Calculus of Variations

Least action principles for incompressible flows and geodesics between shapes

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Abstract

As V. I. Arnold observed in the 1960s, the Euler equations of incompressible fluid flow correspond formally to geodesic equations in a group of volume-preserving diffeomorphisms. Working in an Eulerian framework, we study incompressible flows of shapes as critical paths for action (kinetic energy) along transport paths constrained to have characteristic-function densities. The formal geodesic equations for this problem are Euler equations for incompressible, inviscid potential flow of fluid with zero pressure and surface tension on the free boundary. The problem of minimizing this action exhibits an instability associated with microdroplet formation, with the following outcomes: any two shapes of equal volume can be approximately connected by an Euler spray—a countable superposition of ellipsoidal geodesics. The infimum of the action is the Wasserstein distance squared, and is almost never attained except in dimension 1. Every Wasserstein geodesic between bounded densities of compact support provides a solution of the (compressible) pressureless Euler system that is a weak limit of (incompressible) Euler sprays.

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1 Introduction

1.1 Overview

The geometric interpretation of solutions of the Euler equations of incompressible inviscid fluid flow as geodesic paths in the group of volume-preserving diffeomorphisms was famously pioneered by V. I. Arnold [3]. If we consider an Eulerian description for an incompressible body of constant-density fluid moving freely in space, such geodesic paths correspond to critical paths for the action

$$\mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho |v|^2 \, dx \, dt, \tag{1.1}$$

where $\rho = (\rho_t)_{t \in [0,1]}$ is a path of characteristic-function densities transported by a velocity field $v \in L^2(\rho \, dx \, dt)$ according to the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho v) = 0. \tag{1.2}$$

Such characteristic-function densities ρ_t represent a fluid having *shape* Ω_t at time t:

$$\rho_t = \mathbb{1}_{\Omega_t}, \quad t \in [0, 1]. \tag{1.3}$$

Naturally, the velocity field must be divergence free in the interior of the fluid domain Ω_t, satisfying $\nabla \cdot v = 0$ there. Equation (1.2) holds in the sense of distributions in $\mathbb{R}^d \times [0, 1]$, interpreting ρv as 0 wherever $\rho = 0$.

In this Eulerian framework, it is natural to study the action in (1.1) subject to given endpoint conditions of the form

$$\rho_0 = \mathbb{1}_{\Omega_0}, \quad \rho_1 = \mathbb{1}_{\Omega_1}. \tag{1.4}$$

These endpoint conditions differ from Arnold-style conditions that fix the flow-induced volume-preserving diffeomorphism between Ω_0 and Ω_1, and correspond instead to fixing only the *image* of this diffeomorphism. Imposing endpoint conditions in an Eulerian transport framework as in (1.4) is exactly analogous to the fundamental study of Benamou and Brenier [6] that relates the minimization of the action (1.1) without incompressibility constraints to Wasserstein (Monge–Kantorovich) distance with quadratic cost.

As we show in Sect. 3 below, it turns out that the geodesic equations that result are precisely the Euler equations for *potential flow* of an incompressible, inviscid fluid occupying domain Ω_t, with *zero pressure and zero surface tension* on the free boundary $\partial \Omega_t$. In short, the geodesic equations are classic water wave equations with zero gravity and surface tension. The initial-value problem for these equations has recently been studied in detail—the works [14,15,37] extend the breakthrough works of Wu [57,58] to deal with nonzero vorticity and zero gravity, and establish short-time existence and uniqueness for sufficiently smooth initial data in certain bounded domains.

The problem of *minimizing* the action in (1.1) subject to the constraints above turns out to be ill-posed if the dimension $d > 1$, as we will show in this paper. By this we mean that action-minimizing paths that satisfy all the constraints (1.2), (1.3) and (1.4) do not exist in general, even locally. Nevertheless, the infimum of the action defines a distance between equal-volume sets which we may call *shape distance*, determined by

$$d_s(\Omega_0, \Omega_1)^2 = \inf \mathcal{A}, \tag{1.5}$$

where the infimum is taken subject to the constraints (1.2), (1.3), (1.4) above. By the well-known result of Benamou and Brenier [6], it is clear that

$$d_s(\Omega_0, \Omega_1) \geq d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1}), \tag{1.6}$$

where $d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})$ denotes the usual Wasserstein distance (Monge–Kantorovich distance with quadratic cost) between the measures with densities $\mathbb{1}_{\Omega_0}$ and $\mathbb{1}_{\Omega_1}$. This is so because the result of [6] characterizes the squared Wasserstein distance $d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2$ as the infimum in (1.5) subject to the same transport and endpoint constraints as in (1.2) and (1.4), but *without* the constraint (1.3) that makes ρ a characteristic function.

