TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 368, Number 12, December 2016, Pages 8499–8518
http://dx.doi.org/10.1090/tran/6618
Article electronically published on February 2, 2016

ON GENERATING FUNCTIONS OF HAUSDORFF MOMENT SEQUENCES

JIAN-GUO LIU AND ROBERT L. PEGO

ABSTRACT. The class of generating functions for completely monotone sequences (moments of finite positive measures on $[0,1]$) has an elegant characterization as the class of Pick functions analytic and positive on $(-\infty,1)$. We establish this and another such characterization and develop a variety of consequences. In particular, we characterize generating functions for moments of convex and concave probability distribution functions on $[0,1]$. Also we provide a simple analytic proof that for any real p and r with $p>0$, the Fuss-Catalan or Raney numbers $\frac{r}{pn+r}\binom{pn+r}{n}$, $n=0,1,\ldots$, are the moments of a probability distribution on some interval $[0,\tau]$ if and only if $p\geq 1$ and $p\geq r\geq 0$. The same statement holds for the binomial coefficients $\binom{pn+r-1}{n}$, $n=0,1,\ldots$.

1. Introduction

Given a finite positive (Borel) measure μ on $[0,1]$, its sequence of moments

$$c_j=\int_0^1 t^j\,d\mu(t)\,,\quad j=0,1,\ldots, \tag{1}$$

is *completely monotone*. This means that

$$(I-S)^k c_j\geq 0\quad\text{ for all } j,k\geq 0,$$

where S denotes the backshift operator given by $Sc_j=c_{j+1}$ for $j\geq 0$, so that

$$(I-S)^k c_j=\sum_{n=0}^{k}(-1)^n\binom{k}{n}c_{n+j}\,.$$

A classical theorem of Hausdorff states that complete monotonicity characterizes such moment sequences: A sequence $c=(c_j)_{j\geq 0}$ is the moment sequence of a finite positive measure on $[0,1]$ if and only if it is completely monotone [23, p. 115].

Received by the editors January 29, 2014 and, in revised form, February 26, 2014 and October 13, 2014.

2010 *Mathematics Subject Classification.* Primary 44A60; Secondary 60E99, 62E10, 05A15.

Key words and phrases. Completely monotone sequence, Fuss-Catalan numbers, complete Bernstein function, infinitely divisible, canonical sequence, exchangeable trials, concave distribution function, random matrices.

This material is based upon work supported by the National Science Foundation under grants DMS 1211161 and RNMS11-07444 (KI-Net) and partially supported by the Center for Nonlinear Analysis (CNA) under National Science Foundation grant 0635983.

It is well recognized [17, 18] that the generating function of a completely monotone sequence,

$$F(z) = \sum_{j=0}^{\infty} c_j z^j = \int_0^1 \frac{1}{1-zt}\, d\mu(t)\,, \tag{2}$$

belongs to the set $P(-\infty,1)$, consisting of all *Pick functions* analytic on $(-\infty,1)$. Recall what this means [4]: A function f is a Pick function if f is analytic in the upper half plane $\mathbb{H}=\{z\in\mathbb{C} : \operatorname{Im} z>0\}$ and leaves it invariant, satisfying

$$(\operatorname{Im} z)\operatorname{Im} f(z) \ge 0 \tag{3}$$

for all z in the domain of f. If (a,b) is an open interval in the real line, $P(a,b)$ denotes the set of Pick functions that are analytic on (a,b). This means they take real values on (a,b) and admit an analytic continuation by reflection from the upper half plane across the interval (a,b).

Conditions on F which are equivalent to complete monotonicity of (c_j) have been provided by Ruscheweyh *et al.* [18] and are discussed below. Yet the following simple and useful characterizations, though closely related to known results (see especially Theorems 2.8-2.11 in [2]), appear to have escaped explicit attention in the long literature on the subject.

Theorem 1. *Let $c=(c_j)_{j\ge 0}$ be a real sequence with generating function F and upshifted generating function*

$$F_1(z) = zF(z) = \sum_{j=0}^{\infty} c_j z^{j+1}\,. \tag{4}$$

Then the following are equivalent.

(i) *c is completely monotone.*
(ii) *F is a Pick function that is analytic and nonnegative on $(-\infty,1)$.*
(iii) *F_1 is a Pick function that is analytic on $(-\infty,1)$, with $F_1(0)=0$.*

To explain the terminology, we remark that F_1 is the generating function of the *upshifted* sequence $S^\dagger c$ given by $S^\dagger c_j = c_{j-1}$ for $j\ge 1$ and $S^\dagger c_0 = 0$. The Pick function conditions on F and F_1 in (ii) and (iii) mean that F and F_1 belong to $P(-\infty,1)$, so their domain contains $\mathbb{C}\setminus[1,\infty)$. Whether it is more convenient to verify the Pick property for F_1 or the Pick property and the positivity condition for F seems likely to depend on the context or the application.

It is convenient here to summarize explicit criteria for $(c_j)_{j\ge 0}$ to be the moment sequence for a probability distribution supported on an interval of the form $[0,\tau]$. This is equivalent to saying the generating function satisfies

$$F(z) = \int_0^\tau \frac{1}{1-tz}\, d\mu(t)\,. \tag{5}$$

Corollary 1. *Let $\tau>0$, and let $c=(c_j)_{j\ge 0}$ be a real sequence with $c_0=1$ and generating function F. Then the following are equivalent.*

(i) *c is the moment sequence of a probability measure μ on $[0,\tau]$.*
(ii) *F is a Pick function analytic and nonnegative on $(-\infty,1/\tau)$ with $F(0)=1$.*
(iii) *$F_1(z)=zF(z)$ is a Pick function analytic on $(-\infty,1/\tau)$ with $F_1'(0)=1$.*
(iv) *The sequence $(c_j\tau^{-j})_{j\ge 0}$ is completely monotone.*

We shall omit the simple proof by dilation from Theorem 1. Due to (5), we will call sequences that satisfy the conditions of Corollary 1 *dilated Hausdorff moment sequences*. We note that when $t < 0$, $1/(1-tz)$ is not a Pick function. This has the consequence that the moment generating function F for a probability distribution with nontrivial support in $(-\infty, 0)$ is not a Pick function.

The proof of Theorem 1 is provided in section 2 below. As shown there, the equivalence of (i) and (ii) may be inferred from the proof of Lemma 2.1 of [18]. In section 3 we will describe a number of direct consequences of Theorem 1, including criteria for complete monotonicity of probability distributions obtained by randomization from exchangable trials, and a simple proof of infinite divisibility of completely monotone probability distributions on $\mathbb{N}_0 = \{0, 1, \ldots\}$. In section 4, building on work of Diaconis and Freedman [3] and Gnedin and Pitman [7], we characterize generating functions of moments of distribution functions on $[0, 1]$ that are either convex or concave.

Finally, in section 5 we use Corollary 1 to provide simple analytic proofs that for any real p and r with $p > 0$, the Fuss-Catalan numbers (or Raney numbers)

$$\frac{r}{pn+r}\binom{pn+r}{n}, \quad n = 0, 1, \ldots,$$

form a dilated Hausdorff moment sequence if and only if $p \geq 1$ and $p \geq r \geq 0$, and the same holds for the binomial sequence

$$\binom{pn+r-1}{n}, \quad n = 0, 1, \ldots.$$

These results were proved previously by Młotkowski [11] and Młotkowski and Penson [12] using free probability theory and monotonic convolution arguments. In section 5 we also use the results of section 4 to prove the existence of nonincreasing "canonical densities" associated with Fuss-Catalan sequences. The moments of corresponding canonical distributions turn out to be a binomial sequence with $r = 1$. Moreover, an explicit formula for the inverse of the canonical distribution can be derived from the following integral representation formula for binomial coefficients: For any integer $k > 0$ and real $r \geq k$,

$$\binom{r}{k} = \frac{1}{\pi}\int_0^\pi \left(\frac{\sin x}{\sin^\theta \theta x \, \sin^{1-\theta}(1-\theta)x}\right)^r dx, \qquad \theta = \frac{k}{r}. \tag{6}$$

This formula follows from a statement in the proof of Proposition 2 in [20].

