x_0,t_1+\tau)-2C|y|^{\frac{1}{2}}\right)dyd\tau\\
\geq\ & \frac{1}{\delta}\int_{-\delta}^{\delta}\int_{-\frac{1}{K}(\Delta t)^\alpha}^{\frac{1}{K}(\Delta t)^\alpha} a(x_0+y)\left(h(x_0,t_2+\tau)-h(x_0,t_1+\tau)\right)dyd\tau - C(\Delta t)^{\frac{3\alpha}{2}}.
\end{aligned}$$

By the Lebesgue differentiation theorem, for a.e. $t_1, t_2 \in (0,T)$, we have

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_{-\delta}^{\delta}\int_{-\frac{1}{K}(\Delta t)^\alpha}^{\frac{1}{K}(\Delta t)^\alpha} a(x_0+y)\left(h(x_0,t_2+\tau)-h(x_0,t_1+\tau)\right)dyd\tau\\
=\ & 2\int_{-\frac{1}{K}(\Delta t)^\alpha}^{\frac{1}{K}(\Delta t)^\alpha} a(x_0+y)(h(x_0,t_2)-h(x_0,t_1))\,dx. \qquad (157)
\end{aligned}$$

Therefore from (156) and (157), we know that for any $\varepsilon_0 = (\Delta t)^{\frac{3\alpha}{2}} > 0$, there is a $\delta_0 > 0$ such that when $0 < \delta < \delta_0$, it holds that

$$\begin{aligned}
&2\int_{-\frac{1}{K}(\Delta t)^\alpha}^{\frac{1}{K}(\Delta t)^\alpha} a(x_0+y)(h(x_0,t_2)-h(x_0,t_1))\,dx\\
\leq\ & \frac{1}{\delta}\int_{-\delta}^{\delta}\int_{-\frac{1}{K}(\Delta t)^\alpha}^{\frac{1}{K}(\Delta t)^\alpha} a(x_0+y)\left(h(x_0,t_2+\tau)-h(x_0,t_1+\tau)\right)dyd\tau + \varepsilon_0\\
\leq\ & \langle h(x,t), a(x)b'_\delta(t)\rangle + C(\Delta t)^{\frac{3\alpha}{2}}. \qquad (158)
\end{aligned}$$

Since $h(x_0,t_2) > h(x_0,t_1)$, and from the definition of $a(x)$, we have

$$\frac{1}{K}(\Delta t)^\alpha(h(x_0,t_2)-h(x_0,t_1)) \leq \int_{-\frac{1}{K}(\Delta t)^\alpha}^{\frac{1}{K}(\Delta t)^\alpha} a(x_0+y)(h(x_0,t_2)-h(x_0,t_1))\,dx.$$

Hence we obtain that

$$\frac{2}{K}(\Delta t)^\alpha(h(x_0,t_2)-h(x_0,t_1) \leq \langle h(x,t), a(x)b'_\delta(t)\rangle + C(\Delta t)^{\frac{3\alpha}{2}}, \qquad (159)$$

which implies that (154) holds. □

Theorem 5. *Let $h_\varepsilon(x,t)$ be an entropy weak solution defined in Definition 2 to the regularized problem (36). Then $h_\varepsilon(x,t) \in C^{1/2,1/8}(Q_T)$, and for almost everywhere $x_1, x_2 \in [-L,L]$ and $t_1, t_2 \in [0,T]$, the following uniform in ε estimates hold*

$$|h_\varepsilon(x_2,t) - h_\varepsilon(x_1,t)| \leq C|x_2 - x_1|^{\frac{1}{2}} \quad \textit{for } t \in [0,T], \qquad (160)$$

$$|h_\varepsilon(x_0,t_2) - h_\varepsilon(x_0,t_1)| \leq C|t_1 - t_2|^{\frac{1}{8}} \quad \textit{for } x_0 \in [-L,L], \qquad (161)$$

where $C > 0$ is only dependent of T, L, M_0 and $\|h_0\|_{L^\infty(0,T;H^1(-L,L))}$.

Proof. Since $h_\varepsilon(x,t)$ is an entropy weak solution defined in Definition 2, we have $h_\varepsilon \in L^\infty(0,T;H^1(-L,L))$, and

$$\|h_\varepsilon\|_{L^\infty(0,T;H^1(-L,L))} \leq C,$$

where C is independent of ε. Hence using Lemma 4, we know that (160) holds.