Our objective in this paper is to develop several results that precisely relate the infimum in (1.5) and corresponding geodesics (critical paths for action) on the one hand, to Wasserstein distance and corresponding length-minimizing Wasserstein geodesics—also known as displacement interpolants—on the other hand. Wasserstein geodesic paths typically do not have characteristic-function densities, and thus do not correspond to geodesics for the shape distance d_s. A common theme in our results is the observation that the least-action problem in (1.5) is subject to an instability associated with *microdroplet* formation.

1.2 Main results

Broadly speaking, our aim is to investigate the geometry of the space of shapes (corresponding to characteristic-function densities), focusing on the geodesics for shape distance and the corresponding distance induced by (1.5). Studies of this type have been carried out by many other authors, as will be discussed in Sect. 1.3.

One issue about which we have little to say is that of geodesic completeness, in the sense this term is used in differential geometry. Here this concept corresponds to global existence in time for weak solutions of the free-boundary Euler equations. But in addition to other well-known difficulties for Euler equations, in the present situation there arise further thorny problems, such as collisions of fluid droplets, for example.

Geodesics between shapes. Our principal results instead address the question of determining which targets and sources are connected by geodesics for shape distance, and how these relate to the infimization in (1.5). The general question of determining all exact connecting critical paths is an interesting one that seems difficult to answer. In regard to a related question in a space of smooth enough volume-preserving diffeomorphisms of a fixed manifold, Ebin and Marsden in [22, 15.2(vii)] established a covering theorem showing that the geodesic flow starting from the identity diffeomorphism covers a full neighborhood. By contrast, what our first result will show is that for an arbitrary bounded open source domain Ω_0, targets for shape-distance geodesics are globally dense in the 'manifold' of bounded open sets of the same volume. The idea is to construct geodesics comprised of tiny disjoint droplets (which we call *Euler sprays*) that approximately reach an arbitrarily specified Ω_1 as closely as desired in terms of an optimal-transport distance.

Below, it is convenient to denote the distance between two bounded measurable sets Ω_0, Ω_1 that is induced by Wasserstein distance by the overloaded notation

$$d_W(\Omega_0, \Omega_1) = d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1}), \tag{1.7}$$

and similarly with L^p-Wasserstein distance d_p for any value of $p \in [1, \infty]$.
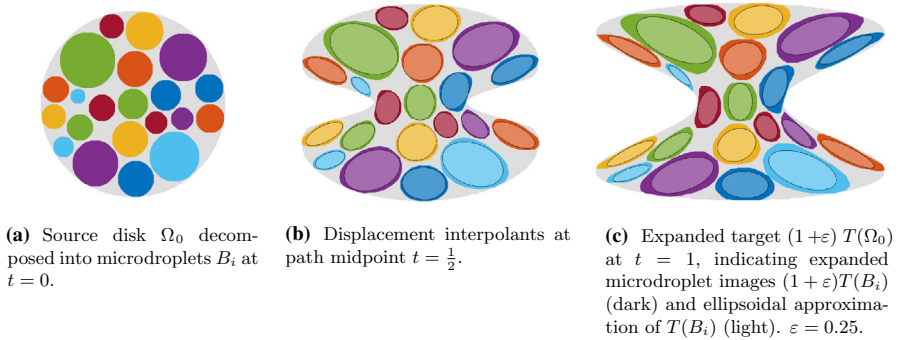
(a) Source disk Ω_0 decomposed into microdroplets B_i at $t = 0$.

(b) Displacement interpolants at path midpoint $t = \frac{1}{2}$.

(c) Expanded target $(1+\varepsilon)\,T(\Omega_0)$ at $t = 1$, indicating expanded microdroplet images $(1+\varepsilon)T(B_i)$ (dark) and ellipsoidal approximation of $T(B_i)$ (light). $\varepsilon = 0.25$.