2. Characterization

The purpose of this section is to prove Theorem 1. First we establish the equivalence of (i) and (iii). Supposing $F_1 \in P(-\infty, 1)$ as stated in (iii), our goal is to prove that (1) holds. We observe that since F_1 is a Pick function, so also is the function defined by

$$F_*(z) = -F_1(1/z). \tag{7}$$

This function is analytic on the cut plane $\mathbb{C} \setminus [0, 1]$, with $F_*(z) \to 0$ as $|z| \to \infty$. The main representation theorems for Pick functions [4, pp. 20 and 24] imply there are real numbers $\alpha_* \geq 0$ and β_*, and a locally finite measure μ on $\mathbb{R}$ with increasing distribution function (denoted the same as is conventional) $\mu\colon \mathbb{R}\to\mathbb{R}$, such that

$$F_*(z) = \alpha_* z + \beta_* + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) d\mu(t). \tag{8}$$

Moreover, μ is characterized by the limits

$$\mu(b)-\mu(a) = \mu(a,b] = \lim_{h\to 0^+} \frac{1}{\pi}\int_a^b \operatorname{Im} F_*(t+ih)\,dt, \qquad a,b\in\mathbb{R}. \tag{9}$$

Because $\operatorname{Im} F_*(t) = 0$ for $t\in\mathbb{R}\setminus[0,1]$, the measure μ has support only on the cut $[0,1]$, and it is a finite measure with total mass $\mu\mathbb{R} = \mu[0,1]$. The fact that $F_*(z)\to 0$ as $z\to-\infty$ forces $\alpha_*=0$ and the cancellation of constant terms, whence

$$F_*(z) = \int_0^1 \frac{1}{t-z}\,d\mu(t). \tag{10}$$

For $|z|<1$, then, geometric series expansion and use of Fubini's theorem yield

$$F_1(z) = -F_*(1/z) = \int_0^1 \frac{z}{1-zt}\,d\mu(t) = \sum_{j=0}^\infty z^{j+1}\int_0^1 t^j\,d\mu(t). \tag{11}$$

Then (1) follows immediately upon comparison with the definition of F_1.

For the converse, given $(c_j)_{j\ge 0}$ completely monotonic, let μ satisfy (1) and define F_* by (7). Then (11) holds for $|z|<1$, and (10) follows for $|z|>1$. But this shows that F_* is Pick and analytic on $\mathbb{C}\setminus[0,1]$, whence F_1 is Pick and analytic on $\mathbb{C}\setminus[1,\infty)$, which means $F_1\in P(-\infty,1)$. Hence (iii) and (i) are equivalent.

Next we show that (i) and (ii) are equivalent. We know that (i) implies (ii) due to (2). Conversely, suppose (ii), and define $\hat F(z) = -F(1/z)$. Then $\hat F$ is a Pick function analytic on $\mathbb{C}\setminus[0,1]$, thus has a representation analogous to (8) as

$$\hat F(z) = \hat\alpha z + \hat\beta + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) d\hat\mu(t). \tag{12}$$

As before, $\hat\mu$ is a finite measure with support in $[0,1]$, and the fact that $\hat F(z)\to -F(0)$ as $z\to-\infty$ forces $\hat\alpha=0$ and

$$\hat F(z) = -F(0) + \int_0^1 \frac{1}{t-z}\,d\hat\mu(t). \tag{13}$$

Since F is a Pick function analytic and nonnegative on $(-\infty,1)$, necessarily $F'(z)\ge 0$ there, and $\lambda = \lim_{z\to-\infty} F(z)$ exists and is nonnegative. Therefore taking $z\to 0$ from the left in (13) implies

$$-\lambda = -F(0) + \int_0^1 \frac{d\hat\mu(t)}{t}\ .$$

In particular this is finite; hence

$$\hat F(z) = -\lambda + \int_0^1 \left(\frac{1}{t-z} - \frac{1}{t}\right) d\hat\mu = \int_0^1 \frac{z}{t-z}\,d\mu(t), \tag{14}$$

where $\mu = \lambda\delta_0 + \hat\mu/t$ is a finite measure on $[0,1]$. By consequence, $F(z) = -\hat F(1/z)$ has an integral representation as in (2), and (i) follows. This finishes the proof of Theorem 1.

Remark 1. The mass that the measure μ assigns to the left end of its support is

$$\mu\{0\} = \lambda = \lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} \frac{F_1(x)}{x}\,. \tag{15}$$

As is classically known [23, p. 164], one has $c_j = f(j)$ for some completely monotonic f defined on $[0,\infty)$ if and only if the term c_0 is minimal, and this is the case if and only if the measure μ has no atom at 0.

The mass that the measure μ assigns to the right end of its support is determined in a similar way. In the scaled expression (5), one has

$$\mu\{\tau\} = \lim_{x\uparrow 1/\tau} (1-\tau x)F(x)\,. \tag{16}$$

Remark 2. Our proof of the equivalence of (i) and (ii) of Theorem 1 basically follows arguments from [18] in a slightly different order. To compare, note that after a trivial normalization to make $c_0 = 1$, Lemma 2.1 of [18] shows that $(c_j)_{j\ge0}$ is completely monotone if and only if $F \in P(-\infty, 1)$ and satisfies the additional conditions:

(a) $\lim_{n\to\infty} F(z_n)/z_n = 0$ for some sequence $z_n \in \mathbb{C}$ with $\operatorname{Im} z_n \to +\infty$, and $\operatorname{Im} z_n \ge \delta \operatorname{Re} z_n$ for some positive constant δ,

(b) $\limsup_{x\to-\infty} F(x) \ge 0$.

If $F \in P(-\infty, 1)$, then due to the Pick property (3), F is increasing in the real interval $(-\infty, 1)$. Consequently, condition (b) is equivalent to nonnegativity of F on $(-\infty, 1)$. Moreover, the additional condition (a) is superfluous, by the lemma below. Thus equivalence of (i) and (ii) follows from the proof of Lemma 2.1 in [18].

Lemma 1. *Suppose* $F \in P(-\infty, 1)$ *and* (b) *holds. Then* (a) *holds.*

Proof. As noted in [18], F has a representation analogous to (8), of the form

$$F(z) = a + bz + \int_1^\infty \frac{1+tz}{t-z}\,d\sigma(t)\ , \tag{17}$$

where $a \in \mathbb{R}$, $b \ge 0$, and σ is a finite measure on $[1,\infty)$. Note that as $z \to -\infty$ along the real axis, we have

$$\frac{1}{z}\int_1^\infty \frac{1+tz}{t-z}\,d\sigma(t) \to 0 \tag{18}$$

by the dominated convergence theorem. Consequently, if $b > 0$, then $F(z) \to -\infty$ as $z \to -\infty$. Thus, if (b) holds, then $b = 0$. Then it follows that (a) holds with $z_n = in$, since with $z = z_n$, (18) holds again. □

3. Consequences

We next develop a few straightforward consequences of Theorem 1. Recall that a composition of Pick functions is a Pick function. Using this and related facts, we get ways of constructing and transforming completely monotone sequences through Taylor expansion of compositions.

3.1. **Dilation.** Suppose $(c_j)_{j\ge 0}$ is completely monotone and $p>0$. Let

$$G(z) = F(pz+1-p) = \sum_{j=0}^\infty c_j \sum_{k=0}^j \binom{j}{k}(1-p)^{j-k}p^k z^k.$$

Then by Theorem 1, G is a Pick function analytic and nonnegative on $(-\infty,1)$, as a consequence of the same properties for F. Provided $0<p<2$ so that $|1-p|<1$, for $|z|$ small we can write

$$G(z) = \sum_{k=0}^\infty b_k z^k, \qquad b_k = \sum_{j=k}^\infty c_j \binom{j}{k}(1-p)^{j-k}p^k.$$

Then it follows that $(b_k)_{k\ge 0}$ is completely monotone.