Now we prove $\frac{1}{8}$-Hölder continuity of h_ε respect to the time t. Multiplying $a(x)b_\delta(t)$ to both sides of the equation of (36), and integrating in $(-L,L)\times(0,T)$, we have

$$\begin{aligned}|\langle\partial_t h_\varepsilon, a(x)b_\delta(t)\rangle| &= \left\langle\sqrt{h_\varepsilon^2+\varepsilon^2}\left(\partial_{xxx}h_\varepsilon+\partial_x(\frac{h_\varepsilon^3}{3})\right), a'(x)b_\delta(t)\right\rangle \\ &\leq \left\|\sqrt{h_\varepsilon^2+\varepsilon^2}\left(\partial_{xxx}h_\varepsilon+\partial_x(\frac{h_\varepsilon^3}{3})\right)\right\|_{L^2((-L,L)\times(0,T))} \\ &\quad \cdot\|a'(x)b_\delta(t)\|_{L^2((-L,L)\times(0,T))}. \end{aligned} \tag{162}$$

Due to (39), we can deduce

$$\left\|\sqrt{h_\varepsilon^2+\varepsilon^2}\left(\partial_{xxx}h_\varepsilon+\partial_x(\frac{h_\varepsilon^3}{3})\right)\right\|_{L^2((-L,L)\times(0,T))}\leq C,$$

where the constant C is independent of ε. Hence we obtain

$$|\langle\partial_t h_\varepsilon, a(x)b_\delta(t)\rangle|\leq C\|a'(x)b_\delta(t)\|_{L^2((-L,L)\times(0,T))}. \tag{163}$$

Notice that

$$\langle h_\varepsilon, a(x)b'_\delta(t)\rangle = -\langle\partial_t h_\varepsilon, a(x)b_\delta(t)\rangle, \tag{164}$$

and

$$\begin{aligned}\|a'(x)b_\delta(t)\|_{L^2((-L,L)\times(0,T))} &= \left(\iint_{(-L,L)\times(0,T)} a'^2(y)b_\delta^2(s)\,dyds\right)^{\frac12} \\ &\leq \left(\int_{x_0-\frac{1}{K}|t_2-t_1|^\alpha}^{x_0-\frac{1}{K}|t_2-t_1|^\alpha} a'^2(y)\,dy\int_0^T b_\delta^2(s)\,ds\right)^{\frac12} \\ &\leq C\left(\int_{-1}^1\frac{K}{|t_2-t_1|^\alpha}\,a_0'^2(y)\,dy\right)^{\frac12}\left(\int_0^T b_\delta^2(s)\,ds\right)^{\frac12} \\ &= C\frac{1}{|t_2-t_1|^{\frac{\alpha}{2}}}\cdot|t_2-t_1+2\delta|^{\frac12} \\ &\leq C\,|t_2-t_1|^{\frac12-\frac{\alpha}{2}}. \end{aligned} \tag{165}$$

Combining (163), (164) and (165), we get

$$|\langle h_\varepsilon, a(x)b'_\delta(t)\rangle|\leq C\,|t_2-t_1|^{\frac12-\frac{\alpha}{2}}. \tag{166}$$

Hence using Lemma 5, we have

$$|\,u(x_0,t_2)-u(x_0,t_1)\,|\leq C\left(|t_2-t_1|^{\frac12-\frac32\alpha}+|t_2-t_1|^{\frac{\alpha}{2}}\right). \tag{167}$$

Taking $\alpha=\frac14$, then (167) indicates that (161) holds. This completes the proof of Theorem 5. □

Acknowledgments. The authors sincerely thank the anonymous referees for their thorough review and highly appreciate their constructive criticisms, very helpful suggestions and advice, which contributed significantly to improving the quality of the manuscript.

REFERENCES

[1] P. Álvarez-Caudevilla and V. A. Galaktionov, Well-posedness of the Cauchy problem for a fourth-order thin film equation via regularization approaches, *Nonl. Anal.*, **121** (2015), 19–35.

[2] E. F. Beckenbach and R. Bellman, *Introduction to Inequalities*, Random House Inc, 1965.

[3] E. Beretta, M. Bertsch and R. Dal Passo, Nonnegative solutions of a fourth order nonlinear degenerate parabolic equation, *Arch. Ration. Mech. Anal.*, **129** (1995), 175–200.

[4] F. Bernis, Finite speed of propagation and continuity of the interface for slow viscous flows, *Adv. Differential Equations*, **1** (1996), 337–368.

[5] F. Bernis and A. Friedman, Higher order nonlinear degenerate parabolic equations, *J. Differential Equations*, **83** (1990), 179–206.

[6] A. L. Bertozzi and M. C. Pugh, The lubrication approximation for thin viscous films, the moving contact line with a porous media cut off of Van der Waals interactions, *Nonlinearity*, **7** (1994), 1535–1564.

[7] A. L. Bertozzi and M. C. Pugh, The lubrication approximation for thin viscous films: Regularity and long time behavior of weak solutions, *Comm. Pure Appl. Math.*, **49** (1996), 85–123.

[8] A. L. Bertozzi and M. C. Pugh, Long-wave instabilities and saturation in thin film equations, *Comm. Pure Appl. Math.*, **51** (1998), 625–661.

[9] A. L. Bertozzi and M. C. Pugh, Finite-time blow-up of solutions of some long-wave unstable thin film equations, *Indiana Univ. Math. J.*, **49** (2000), 1323–1366.

[10] M. Bertsch, L. Giacomelli and G. Karali, Thin-film equations with "partial wetting" energy: Existence of weak solutions, *Physica D*, **209** (2005), 17–27.