Fig. 1 Illustration of Wasserstein geodesic flow from Ω_0 to $\Omega_1 = T(\Omega_0)$, where T is the Brenier map. Source Ω_0 is decomposed into countably many small balls, few shown. Matching shades indicate corresponding droplets transported by displacement interpolation. Euler spray droplets are nested inside Wasserstein ellipsoids and remain disjoint

Theorem 1.1 *Let Ω_0, Ω_1 be any pair of bounded open sets in $\mathbb{R}^d$ with equal volume. Then for any $\varepsilon > 0$, there is an Euler spray which transports the source Ω_0 (up to a null set) to a target Ω_1^ε satisfying $d_\infty(\Omega_1, \Omega_1^\varepsilon) < \varepsilon$. The action $\mathcal{A}^\varepsilon$ of the spray satisfies*

$$d_s(\Omega_0, \Omega_1^\varepsilon)^2 \leq \mathcal{A}^\varepsilon \leq d_W(\Omega_0, \Omega_1)^2 + \varepsilon.$$

The precise definition of an Euler spray and the proof of this result will be provided in Sect. 4. A particular, simple geodesic for shape distance will play a special role in our analysis. Namely, we observe in Proposition 3.4 that a path $t \mapsto \Omega_t$ of ellipsoids determines a critical path for the action (1.1) constrained by (1.2)–(1.4) if and only if the d-dimensional vector $a(t) = (a_1(t), \ldots, a_d(t))$, formed by the principal axis lengths, follows a geodesic curve on the hyperboloid-like surface in $\mathbb{R}^d$ determined by the constraint that corresponds to constant volume,

$$a_1 a_2 \cdots a_d = \text{const.} \tag{1.8}$$

The fluid motions corresponding to such ellipsoids turn out to be ones known to Dirichlet [19].

To prove Theorem 1.1, we decompose the source domain Ω_0, up to a set of measure zero, as a countable union of tiny disjoint open balls using a Vitali covering lemma. These 'microdroplets' are transported by ellipsoidal geodesics that approximate a local linearization of the Wasserstein geodesic (displacement interpolant) which produces straight-line transport of points from the source Ω_0 to the target Ω_1. Crucially, the droplets remain disjoint (essentially due to the convexity of the density along the straight Wasserstein transport paths). The total action or cost along the resulting path of 'spray' densities is then shown to be close to that attained by the Wasserstein geodesic.

The ideas behind the construction of the Euler sprays are illustrated in Fig. 1. The shaded background in panel (c) indicates the target $\Omega_1 = T(\Omega_0)$, expanded by a factor $(1 + \varepsilon)$, where $T: \Omega_0 \to \Omega_1$ is a computed approximation to the Brenier (optimal transport) map. The expanded images $(1 + \varepsilon)T(B_i)$ of balls B_i in the source are shown in dark shades, and (nested inside) ellipsoidal approximations to $T(B_i)$ in corresponding light shades. We show that along Wasserstein geodesics (displacement interpolants), nested images remain nested, and that the ellipsoidal Euler geodesics (not shown) remain nested inside the Wasserstein-transported ellipses indicated in light shades.

The result of Theorem 1.1 directly implies that a natural relaxation of the shape distance d_s—the lower semicontinuous envelope with respect to Wasserstein distance—agrees with the induced Wasserstein distance d_W. (See [7, section 1.7.2] regarding the general notion of relaxation of variational problems.) In fact, by a rather straightforward completion argument we can identify the shape distance in (1.5) as follows.

Theorem 1.2 *For every pair of bounded measurable sets in $\mathbb{R}^d$ of equal volume,*

$$d_s(\Omega_0, \Omega_1) = d_W(\Omega_0, \Omega_1).$$

As is well known, Wasserstein distance between measures of a given mass that are supported inside a fixed compact set induces the topology of weak-$\star$ convergence. In this topology, the closure of the set of such measures with characteristic-function densities is the set of measurable functions $\rho \colon \mathbb{R}^d \to [0, 1]$ with compact support. Theorem 1.2 above is a corollary of the following more general result that indicates how Euler-spray geodesic paths approximately connect arbitrary endpoints in this set. Both theorems are proved in Sect. 5.

Theorem 1.3 *Let $\rho_0, \rho_1 \colon \mathbb{R}^d \to [0, 1]$ be measurable functions of compact support that satisfy*

$$\int_{\mathbb{R}^d} \rho_0 = \int_{\mathbb{R}^d} \rho_1.$$

Then

(a) For any $\varepsilon > 0$ there are open sets Ω_0, Ω_1 which satisfy

$$d_\infty(\rho_0, \mathbb{1}_{\Omega_0}) + d_\infty(\rho_1, \mathbb{1}_{\Omega_1}) < \varepsilon,$$

and are connected by an Euler spray whose total action $\mathcal{A}^\varepsilon$ satisfies