This application has a familiar interpretation in terms of compound probability distributions [5, §XII.1]: Let the random variable S_N be the number of successes in N independent Bernoulli trials, each successful with probability $p\in(0,1)$. Suppose N is itself random, independent of the Bernoulli trials, with

$$\mathbb{P}\{N=j\} = c_j\,, \quad j\ge 0. \tag{19}$$

Then the compound probability of having exactly k successes is

$$\mathbb{P}\{S_N=k\} = \sum_{j=k}^\infty \mathbb{P}\{N=j\}\mathbb{P}\{S_j=k\} = b_k\,, \quad k\ge 0,$$

with b_k as above. Thus, if (c_j) is completely monotone, then so is (b_k).

By averaging over p, we can also treat the case when independent trials are replaced by exchangeable ones, due to a well-known theorem of B. de Finetti. Let S_n denote the number of successes in the first n trials of an infinite exchangeable sequence of successes and failures. Then by de Finetti's theorem [6, VII.4], there is a probability distribution ν on $[0,1]$ such that

$$\mathbb{P}\{S_n=k\} = \binom{n}{k}\int_0^1 t^k(1-t)^{n-k}\,d\nu(t). \tag{20}$$

If N is selected randomly as before, the probability of k successes becomes

$$\hat b_k := \mathbb{P}\{S_N=k\} = \sum_{j=k}^\infty c_j\binom{j}{k}\int_0^1 (1-t)^{j-k}t^k\,d\nu(t). \tag{21}$$

By approximating ν using discrete measures and using the fact that convex combinations and pointwise limits of completely monotone sequences are completely monotone, it follows that $(\hat b_k)_{k\ge 0}$ is completely monotone if $(c_j)_{j\ge 0}$ is.

We remark that one can show that complete monotonicity of $(\hat b_k)_{k\ge 0}$ persists (though with loss of the probabilistic interpretation) if the averaging in (21) over $[0,1]$ is replaced by one over a compact interval $[0,\tau]\subset[0,2)$:

$$\hat b_k = \sum_{j=k}^\infty c_j\binom{j}{k}\int_0^\tau (1-t)^{j-k}t^k\,d\nu(t). \tag{22}$$

3.2. **Reflection.** Let U be any Pick function analytic on $(0,\infty)$. If U has the additional property that the right limit $U(0^+)$ exists, then it is a *complete Bernstein function*. (The theory of these functions, including an extensive table of examples, is developed in [19]. Our Theorem 1 may be regarded as a discrete analog of Theorem 6.2 in [19], which is the main characterization theorem for complete Bernstein functions.) Now, fix $r > 0$ and let

$$F_1(z) = U(r) - U(r(1-z)). \tag{23}$$

Then for $|z| < 1$ we may write

$$F_1(z) = \sum_{n=1}^{\infty} c_{n-1} z^n, \qquad c_{n-1} = (-1)^{n-1} U^{(n)}(r) \frac{r^n}{n!}, \quad n \geq 1. \tag{24}$$

Then F_1 is in $P(-\infty, 1)$; hence the sequence $(c_n)_{n\geq 0}$ is completely monotone by Theorem 1. Conversely, if $(c_n)_{n\geq 0}$ is completely monotone and $U(rz) = -F_1(1-z)$, then U is a Pick function analytic on $(0,\infty)$, and U is completely Bernstein if the left limit $F_1(1^-) < \infty$.

3.3. **Composition I.** Next, consider more general types of compositions. Let c_k^{*j} denote the k-th term of the standard j-fold convolution of the sequence $c = (c_n)_{n\geq 0}$ with itself, which we can write in terms of the Kronecker delta function δ_k^n and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as

$$c_k^{*j} = \sum_{n \in \mathbb{N}_0^j} \delta_k^{n_1 + \ldots + n_j} c_{n_1} \ldots c_{n_j} .$$

Recall that if $b = (b_j)_{j\geq 0}$ is a sequence with generating function G, then $A = G \circ F$ is the generating function for the sequence $a = (a_k)_{k\geq 0}$ defined by

$$a_k = \sum_{j=0}^{\infty} b_j c_k^{*j} , \quad k \geq 0. \tag{25}$$

By Theorem 1, a is completely monotone if and only if $G \circ F$ is a Pick function analytic and nonnegative on $(-\infty, 1)$, or equivalently $zG \circ F(z)$ is a Pick function analytic on $(-\infty, 1)$.

If b and c are probability distributions on $\mathbb{N}_0$, then a is a compound b distribution with compounding distribution c. In this case we have the following.

Proposition 1. *Suppose that b and c are probability distributions on $\mathbb{N}_0$ and that c and $Sb = (b_{j+1})_{j\geq 0}$ are completely monotone, with $b_0 \geq 0$. Then a, given by* (25), *is completely monotone.*

Proof. By Theorem 1, F is a Pick function analytic and nonnegative on $(-\infty, 1)$ with $F(1) = 1$, and G is a Pick function analytic and nonnegative on $[0, 1)$ with $G(1) = 1$. Then $G \circ F$ is a Pick function analytic and nonnegative on $(-\infty, 1)$; hence a is completely monotone. □

3.4. **Infinite divisibility.** The concept of infinite divisibility has been called "the core of the now classical limit theorems of probability theory — the reservoir created by the contributions of innumerably many individual streams and developments" [6, Chap. XVII]. Recall, for example, that a probability distribution μ on $\mathbb{R}$ is infinitely divisible if and only if it can arise as the weak limit of the law of the sum of n independent random variables with common distribution ν_n (see [6, Thm. XVII.1.1] or [22, Thm. 5.2]). This means that the distribution function μ is the pointwise limit of n-fold convolutions: At each point of continuity of μ,

$$\mu(x) = \lim_{n\to\infty} \nu_n^{*n}(x) \ . \tag{26}$$

Theorem 1 provides an alternative route to the following long-known result for distributions supported on $\mathbb{N}_0$ (see [21] and Theorem 10.4 of [22]).

Proposition 2. *If (c_j) is a probability distribution on $\mathbb{N}_0$ and (c_j) is completely monotone, then it is infinitely divisible.*

Proof. For $0<r<1$ the function $G(z)=z^r$ is a Pick function, positive for $z>0$. If $c=(c_j)_{j\ge 0}$ is completely monotone, then $F_r(z)=F(z)^r$ is a Pick function analytic and nonnegative on $(-\infty,1)$; hence F_r is the generating function for a completely monotone sequence $(a_k)_{k\ge 0}$ by Theorem 1. For $r=1/n$ we have $a_k^{*n}=c_k$, because $(F_{1/n})^n=F$. It follows that (c_j) is infinitely divisible. □

3.5. **Convolution groups and canonical sequences.** It is interesting to note that for any nontrivial completely monotone c and any real r, there is a simple algorithm for computing the sequence $a^{(r)}$ with generating function $F(z)^r$, arising from the characterization theorems of Hansen and Steutel [10] (see [22, Thm. 10.5]). Suppose $c_0=1$ for convenience. First compute the *canonical sequence* $(b_k)_{k\ge 0}$ so that

$$(n+1)c_{n+1} = \sum_{k=0}^{n} c_{n-k}b_k \,, \quad n=0,1,\dots; \tag{27}$$

then determine $a^{(r)}$ by setting $a_0^{(r)}=c_0^r=1$ and requiring

$$(n+1)a_{n+1}^{(r)} = r\sum_{k=0}^{n} a_{n-k}^{(r)}b_k \,, \quad n=0,1,\dots. \tag{28}$$

The generating functions F, G, and A_r, for c, b, and $a^{(r)}$ respectively, satisfy

$$F'(z)=F(z)G(z), \quad F(0)=1\,, \tag{29}$$

$$A_r'(z)=rA_r(z)G(z), \quad A_r(0)=1 \ , \tag{30}$$

whence it follows $A_r(z)=F(z)^r$, with

$$F(z)=\exp\tilde G_1(z), \qquad \tilde G_1(z)=\int_0^z G(t)\,dt=\sum_{n=0}^{\infty}\frac{b_n z^{n+1}}{n+1} \ . \tag{31}$$

The resulting set $\{a^{(r)}: r\in\mathbb{R}\}$ forms a convolution group: $a^{(r+s)}=a^{(r)}*a^{(s)}$.