[11] M. Bertsch, R. Dal Passo, H. Garcke and G. Grün, The thin viscous flow equation in higher space dimensions, *Adv. Dierential Equations*, **3** (1998), 417–440.

[12] S. Bian and J.-G. Liu, Dynamic and steady states for multi-dimensional Keller-Segel model with diffusion exponent $m > 0$, *Comm. Math. Phys.*, **323** (2013), 1017–1070.

[13] M. Chugunova, M. C. Pugh and R. M. Taranets, Research Announcement: Finite-time blow up and long-wave unstable thin film equations, `arXiv:1008.0385v1`, (2010).

[14] M. Chugunova and R. M. Taranets, Blow-up with mass concentration for the long-wave unstable thin-film equation, *Appl. Anal.*, **95** (2016), 944–962.

[15] R. Dal Passo and H. Garcke, Solutions of a fourth order degenerate parabolic equation with weak initial trace, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **28** (1999), 153–181.

[16] R. Dal Passo, H. Garcke and G. Grün, On a fourth-order degenerate parabolic equation: Global entropy estimates, existence, and qualitative behavior of solutions, *SIAM J. Math. Anal.*, **29** (1998), 321–342.

[17] L. Giacomelli, M. V. Gnann and F. Otto, Regularity of source-type solutions to the thin-film equation with zero contact angle and mobility exponent between 3/2 and 3, *uropean J. Appl. Math.*, **24** (2013), 735–760.

[18] L. Giacomelli, H. Knüpfer and F. Otto, Smooth zero-contact-angle solutions to a thin-film equation around the steady state, *J. Differential Equations*, **245** (2008), 1454–1506.

[19] M. V. Gnann, Well-posedness and self-similar asymptotics for a thin-film equation, *SIAM J. Math. Anal.*, **47** (2015), 2868—2902.

[20] G. Grün, Droplet spreading under weak slippage: The optimal asymptotic propagation rate in the multi-dimensional case, *Interfaces Free Bound.*, **4** (2002), 309–323.

[21] G. Grün, Droplet spreading under weak slippage: A basic result on nite speed of propagation, *SIAM J. Math. Anal.*, **34** (2003), 992–1006.

[22] G. Grün, Droplet spreading under weak slippage-existence for the Cauchy problem, *Comm. Partial Differential Equations*, **29** (2004), 1697–1744.

[23] D. John, On uniqueness of weak solutions for the thin-film equation, *J. Differential Equations*, **259** (2015), 4122–4171.

[24] H. Knüpfer, Well-posedness for the Navier slip thin film equation in the case of partial wetting, *Comm. Pure Appl. Math.*, **64** (2011), 1263–1296.

[25] H. Knüpfer and N. Masmoudi, Darcy flow on a plate with prescribed contact angle well-posedness and lubrication approximation, *Arch. Rational Mech. Anal.*, **218** (2015), 589–646.

[26] R. S. Laugesen and M. C. Pugh, Properties of steady states for thin film equations, *European J. Appl. Math.*, **11** (2000), 293–351.

[27] J.-L. Lions, *Quelques Méthodes de Résolution Des Problèmes Aux Limites Non Linéaires*, Paris, Dunod, 1969.

[28] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Vol. 27, Cambridge University Press, 2002.

[29] D. Matthes , R. J. McCann and G. Savaré, A family of nonlinear fourth order equations of gradient flow type, *Comm. Partial Differential Equations*, **34** (2009), 1352–1397.

[30] A. Mellet, The thin film equation with non zero contact angle: A singular perturbation approach, *Comm. Partial Differential Equations,* **40** (2015), 1C-39.

[31] T. G. Myers, Thin films with high surface tension, *SIAM Rev.*, **40** (1998), 441–462.

[32] B. V. Sz. Nagy, Über Integralungleichungen zwischen einer Funktion und ihrer Ableitung (German), *Acta Univ. Szeged. Sect. Sci. Math.*, **10** (1941), 64–74.

[33] F. Otto, Lubrication approximation with prescribed nonzero contact angle, *Comm. Partial Differential Equations*, **23** (1998), 2077–2164.

[34] D. Slepčev and M. C. Pugh, Self-similar blow-up of unstable thin-film equations, *Indiana Univ. Math. J.*, **54** (2005), 1697–1738.

[35] R. M. Taranets and J. R. King, On an unstable thin-film equation in multi-dimensional domains, *NoDEA Nonlinear Differential Equations Appl.*, **21** (2014), 105–128.

[36] T. P. Witelski, A. J. Bernoff and A. L. Bertozzi, Blow-up and dissipation in a critical-case unstable thin film equation, *European J. Appl. Math.*, **15** (2004), 223–256.

[37] Z. Q. Wu, J. N. Zhao, J. X. Yin and H. L. Li, *Nonlinear Diffusion Equations*, 2^{nd} edition, Singapore, World Scientific, 2001.

Received December 2015; revised November 2016.

E-mail address: jliu@phy.duke.edu
E-mail address: wjh800415@163.com