$$\mathcal{A}^\varepsilon \leq d_W(\rho_0, \rho_1)^2 + \varepsilon.$$

(b) For any $\varepsilon > 0$ there is a path $\rho^\varepsilon = (\rho_t^\varepsilon)_{t \in (0,1)}$ on $(0, 1)$ consisting of a countable concatenation of Euler sprays, such that

$$\rho_t^\varepsilon \overset{\star}{\rightharpoonup} \rho_0 \ \ as \ t \to 0^+, \qquad \rho_t^\varepsilon \overset{\star}{\rightharpoonup} \rho_1 \ \ as \ t \to 1^-,$$

and the total action $\mathcal{A}^\varepsilon$ of the path satisfies

$$\mathcal{A}^\varepsilon = \int_0^1 \int_{\mathbb{R}^d} \rho_t^\varepsilon |v^\varepsilon|^2 \, dx \, dt \leq d_W(\rho_0, \rho_1)^2 + \varepsilon.$$

The results of Theorems 1.1 and 1.3 concern geodesics for shape distance that only approximately connect arbitrary sources Ω_0 and targets Ω_1. A uniqueness property of Wasserstein geodesics allows us to establish the following sharp criterion for existence and non-existence of *length-minimizing* shape geodesics that exactly connect source to target.

Theorem 1.4 *Let Ω_0, Ω_1 be bounded open sets in $\mathbb{R}^d$ with equal volume, and let $\rho = (\rho_t)_{t \in [0,1]}$ be the density along the Wasserstein geodesic path that connects $\mathbb{1}_{\Omega_0}$ and $\mathbb{1}_{\Omega_1}$. Then the infimum for shape distance in (1.5) is achieved by some path satisfying the constraints (1.2), (1.3), (1.4) if and only if ρ is a characteristic function.*

For dimension $d = 1$ the Wasserstein density is always a characteristic function. For dimension $d > 1$ however, this property of being a characteristic function, together with convexity of the density along transport lines, requires that the Wasserstein geodesic is given piecewise by rigid translation. See Corollary 5.8 and Remark 5.9 in Sect. 5.1.

Limits of Euler sprays. For the Euler sprays constructed in the proof of Theorem 1.1, the fluid domains Ω_t do not typically have smooth boundary, due to the presence of cluster points of the countable set of microdroplets. The geodesic equations that they satisfy, then, are not quite classical free-boundary water-wave equations. Rather, our Euler sprays provide a family of weak solutions $(\rho^\varepsilon, v^\varepsilon, p^\varepsilon)$ to the following system of Euler equations:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \tag{1.9}$$

$$\partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla p = 0, \tag{1.10}$$

with the "incompressibility" constraint that ρ^ε is a characteristic function as in (1.3). Both of these equations hold in the sense of distributions on $\mathbb{R}^d \times [0, 1]$, which means the following: For any smooth test functions $q \in C_c^\infty(\mathbb{R}^d \times [0, 1], \mathbb{R})$ and $\tilde{v} \in C_c^\infty(\mathbb{R}^d \times [0, 1], \mathbb{R}^d)$,

$$\int_0^1 \int_{\mathbb{R}^d} \rho(\partial_t q + v \cdot \nabla q) \, dx \, dt = \int_{\mathbb{R}^d} \rho q \, dx \Big|_{t=0}^{t=1}, \tag{1.11}$$

$$\int_0^1 \int_{\mathbb{R}^d} \rho v \cdot (\partial_t \tilde{v} + v \cdot \nabla \tilde{v}) + p \nabla \cdot \tilde{v} \, dx \, dt = \int_{\mathbb{R}^d} \rho v \cdot \tilde{v} \, dx \Big|_{t=0}^{t=1}. \tag{1.12}$$

Now, limits as $\varepsilon \to 0$ of these Euler-spray geodesics can be considered. We find it is possible to approximate a general family of Wasserstein geodesic paths that connect any two equal-mass measures having bounded densities and compact support. Scaling so the densities are bounded by 1, we can approximate in the weak-$\star$ sense by a sequence of characteristic-function initial and final data $\rho_0^k = \mathbb{1}_{\Omega_0^k}$, $\rho_1^k = \mathbb{1}_{\Omega_1^k}$, and obtain the following.