By [10, Thm. 4], the sequence $\tilde b=(b_n/(n+1))_{n\ge 0}$ is completely monotone and has an interesting characterization in terms of moments of contractive distribution functions: namely, $\tilde G_1$ is the upshifted generating function for $\tilde b$, and there is a *canonical density* $w : [0,1] \to [0,1]$ (measurable) such that

$$\operatorname{Log} F(z) = \tilde{G}_1(z) = \int_0^1 \frac{z}{1-tz}\, w(t)\, dt\,. \tag{32}$$

(The distribution function $\nu(t) = \int_0^t w(s)\, ds$ is contractive: $|\nu(t+s) - \nu(t)| \le |s|$.)

We note a curious formula for (b_n). By Taylor expansion of Log,

$$\tilde{G}_1(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} (F(z)-1)^j,$$

and the coefficient of z^n in $(F(z)-1)^j$ vanishes for $j > n$. Binomial expansion, $F(z)^k = A_k(z)$, and use of a well-known identity [8, (5.10)] yield

$$\frac{b_{n-1}}{n} = \sum_{j=1}^{n} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{j} \binom{j}{k} a_n^{(k)} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} a_n^{(k)}. \tag{33}$$

The algorithm above extends without change to the case of any dilated Hausdorff moment sequence (c_j), i.e., a sequence satisfying the conditions of Corollary 1. Since $F^r = (F^s)^{r/s}$ is a Pick function whenever F^s is Pick and $0 \le r \le s$ and pointwise limits of Pick functions are Pick, $a^{(r)}$ is guaranteed to be a dilated Hausdorff moment sequence for r in some *maximal interval*, typically of the form $[0, r_*]$. We may characterize this interval in terms of a canonical density as follows. Dilation of (32) shows that the sequence $c = a^{(1)}$ has generating function $F = A_1$ given by a canonical density $w : [0,\tau] \to [0,1]$ according to

$$\operatorname{Log} A_1(z) = \int_0^{\tau} \frac{z}{1-tz}\, w(t)\, dt\,. \tag{34}$$

Proposition 3. *Let $c = (c_n)_{n\ge 0}$ be a dilated Hausdorff sequence, satisfying the conditions of Corollary 1. Let F be its generating function, and let $A_r = F^r$ be the generating function of $a^{(r)}$ as above. Then $a^{(r)}$ is a dilated Hausdorff sequence if and only if $0 \le r \operatorname{ess\,sup} w(t) \le 1$.*

Proof. By the corollary to Lemma 5 in [10], one has the following. Recall $\mathbb{H} = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$.

Lemma 2. *$\arg A_1(z) \in (0, \pi\rho)$ for all $z \in \mathbb{H}$ if and only if $|w(t)| \le \rho$ for a.e. t.*

By consequence, we find that since $\arg A_1(z) = 0$ for $z < 1$, the image

$$\arg A_1(\mathbb{H}) = (0, \pi\rho_*), \quad \rho_* = \operatorname{ess\,sup} |w(t)|. \tag{35}$$

It follows that $A_r = A_1^r$ is a Pick function if and only if $0 \le r\rho_* \le 1$. Then the result follows from Corollary 1. □

3.6. **Composition II.** If instead of composing generating functions directly, we compose their upshifted versions, we obtain the following.

Proposition 4. *Let b and c be real sequences with upshifted generating functions G_1 and F_1 respectively. Suppose $G_1 \circ F_1$ is a Pick function analytic on $(-\infty, 1)$. Then the sequence $\hat{a}$ defined by*

$$\hat{a}_{k-1} = \sum_{j=1}^{k} b_{j-1} c_{k-j}^{*j}\,, \quad k \ge 1,$$

has upshifted generating function $\hat{A}_1 = G \circ F_1$ and is completely monotone.

Proof. Compute $G_1 \circ F_1(z) = \sum_{k=1}^\infty \hat{a}_{k-1} z^k$ via

$$G_1 \circ F_1(z) = \sum_{j=1}^\infty b_{j-1} z^j \left(\sum_{n=0}^\infty c_n z^n \right)^j = \sum_{j=1}^\infty b_{j-1} z^j \sum_{n \in \mathbb{N}_0^j} c_{n_1} \cdots c_{n_j} z^{n_1 + \ldots + n_j},$$

collecting terms to obtain the coefficient of z^k, $k \geq 1$, which we label as a_{k-1}. $\square$

Example 1. Let $c = (c_j)_{j \geq 0}$ be any completely monotone sequence. Then the sequence $(\hat{c}_k)_{k \geq 0}$ of its *leading differences*, given by

$$\hat{c}_k = (I - S)^k c_0 = \sum_{j=0}^k \binom{k}{j} (-1)^j c_j \ , \quad k \geq 0, \tag{36}$$

is completely monotone. Actually, this is easiest to establish directly from the Hausdorff representation (1), since binomial expansion yields

$$\sum_{j=0}^k \binom{k}{j} (-1)^j \int_0^1 t^j \, d\mu(t) = \int_0^1 (1-t)^k \, d\mu(t) = \int_0^1 t^k \, d\hat{\mu}(t),$$

where $\hat{\mu}(t) = \mu(1) - \mu(1-t)$ is obtained by reflection. It is a charming fact that taking leading differences of leading differences gives back the original sequence. (For a more general inversion formula see [6, VII.1].) Here, though, for later use in section 4 we wish to point out how this is related to Theorem 1 and Proposition 4: The sequence $(\hat{c}_k)_{k \geq 0}$ in (36) has upshifted generating function given by

$$\hat{F}_1(z) = -F_1(H(z)), \qquad H(z) = \frac{-z}{1-z} = 1 - \frac{1}{1-z} \ . \tag{37}$$

This is true because

$$(1-z)^{-j-1} = \sum_{n=0}^\infty \binom{n+j}{n} z^n \ ,$$

hence

$$\begin{aligned} -F_1(H(z)) &= -\sum_{j=0}^\infty c_j \left(\frac{-z}{1-z} \right)^{j+1} = \sum_{j=0}^\infty \sum_{n=0}^\infty (-1)^j c_j \binom{n+j}{n} z^{n+j+1} \\ &= \sum_{k=0}^\infty \sum_{j=0}^k \binom{k}{j} (-1)^j c_j z^{k+1} = \sum_{k=0}^\infty \hat{c}_k z^{k+1} = \hat{F}_1(z) \ . \end{aligned}$$

Because $-H$ is a Pick function in $P(-\infty, 1)$ which maps $(-\infty, 1)$ onto $(-1, \infty)$, and $z \mapsto -F_1(-z)$ is a Pick function analytic on $\mathbb{C} \setminus (-\infty, -1]$, it follows that $\hat{F}_1$ is in $P(-\infty, 1)$. Thus $(\hat{c}_k)_{k \geq 0}$ is completely monotone by Theorem 1.

4. Moments of convex and concave distribution functions

Work of Diaconis and Freedman [3] included a characterization of moments of probability distributions admitting monotone densities in terms of the triangular array given by

$$c_{n,m} = \binom{n}{m} (I - S)^{n-m} c_m \ , \quad 0 \leq m < n \, .$$

Subsequently, Gnedin and Pitman [7] established the following criterion that characterizes moments of increasing densities in terms of complete monotonicity. Such densities correspond to distribution functions μ that are convex. We will work with distribution functions that in general satisfy $\mu(0) = 0 \le \mu(0^+)$ and are right continuous on $(0,1]$. If the distribution function μ is convex, necessarily $\mu(0^+) = 0$ and the measure μ has no atoms except possibly at $t = 1$.

Definition. A sequence $a = (a_n)_{n\ge0}$ is *completely alternating* if the sequence given by $(S-I)a = (a_{n+1} - a_n)_{n\ge0}$ is completely monotone.

Theorem 2 ([7]). *A sequence $(c_n)_{n\ge0}$ is the sequence of moments of a probability distribution on $[0,1]$ having a convex distribution function μ if and only if $c_0 = 1$ and the sequence $(a_n)_{n\ge0}$ defined by*

$$a_0 = 0, \qquad a_n = nc_{n-1}, \quad n = 1, 2, \ldots, \tag{38}$$

is completely alternating.

From Theorem 1, we directly obtain the following characterizations of completely alternating sequences, and moments of a convex distribution function, in terms of the corresponding generating functions. We find it convenient to consider distribution functions that are not necessarily normalized to be probability distribution functions. Note that if the sequence a has generating function A, then $\hat a = (S-I)a$ has upshifted generating function

$$\hat A_1(z) = (1-z)A(z) - A(0)\ . \tag{39}$$

Also, if a is determined as in Theorem 2, from c with upshifted generating function F_1, then $A(z) = zF_1'(z)$.

Proposition 5. *A sequence $(a_n)_{n\ge0}$ is completely alternating if and only if its generating function A has the property that the function $(1-z)A(z)$ is a Pick function analytic on $(-\infty, 1)$.*

Theorem 3. *Let $c = (c_n)_{n\ge0}$ be a real sequence with upshifted generating function F_1. Then the following are equivalent.*

(i) *c is the sequence of moments of a convex distribution function μ on $[0,1]$ with $\mu(0) = 0$.*
(ii) *The sequence a determined from c by (38) is completely alternating.*
(iii) *The function $\hat A_1(z) = (1-z)zF_1'(z)$ is a Pick function analytic on $(-\infty, 1)$.*
(iv) *The function $\hat A(z) = (1-z)F_1'(z)$ is a Pick function analytic and nonnegative on $(-\infty, 1)$.*

Criteria for $(c_n)_{n\ge0}$ to be the moments of a concave distribution function μ, whose corresponding measure has a decreasing density on $(0,1]$ with possible atom at 0, turn out to be slightly simpler.

Theorem 4. *Let $c = (c_n)_{n\ge0}$ be a real sequence with upshifted generating function F_1. Then the following are equivalent.*

(i) *c is the sequence of moments of a concave distribution function μ on $[0,1]$.*
(ii) *The sequence $((n+1)c_n)_{n\ge0}$ is completely monotone.*
(iii) *The function $F_2(z) = zF_1'(z)$ is a Pick function analytic on $(-\infty, 1)$.*
(iv) *The function F_1' is a Pick function analytic and nonnegative on $(-\infty, 1)$.*

Although this result can be derived from Theorem 3 by using reflection as in Example 1 above, we prefer to illustrate the use of Theorem 1 by providing a self-contained proof. Essentially the idea is to consider mixtures of uniform distributions on $[0,s]$, similar to [7]. In particular we make use of the following identity valid for $0<s\le 1$ and $z\in\mathbb{C}\setminus[1,\infty)$:

$$\frac{1}{1-sz}=\frac{d}{dz}\int_0^1 \frac{z}{1-tz}\left(\frac{1}{s}\mathbb{1}_{0<t<s}\right)dt. \tag{40}$$

Proof. Since (ii), (iii) and (iv) are equivalent by Theorem 1, it remains to prove (iv) is equivalent to (i). Assume (iv). Then by Theorem 1, there is a finite positive measure ν on $[0,1]$ with $\nu\{0\}=0$, and $a_0\ge 0$, such that

$$F_1'(z)=a_0+\int_0^1 \frac{1}{1-sz}\,d\nu(s). \tag{41}$$

(Here we have separated out the mass a_0 of any atom at $s=0$.) Using (40), integration, division by z, and Fubini's theorem we obtain

$$F(z)-a_0=\int_0^1\left(\int_0^1 \frac{1}{1-tz}\left(\frac{1}{s}\mathbb{1}_{0<t<s}\right)dt\right)d\nu(s)=\int_0^1 \frac{1}{1-tz}w(t)\,dt\,, \tag{42}$$

where

$$w(t)=\int_0^1\left(\frac{1}{s}\mathbb{1}_{0<t<s}\right)d\nu(s)=\int_t^1 \frac{d\nu(s)}{s}\,. \tag{43}$$

This function w is decreasing on $(0,1]$, and taking $z=0$ in (42) shows that w is integrable on $(0,1)$, with

$$F(0)=a_0+\int_0^1 w(t)\,dt\,.$$

Then the distribution function $\mu(t)=a_0+\int_0^t w(r)\,dr$, $t>0$, is concave, and (2) holds as desired, proving (i).

For the converse, assume (i). Then (42) holds for some $a_0\ge 0$ and w decreasing and integrable on $(0,1]$. Extend w by zero for $t>1$ and define the measure ν such that $d\nu(s)=-s\,dw(s)$ on $(0,1]$. Then (43) holds, and $\int_0^1 w(t)\,dt=\int_0^1 d\nu(s)$ by Fubini's theorem, and one deduces (41) by reversing the steps above. Thus (iv) follows. $\square$

Finally, we derive Theorem 3 from Theorem 4, using reflection as in Example 1.

Proof of Theorem 3. Suppose (i) $(c_n)_{n\ge 0}$ has the moment representation (1) with convex distribution function μ, corresponding to an increasing density (with a possible atom at 1). Then by Example 1, the sequence of leading differences $(\hat{c}_k)_{k\ge 0}$ is represented by the reflected density (with possible atom at 0). This density is decreasing, implying the leading differences are moments of a concave distribution function. The upshifted generating function $\hat{F}_1=-F_1\circ H$ given by (37) therefore has the property that

$$\hat{F}_2(z)=z\hat{F}_1'(z)=\frac{z}{(1-z)^2}F_1'(H(z))\,,$$

and this is a Pick function analytic on $(-\infty,1)$. But since $H(H(z))=z$,

$$-\hat{F}_2(H(z))=-H(z)(1-z)^2F_1'(z)=z(1-z)F_1'(z)=\hat{A}_1(z),$$

and this is a Pick function analytic on $(-\infty,1)$. This yields (iii), and (ii) and (iv) are equivalent by Theorem 1.

In the converse direction, if we assume the sequence a derived from c in (38) is completely alternating, then $\hat{A}_1$ is Pick. After reversing the arguments we deduce that the leading difference sequence $\hat{c} = (\hat{c}_k)_{k\geq 0}$ is represented by a decreasing density. By Example 1, c itself is represented by the reflected, increasing density, hence by a convex distribution function. $\square$

5. Fuss-Catalan and binomial sequences

For any real p and r, the general *Fuss-Catalan* numbers [8, 11], also called *Raney* numbers [13, 14], are defined by $A_0(p,r) = 1$ and

$$A_n(p,r) = \frac{r}{n!}\prod_{j=1}^{n-1}(pn+r-j) = \frac{r}{pn+r}\binom{pn+r}{n}, \quad n = 1, 2, \dots . \tag{44}$$

For $r = 1$ one has

$$A_n(p,1) = \frac{(pn)!}{n!(pn+1-n)!} = \frac{1}{pn-n+1}\binom{pn}{n},$$

showing that in particular, $A_n(2,1)$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. The Fuss-Catalan numbers have a long history and a very large number of fascinating interpretations and applications, of which we only mention a couple. For example, $A_n(m,1)$ counts the number of m-ary trees with n nodes, and $A_n(m,k)$ counts the number of sequences of $mn+k$ terms selected from $\{1, 1-m\}$ which sum to 1 while having partial sums that are all positive [8, eq. (7.70)].

5.1. **Fuss-Catalan moments.** Recently, Alexeev *et al.* [1] proved that $A_n(m,1)$ arises in random matrix theory as the n-th moment of the limiting distribution of scaled squared singular values of products of m complex matrices with random i.i.d. entries having zero mean, unit variance, and bounded fourth moments. Here, we make use of Corollary 1 to provide a simple analytic proof of the following characterization of those real p and r with $p > 0$ for which (44) yields a dilated Hausdorff moment sequence.