Theorem 1.5 *Let* $\rho_0, \rho_1 \colon \mathbb{R}^d \to [0, 1]$ *be measurable functions of compact support that satisfy*

$$\int_{\mathbb{R}^d} \rho_0 = \int_{\mathbb{R}^d} \rho_1.$$

Let (ρ, v) *be the density and transport velocity determined by the unique Wasserstein geodesic that connects the measures with densities* ρ_0 *and* ρ_1 *as described in Sect. 2.*

Then there is a sequence of weak solutions (ρ^k, v^k, p^k) *to* (1.11)–(1.12), *associated to Euler sprays as provided by Theorem 1.1, that converge to* $(\rho, v, 0)$, *and* (ρ, v) *is a weak solution of the pressureless Euler system*

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \tag{1.13}$$

$$\partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) = 0. \tag{1.14}$$

The convergence holds in the the following sense: $p^k \to 0$ *uniformly, and*

$$\rho^k \overset{\star}{\rightharpoonup} \rho, \qquad \rho^k v^k \overset{\star}{\rightharpoonup} \rho v, \qquad \rho^k v^k \otimes v^k \overset{\star}{\rightharpoonup} \rho v \otimes v, \tag{1.15}$$

weak-$\star$ in L^∞ *on* $\mathbb{R}^d \times [0, 1]$.

This result, proved in Sect. 6, shows that one can approximate a large family of solutions of pressureless Euler equations, ones coming from Wasserstein geodesics having bounded densities of compact support, by solutions of incompressible Euler equations with vacuum.

The weak-$\star$ convergence results stated in (1.15) indicate (correctly) the presence of oscillations in the approximating sequence. This convergence can be strengthened, however, in terms of the TL^p metric that was introduced in [28] to compare two functions that are absolutely continuous with respect to different probability measures—see Sect. 6.2. This provides an appropriate framework to compare the velocity fields of Wasserstein geodesics to those of the approximating Euler sprays.

Theorem 1.6 *Under the same hypotheses as Theorem 1.5, we have the following. There is a sequence of weak solutions (ρ^k, v^k, p^k) to (1.11)–(1.12), associated to Euler sprays as provided by Theorem 1.1, which in addition to conclusions of Theorem 1.5 satisfies*

$$\sup_{t \in [0,1]} d_{TL^2}((\rho_k(t), v_k(t)), (\rho(t), v(t))) \to 0 \ as \ k \to \infty, \tag{1.16}$$

$$\sup_{t \in [0,1]} d_{TL^1}((\rho_k(t), v_k(t) \otimes v_k(t)), (\rho(t), v(t) \otimes v(t))) \to 0 \ \ as \ k \to \infty. \tag{1.17}$$

The result of Theorem 1.6 essentially shows that while oscillations exist in space and time for the densities ρ^k and velocities v^k in Theorem 1.5, there are *no oscillations following appropriately matched flow lines*. (For a further manifestation of this see Corollary 6.7, which establishes TL^2 convergence of the Lagrangian flow maps for the Euler sprays.)

Our analysis of convergence in the TL^p topology is based upon an improved stability result regarding the stability of transport maps. We describe and establish this stability result in Theorem B.1 in "Appendix 1". We believe this result is of independent interest, as the TL^p metric allows one to quantify the stability of displacement interpolation in a stronger way than weak convergence.

Shape distance without volume constraint. Our investigations in this paper were motivated in part by an expanded notion of shape distance that was introduced and examined by Schmitzer and Schnörr in [48]. These authors considered a shape distance determined by restricting the Wasserstein metric to smooth paths of 'shape measures' consisting of uniform distributions on bounded open sets in $\mathbb{R}^2$ with connected smooth boundary. This allows one to naturally compare shapes of different volume. In our present investigation, the only smoothness properties of shapes and paths that we require are those intrinsically associated with Wasserstein distance. Thus, we investigate the geometry of a 'submanifold' of the Wasserstein space consisting of uniform distributions on shapes regarded as arbitrary bounded measurable sets in $\mathbb{R}^d$. As we will see in Sect. 7 below, geodesics for this extended shape distance correspond to a modified water-wave system with spatially uniform compressibility and zero *average* pressure. In Theorem 7.1 below we extend the result of Theorem 1.2, for volume-constrained paths of shapes, to deal with paths of uniform measures connecting two arbitrary bounded measurable sets. We show that the extended shape distance again agrees with the Wasserstein distance between the endpoints. The proof follows directly from the construction of concatenated Euler sprays used to prove Theorem 1.3(b).

1.3 Related work on geometry of image and shape spaces

The shape distance that we defined in (1.5) is related to a large body of work in imaging science and signal processing.

The general problem of finding good ways to compare two signals (such as time series, images, or shapes) is important in a number of application areas, including computer vision, machine learning, and computational anatomy. The idea to use deformations as a means of comparing images goes back to pioneering work of D'Arcy Thompson [50].