Theorem 5. *Let p and r be real with $p > 0$. Then $(A_n(p,r))_{n\geq 0}$ is the sequence of moments of some probability distribution $\mu_{p,r}$ having compact support in $[0,\infty)$ if and only if $p \geq 1$ and $p \geq r \geq 0$. In this case, $\mu_{p,r}$ is supported in the minimal interval $[0, \tau_p]$ with $\tau_p = p^p/(p-1)^{p-1}$ for $p > 1$, $\tau_1 = 1$.*

Młotkowski [11] proved the *if* part of this theorem using techniques from free probability theory, and the *only if* part was subsequently established by Młotkowski and Penson [12] using arguments that involve monotonic convolution. Explicit representations of densities $W_{p,r}(t)$ for $\mu_{p,r}$ and more general probability distributions have been provided by Penson and Zyczkowski [14] and Młotkowski *et al.* [11] in terms of the Meijer G function and hypergeometric functions. Haagerup and Möller [9] have derived parametrized forms of the densities $W_{p,1}(t)$ in terms of trigonometric functions. These distributions provide generalizations of previously known distributions such as the Marchenko-Pastur distribution

$$d\mu_{2,1}(t) = \frac{1}{2\pi}\sqrt{\frac{4-t}{t}}\,dt,$$

and $d\mu_{2,2}(t) = t\,d\mu_{2,1}(t)$, given by the Wigner semi-circle law centered on $t = 2$.

Due to Corollary 1, Theorem 5 is directly implied by the following.

Lemma 3. *Let p and r be real with $p > 0$, and let $B_{p,r}$ denote the generating function for the sequence $(A_n(p,r))_{n\ge 0}$. Then $B_{p,r}$ is a Pick function analytic and nonnegative on some interval $(-\infty, 1/\tau)$, $0 < \tau < \infty$, if and only if $p \ge 1$ and $p \ge r \ge 0$. If this is the case, the minimal τ is τ_p.*

Proof. The proof breaks into various cases.

1. *The case $p = 1 = r$.* Since $A_n(1,1) = 1$ for all n, $B_{1,1}(z) = 1/(1-z)$, and this is a Pick function analytic and positive on the maximal interval $(-\infty, 1)$.

2. *The case $p > 1 = r$.* Let $B_p = B_{p,1}$ denote the generating function for $(A_n(p,1))_{n\ge 0}$. It is well known [8, 11] that $B_p(z)$ satisfies the functional relation

$$B_p(z) = 1 + zB_p(z)^p \ , \tag{45}$$

as can be checked using the Lagrange inversion formula. Note $B_p(z)$ cannot vanish, so (45) is equivalent to

$$z = \psi_p(B_p(z)), \qquad \psi_p(c) = \frac{c-1}{c^p} = c^{1-p} - c^{-p}. \tag{46}$$

The function ψ_p is analytic and strictly increasing on the interval $(0, p/(p-1))$, rising from $-\infty$ to the value

$$z_p := 1/\tau_p = (p-1)^{p-1}/p^p \ .$$

By consequence B_p is analytic, positive, and strictly increasing on $(-\infty, 1)$, hence satisfies the Pick property in a neighborhood of this interval.

We continue B_p analytically to the domain $\mathbb{C} \setminus [z_p, \infty)$ by using a differential equation that B_p satisfies; namely, (46) implies

$$B_p'(z) = \frac{1}{z} \frac{B_p(B_p - 1)}{p - (p-1)B_p} \ . \tag{47}$$

We integrate along rays $t \mapsto te^{i\theta}$ from $t = t_0$ near the origin, for fixed $\theta \in (0, 2\pi)$. By continuation theory for ordinary differential equations, the solution exists for t in some maximal interval $[t_0, T)$ with the property that if $T < \infty$, then as $t \uparrow T$, either $|B_p| \to \infty$ or $B_p \to p/(p-1)$. The first case is not possible since by (46), $|B_p| \to \infty$ implies $te^{i\theta} \to 0$. And the second case is not possible since by (46), $B_p \to p/(p-1)$ implies $te^{i\theta} \to z_p$. Therefore $T = \infty$ for every θ.

This provides an analytic continuation of B_p to $\mathbb{C} \setminus [z_*, \infty)$. Necessarily $\operatorname{Im} B_p > 0$ everywhere in the upper half plane $\operatorname{Im} z > 0$, since $\operatorname{Im} B_p$ cannot vanish due to (46). Hence B_p is a Pick function and is analytic and positive on the maximal interval $(-\infty, z_p)$. This finishes the proof for $p > r = 1$.

3. *The case $p \ge 1$, $p \ge r \ge 0$.* For any real p and r, the generating function $B_{p,r}$ satisfies the Lambert equation

$$B_{p,r}(z) = B_p(z)^r \ . \tag{48}$$

(See [8, eq. (5.60)] and [11, eq. (3.2)], which is based on [16, p. 148].) Since B_p is analytic and never vanishes or takes negative values, $\operatorname{Log} \circ B_p$ is a Pick function, and $B_p(z)^r = \exp(r \operatorname{Log} B_p(z))$. Thus $B_p(z)^r$ is analytic in the upper half plane, and positive and increasing on the maximal interval $(-\infty, z_p)$, with limit 0 at $-\infty$ and value 1 at $z = 0$.

We claim $B_{p,r}$ is a Pick function: note $zB_p^p(z) = B_p(z) - 1$, and this is a nontrivial Pick function analytic on $(-\infty, z_p)$ that vanishes at $z = 0$. By Corollary 1, then,

$B_p(z)^p$ itself must be a Pick function. Since $z \mapsto z^{r/p}$ is a Pick function, it follows that $(B_p(z)^p)^{r/p} = B_p(z)^r$ is a Pick function.

4. *The case $r > p \geq 1$.* Recall $zB_p(z)^p$ is a Pick function analytic on $(-\infty, 1)$. This function is negative for $z < 0$, so necessarily

$$\arg zB_p(z)^p = \pi\,, \quad z < 0.$$

Since $0 < \arg z < \pi$ implies $0 < \arg B_p(z) < \pi$, it follows that by taking $z = e^{i\theta}$, the quantity

$$\arg zB_p(z)^r = \arg zB_p(z)^p + (r-p)\arg B_p(z)$$

takes values ranging from 0 to more than π as θ varies from 0 to π. Consequently $zB_p(z)^r$ cannot be a Pick function.

5. *The case $0 < p < 1$, $r > 0$.* In this case, ψ_p is globally strictly monotone on $(0, \infty)$ and maps this interval analytically onto $\mathbb{R}$. This means the inverse function B_p is globally real analytic on $\mathbb{R}$, and the same is true for $B_{p,r}$ for any $r > 0$ by (48). In this case, $B_{p,r}$ cannot be a Pick function. Indeed, the only Pick functions analytic and positive globally on $\mathbb{R}$ are constant. This is because they have a general representation as in (8), with μ as in (9). Hence $d\mu = 0$ for a Pick function globally analytic on $\mathbb{R}$.

6. *The case $p > 0 > r$.* Due to the Lambert equation (48), $B_{p,r}$ is decreasing on $(-\infty, z_p)$ in this case; hence $B_{p,r}$ cannot be a Pick function. □

We next apply Theorem 4 to deduce that the distribution functions $\mu_{p,1}$ have decreasing densities for $p \geq 2$. Numerical plots in [13] indicate that this condition is not sharp for noninteger values of p. See [13, 14] for detailed information concerning densities $W_{p,r}$ for all $p \geq 1$, $0 < r \leq p$. Note, however, that for $r = p$ one has $A_n(p,p) = A_{n+1}(p,1)$, whence follows the simple relation

$$d\mu_{p,p}(t) = t\, d\mu_{p,1}(t)\,. \tag{49}$$

Proposition 6. *If $p \geq 2$, then the probability distribution function $\mu_{p,1}$ is concave and continuous at 0. We have*

$$d\mu_{p,1}(t) = W_{p,1}(t)\,dt, \qquad d\mu_{p,p}(t) = t\,W_{p,1}(t)\,dt,$$

where $W_{p,1}$ is a decreasing, integrable function on $(0,1]$.

Proof. The function $F_1 = zB_p(z)$ satisfies

$$F_1' = \frac{p-2}{p-1}B_p + \frac{1}{p-1}\left(\frac{p}{p-(p-1)B_p} - 1\right)\,, \tag{50}$$

and if $p \geq 2$, this is a Pick function analytic and positive on $(-\infty, 1)$, fulfilling condition (iv) of Theorem 4. By Remark 1, the measure $\mu_{p,1}$ has no atom at 0, since $B_p(z) \to 0$ as $z \to -\infty$. □

5.2. **Fuss-Catalan canonical sequences and densities.** Next we study the canonical sequences and densities that are associated with Fuss-Catalan sequences according to Theorem 5 and Proposition 3.

Theorem 6. *For every $p \geq 1$, there are a canonical sequence $(b_n^{(p)})$, given explicitly by*

$$b_{n-1}^{(p)} = n\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\binom{n}{k}A_n(p,k) = \binom{pn-1}{n-1}\,, \quad n = 1, 2, \ldots, \tag{51}$$

and a nonincreasing density $w_p\colon [0,\tau_p] \to [0,1/p]$ *satisfying* $pw_p(0^+) = 1$ *and* $w_p(\tau_p^-) = 0$ *such that*

$$\frac{1}{r}\,\mathrm{Log}\,B_{p,r}(z) = \sum_{n=0}^{\infty} \frac{b_n^{(p)}}{n+1} z^{n+1} = \int_0^{\tau_p} \frac{z}{1-tz}\, w_p(t)\,dt\,, \quad 0 < r \le p. \tag{52}$$

Moreover, $(b_n^{(p)}\tau_p^{-n})_{n\ge 0}$ *is a completely monotone sequence.*

Proof. For $r > 0$, the sequence $a^{(r)} = (A_n(p,r))_{n\ge 0}$ is a dilated Hausdorff sequence if and only if $r \le p$. By Proposition 3, we infer (52), with $p\,\mathrm{ess\,sup}\, w_p(t) = 1$. (Note: $w_1(t) \equiv 1$.) To prove that w_p is nonincreasing, observe that (47) implies

$$(p-1)\frac{zB_p'(z)}{B_p(z)} = \frac{1}{p-(p-1)B_p(z)} - 1, \tag{53}$$

and this is a Pick function analytic on $(-\infty, 1/\tau_p)$. Let

$$\tilde G_1(z) = \mathrm{Log}\,B_p(z/\tau_p) = \int_0^1 \frac{z}{1-sz}\, w_p(\tau_p s)\,ds. \tag{54}$$

Then $z\tilde G_1'(z) = \hat z B_p'(\hat z)/B_p(\hat z)$ ($\hat z = z/\tau_p$) is a Pick function analytic on $(-\infty,1)$. Theorem 4 implies the density $s \mapsto w_p(\tau_p s)$ is nonincreasing on $(0,1]$. Therefore $pw_p(0^+) = 1$. (See (58) below for the proof that $w_p(\tau_p^-) = 0$.)

The first expression in (51) follows from (33). Noticing that (53) implies

$$\sum_{n=0}^{\infty} b_n^{(p)} z^n = \frac{B_p'(z)}{B_p(z)} = \frac{B_p(z)^{p-1}}{1-p+pB_p(z)^{-1}}\ , \tag{55}$$

the second expression follows from identity (5.61) in [8] (setting $r = p-1$). Finally, plugging $\hat z = z/\tau_p$ into the power series in (52) and using Theorem 4, part (ii), we infer $(b_n^{(p)}\tau_p^{-n})$ is completely monotone as claimed. □

5.3. **Binomial sequences.** In a recent paper, Młotkowski and Penson [12] established necessary and sufficient conditions for the binomial sequence

$$\binom{pn+r-1}{n}, \qquad n = 0,1,\dots\,, \tag{56}$$

to be the moment sequence of some probability distribution on some interval $[0,\tau]$. In this subsection we will provide an alternative proof of this characterization based on Corollary 1. First, however, we note that the case $r = 1$ is connected with the canonical density w_p described in Theorem 6.

Corollary 2. *For every real* $p > 1$*, the binomial sequence* $\binom{pn}{n}$*,* $n = 0,1,\dots,$ *is the moment sequence for the probability distribution function* $1 - pw_p(t)$ *on* $[0,\tau_p]$ *having generating function*

$$\sum_{n=0}^{\infty} \binom{pn}{n} z^n = \int_0^{\tau_p} \frac{1}{1-tz}\, d(1-pw_p(t)). \tag{57}$$

Proof. Noting that $\binom{pn}{n} = pb^{(p)}_{n-1}$ for $n \geq 1$, we use (52) to compute that

$$\begin{aligned}\sum_{n=0}^{\infty} \binom{pn}{n} z^n &= 1 + pz \int_0^{\tau_p} \frac{\partial}{\partial z}\left(\frac{z}{1-tz}\right) w_p(t)\,dt \\ &= 1 + \int_0^{\tau_p} \frac{\partial}{\partial t}\left(\frac{1}{1-tz}\right) pw_p(t)\,dt \\ &= \int_0^{\tau_p} \frac{1}{1-tz}\, d(1-pw_p(t)) + \frac{pw_p(\tau_p^-)}{1-\tau_p z}\,.\end{aligned}$$

Comparing the last line of this calculation to the first and using (52), (53), and (46), we find that

$$\begin{aligned}w_p(\tau_p^-) &= \lim_{z\uparrow 1/\tau_p} (1-\tau_p z) z B_p'(z)/B_p(z) \\ &= \frac{1}{p-1} \lim_{z\uparrow 1/\tau_p} \frac{1-\tau_p z}{p-(p-1)B_p(z)} \\ &= \frac{\tau_p}{(p-1)^2}\psi_p'\left(\frac{p}{p-1}\right) = 0.\end{aligned} \tag{58}$$

This finishes the proof. □

In the case $p = 2$ one can obtain an explicit formula for $w_2(t)$ by elementary means. From the formula

$$\binom{2n}{n} = \frac{1}{\pi}\int_0^{\pi} (4\cos^2(u/2))^n\,du\,, \tag{59}$$

we find

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\pi}\int_0^{\pi} \frac{1}{1-4z\cos^2(u/2)}\,du\,. \tag{60}$$

By comparing with (57), we deduce that

$$w_2(t) = \frac{1}{\pi}\arccos\sqrt{\frac{t}{4}}. \tag{61}$$

Remark 3. For an arbitrary $p > 1$, an explicit formula for the inverse of w_p may be obtained in a similar way, based upon the integral representation formula (6) for binomial coefficients. Set $k = n$, $r = pn$ and

$$f_p(u) = \frac{\sin^p \pi u}{\sin(\pi u/p)\sin^{p-1}((1-1/p)\pi u)}\,.$$

Then $f_p(u)$ decreases from τ_p to 0 as u increases from 0 to 1, and we deduce that

$$\sum_{n=0}^{\infty} \binom{pn}{n} z^n = \int_0^1 \frac{1}{1-f_p(u)z}\,du = \int_0^{\tau_p} \frac{1}{1-tz} d(1-f_p^{-1}(t))\,. \tag{62}$$

By comparing with (57), it follows that

$$pw_p(t) = f_p^{-1}(t), \quad 0 < t < \tau_p\,. \tag{63}$$

Theorem 7 ([12])**.** *Let p and r be real with $p > 0$. The sequence (56) is the sequence of moments of some probability distribution $\nu_{p,r}$ having compact support in $[0,\infty)$ if and only if $p \geq 1$ and $p \geq r \geq 0$. In this case, $\nu_{p,r}$ is supported in the minimal interval $[0,\tau_p]$ with $\tau_p = p^p/(p-1)^{p-1}$ for $p > 1$, $\tau_1 = 1$.*

Proof. The generating function $E_{p,r}(z)$ for the sequence (56) is known to satisfy (see [12] and [8, eq. (5.61)])

$$E_{p,r}(z) = \frac{B_p(z)^r}{p-(p-1)B_p(z)}\,. \tag{64}$$

From what was proved before, $E_{p,r}$ (like B_p) is analytic and nonnegative on $(-\infty, z_p)$ and analytic in the upper half plane. By Corollary 1, it suffices to show that $E_{p,r}$ is a Pick function if and only if $p \geq 1$ and $p \geq r \geq 0$. Similarly to the proof of Theorem 5, the proof breaks into several cases.