Distances derived from optimal transport theory (Monge–Kantorovich, Wasserstein, or earth-mover's distance) have been found useful in analyzing images by a number of workers [27,31,45,49,54,55]. The transport distance with quadratic cost (Wasserstein distance) is special as it provides a (formal) Riemannian structure on spaces of measures with fixed total mass [2,44,52].

Methods which endow the space of signals with the metric structure of a Riemannian manifold are of particular interest, as they facilitate a variety of image processing tasks. This geometric viewpoint, pioneered by Dupuis et al. [21,30], Trouvé [51], Younes [59] and collaborators, has motivated the study of a variety of metrics on spaces of images over a number of years—see [21,29,33,48,60] for a small selection.

The main thrust of these works is to study Riemannian metrics and the resulting distances in the space of image deformations (diffeomorphisms). Connections with the Arnold viewpoint of fluid mechanics were noted from the outset [59], and have been further explored by Holm, Trouvé, Younes and others [29,33,60]. This work has led to the *Euler-Poincaré theory of metamorphosis* [33], which sets up a formalism for analyzing least-action principles based on Lie-group symmetries generated by diffeomorphism groups.

A different way to consider shapes is to study them only via their boundary, and consider Riemannian metrics defined in terms of normal velocity of the boundary. Such a point of view has been taken by Michor, Mumford and collaborators [11,41,42,61]. As they show in [41], a metric given by only the L^2 norm of normal velocity does not lead to a viable geometry, as any two states can be connected by an arbitrarily short curve. On the other hand it is shown in [11] that if two or more derivatives of the normal velocity are penalized, then the resulting metric on the shape space is geodesically complete.

In this context, we note that what our work shows is that if the metric is determined by the L^2 norm of the transport velocity in the bulk, then the global metric distance is not zero, but that it is still degenerate in the sense that a length-minimizing geodesic typically may not exist in the shape space. While our results do not directly involve smooth deformations of smooth shapes, it is arguably interesting to consider shape spaces which permit 'pixelated' approximations, and our results apply in that context.

We speculate that to create a shape distance that (even locally) admits length-minimizing paths in the space of shapes, one needs to prevent the creation a large perimeter at negligible cost. This is somewhat analogous to the motivation for the metrics on the space of curves considered by Michor and Mumford [41]. Possibilities include introducing a term in the metric which penalizes deforming the boundary, or a term which enforces greater regularity for the vector fields considered.

A number of existing works obtain regularity of geodesic paths and resulting diffeomorphisms by considering Riemannian metrics given in terms of (second-order or higher) derivatives of velocities, as in the Large Deformation Diffeomorphic Metric Mapping (LDDMM) approach of [5], see [12]. Metrics based on conservative transport which penalize only one derivative of the velocity field are connected with viscous dissipation in fluids and have been considered by Fuchs et al. [26], Rumpf, Wirth and collaborators [46,56], as well as by Brenier, Otto, and Seis [9], who established a connection to optimal transport.

1.4 Outline

The plan of this paper is as follows. In Sect. 2 we collect some basic facts and estimates that concern geodesics for Monge–Kantorovich/Wasserstein distance. In Sect. 3 we derive formally the geodesic equations for paths of shape densities and describe the special class

of ellipsoidal solutions. The construction of Euler sprays and the proof of Theorem 1.1 is carried out in Sect. 4. Theorems 1.2 and 1.3 are proved in Sect. 5. In Sect. 6 we study weak convergence of Euler sprays and provide the proofs of Theorem 1.5 and Theorem 1.6. The main part of the paper concludes in Sect. 7 with a treatment of the extended notion of shape distance related to that examined by Schmitzer and Schnörr in [48]. Two appendices provide (a) proofs of a few basic facts about subgradients, and (b) a treatment of the TL^p topology used in Sect. 6.2.

2 Preliminaries: Wasserstein geodesics between open shapes

In this section we recall some basic properties of the standard minimizing geodesic paths (displacement interpolants) for the Wasserstein or Monge–Kantorovich distance between shape densities on open sets, and establish some basic estimates. Two properties that are key in the sequel are that the density ρ is (i) smooth on an open subset of full measure, and (ii) it is *convex* along the corresponding particle paths, see Lemma 2.1.