1. *The case* $r < 0$. In this case, $E_{p,r}(z)$ is decreasing in z for large $z < 0$; hence $E_{p,r}$ cannot be a Pick function.

2. *The case* $0 < p < 1$, $r \geq 0$. As in case 5 of the proof of Theorem 5, B_p, and hence $E_{p,r}$, is globally analytic and positive on $\mathbb{R}$. The only Pick functions with this property are constant, so $E_{p,r}$ is not a Pick function.

3. *The case* $p > 1$, $0 \leq r \leq p$. By (55), $(p-1)zE_{p,p}(z)$ equals the right-hand side of (53) and therefore is a Pick function. Now, for $0 < \arg z < \pi$ we have, on the one hand, that

$$\arg zE_{p,r}(z) = \arg z + \arg E_{p,0}(z) + r \arg B_p(z) > 0, \tag{65}$$

and, on the other hand, that

$$\arg zE_{p,r}(z) = \arg zE_{p,p}(z) + (r-p)\arg B_p(z) < \pi\ . \tag{66}$$

This shows $zE_{p,r}(z)$ is a Pick function; hence $E_{p,r}$ is a Pick function by Corollary 1.

4. *The case* $p > 1$, $r > p$. In this case we claim $zE_{p,r}(z)$ is not a Pick function. To see this, take $z = e^{i\theta}$ for $0 \leq \theta \leq \pi$ and note that in (66), the last term is positive and the first term varies from 0 to (at least) π. Hence the sum is somewhere more than π for some z in the upper half plane.

5. *The case* $p = 1$. In this case, $E_{1,r}(z) = (1-z)^{-r}$, and this is a Pick function if and only if $0 \leq r \leq 1$. □

Remark 4. For rational $p \geq 1$, with $p \geq 1 + r > 0$, explicit formulae in terms of the Meijer G function have been derived by Młotkowski and Penson [12] for a density denoted $V_{p,r}(t)$ with the property that

$$\sum_{n=0}^{\infty} \binom{pn+r}{n} z^n = \int_0^{\tau_p} \frac{1}{1-tz} V_{p,r}(t)\,dt\,. \tag{67}$$

By comparing with the above, it follows that

$$-pw_p'(t) = V_{p,0}(t). \tag{68}$$

Acknowledgements

The authors thank Tewodros Amdeberhan (private communication) for finding the second expression in (51) from the first by using the Zeilberger algorithm [15]. The authors are grateful to Pierre Degond for initiating a collaboration which led to this work. The second author is grateful to Govind Menon for discussions on Pick functions.

References

[1] N. Alexeev, F. Götze, and A. Tikhomirov, *Asymptotic distribution of singular values of powers of random matrices*, Lith. Math. J. **50** (2010), no. 2, 121–132, DOI 10.1007/s10986-010-9074-4. MR2653641 (2011c:15102)

[2] Julius Bendat and Seymour Sherman, *Monotone and convex operator functions*, Trans. Amer. Math. Soc. **79** (1955), 58–71. MR0082655 (18,588b)

[3] Persi Diaconis and David Freedman, *The Markov moment problem and de Finetti's theorem. I*, Math. Z. **247** (2004), no. 1, 183–199, DOI 10.1007/s00209-003-0636-6. MR2054525 (2005m:60004)

[4] William F. Donoghue Jr., *Monotone matrix functions and analytic continuation,* Die Grundlehren der mathematischen Wissenschaften, Band 207, Springer-Verlag, New York-Heidelberg, 1974. MR0486556 (58 #6279)

[5] William Feller, *An introduction to probability theory and its applications. Vol. I*, Third edition, John Wiley & Sons, Inc., New York-London-Sydney, 1968. MR0228020 (37 #3604)

[6] William Feller, *An introduction to probability theory and its applications. Vol. II*, Second edition, John Wiley & Sons Inc., New York, 1971. MR0270403

[7] Alexander Gnedin and Jim Pitman, *Moments of convex distribution functions and completely alternating sequences*, Probability and statistics: essays in honor of David A. Freedman, Inst. Math. Stat. Collect., vol. 2, Inst. Math. Statist., Beachwood, OH, 2008, pp. 30–41, DOI 10.1214/193940307000000374. MR2459948 (2010d:60083)

[8] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete mathematics,* A foundation for computer science, 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994. MR1397498 (97d:68003)

[9] Uffe Haagerup and Sören Möller, *The law of large numbers for the free multiplicative convolution*, Operator algebra and dynamics, Springer Proc. Math. Stat., vol. 58, Springer, Heidelberg, 2013, pp. 157–186, DOI 10.1007/978-3-642-39459-1_8. MR3142036

[10] B. G. Hansen and F. W. Steutel, *On moment sequences and infinitely divisible sequences*, J. Math. Anal. Appl. **136** (1988), no. 1, 304–313, DOI 10.1016/0022-247X(88)90133-3. MR972601 (90a:60155)

[11] Wojciech Młotkowski, *Fuss-Catalan numbers in noncommutative probability*, Doc. Math. **15** (2010), 939–955. MR2745687 (2012c:46170)

[12] Wojciech Młotkowski and Karol A. Penson, *Probability distributions with binomial moments*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **17** (2014), no. 2, 1450014, 32, DOI 10.1142/S0219025714500143. MR3212684

[13] Wojciech Młotkowski, Karol A. Penson, and Karol Życzkowski, *Densities of the Raney distributions*, Doc. Math. **18** (2013), 1573–1596. MR3158243

[14] K. A. Penson and K. Życzkowski, *Product of Ginibre matrices: Fuss-Catalan and Raney distributions*, Phys. Rev. E **83** (2011), 061118.

[15] Marko Petkovšek, Herbert S. Wilf, and Doron Zeilberger, *$A = B$*, A K Peters, Ltd., Wellesley, MA, 1996. With a foreword by Donald E. Knuth; With a separately available computer disk. MR1379802 (97j:05001)

[16] John Riordan, *Combinatorial identities*, John Wiley & Sons, Inc., New York-London-Sydney, 1968. MR0231725 (38 #53)

[17] Oliver Roth, Stephan Ruscheweyh, and Luis Salinas, *A note on generating functions for Hausdorff moment sequences*, Proc. Amer. Math. Soc. **136** (2008), no. 9, 3171–3176, DOI 10.1090/S0002-9939-08-09460-4. MR2407081 (2009a:30077)

[18] Stephan Ruscheweyh, Luis Salinas, and Toshiyuki Sugawa, *Completely monotone sequences and universally prestarlike functions*, Israel J. Math. **171** (2009), 285–304, DOI 10.1007/s11856-009-0050-9. MR2520111 (2010e:30012)

[19] René L. Schilling, Renming Song, and Zoran Vondraček, *Bernstein functions,* Theory and applications, de Gruyter Studies in Mathematics, vol. 37, Walter de Gruyter & Co., Berlin, 2010. MR2598208 (2011d:60060)

[20] Thomas Simon, *Comparing Fréchet and positive stable laws*, Electron. J. Probab. **19** (2014), no. 16, 25, DOI 10.1214/EJP.v19-3058. MR3164769

[21] F. W. Steutel, *Note on completely monotone densities*, Ann. Math. Statist. **40** (1969), 1130–1131. MR0254949 (40 #8156)

[22] Fred W. Steutel and Klaas van Harn, *Infinite divisibility of probability distributions on the real line*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 259, Marcel Dekker, Inc., New York, 2004. MR2011862 (2005j:60004)

[23] David Vernon Widder, *The Laplace Transform*, Princeton Mathematical Series, v. 6, Princeton University Press, Princeton, N. J., 1941. MR0005923 (3,232d)

Departments of Physics and Mathematics, Duke University, Durham, North Carolina 27708

E-mail address: jliu@phy.duke.edu

Department of Mathematical Sciences and Center for Nonlinear Analysis, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213

E-mail address: rpego@cmu.edu