Let Ω_0 and Ω_1 be two bounded open sets in $\mathbb{R}^d$ with equal volume. Let μ_0 and μ_1 be measures with respective densities

$$\rho_0 = \mathbb{1}_{\Omega_0}, \qquad \rho_1 = \mathbb{1}_{\Omega_1}.$$

As is well known [8,35], there exists a convex function ψ such that the a.e.-defined map $T = \nabla\psi$ (called the *Brenier map* in [52]) is the optimal transportation map between Ω_0 and Ω_1, pushing μ_0 forward to μ_1, corresponding to the quadratic cost. Moreover, this map is unique a.e. in Ω_0; see [8] or [52, Thm. 2.32].

McCann [40] later introduced a natural curve $t \mapsto \mu_t$ that interpolates between μ_0 and μ_1, called the *displacement interpolant*, which can be described as the push-forward of the measure μ_0 by the interpolation map T_t given by

$$T_t(z) = (1-t)z + t\nabla\psi(z), \qquad 0 \le t \le 1. \tag{2.1}$$

Because ψ is convex, $\nabla\psi$ is monotone, satisfying $\langle \nabla\psi(z) - \nabla\psi(\hat{z}), z - \hat{z} \rangle \ge 0$ for all $z, \hat{z}$. Hence the interpolating maps T_t are injective for $t \in [0, 1)$, satisfying

$$|T_t(z) - T_t(\hat{z})| \ge (1-t)|z - \hat{z}|. \tag{2.2}$$

Note that particle paths $z \mapsto T_t(z)$ follow straight lines with constant velocity:

$$v(T_t(z), t) = \nabla\psi(z) - z. \tag{2.3}$$

Furthermore [6], μ_t has density ρ_t that satisfies the continuity equation

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \tag{2.4}$$

and in terms of these quantities, the Wasserstein distance satisfies

$$d_W(\mu_0, \mu_1)^2 = \int_{\Omega_0} |\nabla\psi(z) - z|^2 \, dz = \int_0^1 \int_{\Omega_t} \rho|v|^2 \, dx \, dt. \tag{2.5}$$

The displacement interpolant has the property that

$$d_W(\mu_s, \mu_t) = (t-s)d_W(\mu_0, \mu_1), \quad 0 \le s \le t \le 1. \tag{2.6}$$

The property (2.6) implies that the displacement interpolant is a *constant-speed geodesic* (length-minimizing path) with respect to Wasserstein distance. The displacement interpolant

$t \mapsto \mu_t$ is the *unique* constant-speed geodesic connecting μ_0 and μ_1, due to the uniqueness of the Brenier map and Proposition 5.32 of [47] (or see [1, Thm. 3.10]). For brevity the path $t \mapsto \mu_t$ is called the *Wasserstein geodesic* from μ_0 to μ_1.

At this point it is convenient to mention that the result of Theorem 1.4, providing a sharp criterion for the existence of a minimizer for the shape distance in (1.5), will be derived by combining the uniqueness property of Wasserstein geodesics with the result of Theorem 1.2— see the end of Sect. 5 below.

We note here that the L^∞ transport distance may be defined as a minimum over maps S that push forward the measure μ_0 to μ_1 [47, Thm. 3.24] and satisfies the estimate

$$
\begin{aligned}
d_\infty(\mu_0, \mu_1) &= \min\{\|S - \mathrm{id}\|_{L^\infty(\mu_0)} : S_\sharp \mu_0 = \mu_1\} \\
&\geq |\Omega_0|^{-1/2} \min\{\|S - \mathrm{id}\|_{L^2(\mu_0)} : S_\sharp \mu_0 = \mu_1\} \\
&= |\Omega_0|^{-1/2} d_W(\Omega_0, \Omega_1).
\end{aligned}
\tag{2.7}
$$

3 Geodesics and incompressible fluid flow

3.1 Incompressible Euler equations for smooth critical paths

In this subsection, for completeness we derive the Euler fluid equations that formally describe smooth geodesics (paths with stationary action) for the shape distance in (1.5). To cope with the problem of moving domains we work in a Lagrangian framework, computing variations with respect to flow maps that preserve density and the endpoint shapes Ω_0 and Ω_1.

Toward this end, suppose that

$$
Q = \bigcup_{t \in [0,1]} \Omega_t \times \{t\} \ \subset \mathbb{R}^d \times [0, 1]
\tag{3.1}
$$

is a space–time domain generated by deformation of Ω_0 due to a velocity field $v \colon \bar{Q} \to \mathbb{R}^d$ that is smooth up to the boundary. That is, the t-cross section of Q is given by

$$
\boxed{\Omega_t = X(\Omega_0, t),}
\tag{3.2}
$$

where X is the Lagrangian flow map associated to v, satisfying

$$
\dot{X}(z, t) = v(X(z, t), t), \qquad X(z, 0) = z,
\tag{3.3}
$$

for all $(z, t) \in \Omega_0 \times [0, 1]$.

For any (smooth) extension of v to $\mathbb{R}^d \times [0, 1]$, the solution of the mass-transport equation in (1.2) with given initial density ρ_0 supported in $\bar{\Omega}_0$ is

$$
\rho(x, t) = \rho_0(z) \det\left(\frac{\partial X}{\partial z}(z, t)\right)^{-1}, \quad x = X(z, t) \in \Omega_t,
$$

with $\rho = 0$ outside $\bar{Q}$.

Considering a family $\varepsilon \mapsto X_\varepsilon$ of flow maps smoothly depending on a variational parameter ε, the variation $\delta X = (\partial X_\varepsilon / \partial \varepsilon)|_{\varepsilon=0}$ induces a variation in density $\delta\rho = (\partial \rho_\varepsilon / \partial \varepsilon)|_{\varepsilon=0}$ satisfying

$$
-\frac{\delta\rho}{\rho} = \delta \log \det\left(\frac{\partial X}{\partial z}(z, t)\right) = \mathrm{tr}\left(\frac{\partial \delta X}{\partial z}\left(\frac{\partial X}{\partial z}\right)^{-1}\right)
\tag{3.4}
$$

Introducing $\tilde{v}(x, t) = \delta X(z, t)$, $x = X(z, t)$, it follows

$$-\frac{\delta\rho}{\rho} = \sum_j \frac{\partial \tilde{v}_j}{\partial x_j} = \nabla \cdot \tilde{v}. \tag{3.5}$$

For variations that leave the density invariant, necessarily $\nabla \cdot \tilde{v} = 0$ inside Q.

We now turn to consider the variation of the action or transport cost as expressed in terms of the flow map:

$$\mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho(x, t)|v(x, t)|^2 dx\, dt = \int_0^1 \int_{\Omega_0} |\dot{X}(z, t)|^2 dz\, dt. \tag{3.6}$$

For flows preserving $\rho = 1$ in $\bar{Q}$, of course $\nabla \cdot v = 0$. Computing the first variation of $\mathcal{A}$ about such a flow, after an integration by parts in t and changing to Eulerian variables, we find

$$\begin{aligned}
\frac{\delta\mathcal{A}}{2} &= \int_0^1 \int_{\Omega_0} \dot{X} \cdot \delta\dot{X} \, dz\, dt \\
&= \int_{\Omega_0} \dot{X} \cdot \delta X \, dz \Big|_{t=1} - \int_0^1 \int_{\Omega_0} \ddot{X} \cdot \delta X \, dz\, dt \\
&= \int_{\Omega_t} v \cdot \tilde{v} \, dx \Big|_{t=1} - \int_0^1 \int_{\Omega_t} (\partial_t v + v \cdot \nabla v) \cdot \tilde{v} \, dx\, dt. \tag{3.7}
\end{aligned}$$

Recall that any L^2 vector field u on Ω_t has a unique (standard) Helmholtz decomposition, as the sum of a gradient and a field L^2-orthogonal to all gradients, which is divergence-free with zero normal component at the boundary:

$$u = \nabla p + w, \quad \nabla \cdot w = 0 \text{ in } \Omega_t, \quad w \cdot n = 0 \text{ on } \partial\Omega_t. \tag{3.8}$$

We make use of a variant of this decomposition [16, p. 215], which states that u has a unique L^2-orthogonal decomposition as the sum of a divergence-free field and a gradient of a function that vanishes on the boundary:

$$u = w + \nabla p, \quad \nabla \cdot w = 0 \text{ in } \Omega_t, \quad p = 0 \text{ on } \partial\Omega_t. \tag{3.9}$$

Requiring $\delta\mathcal{A} = 0$ for arbitrary virtual displacements having $\nabla \cdot \tilde{v} = 0$ (and $\tilde{v} = 0$ at $t = 1$ at first), we find that necessarily $u = -(\partial_t v + v \cdot \nabla v)$ has a representation as in (3.9) with $w = 0$. Thus the incompressible Euler equations hold in Q:

$$\partial_t v + v \cdot \nabla v + \nabla p = 0, \quad \nabla \cdot v = 0 \text{ in } Q, \tag{3.10}$$

where $p : \bar{Q} \to \mathbb{R}$ is smooth and satisfies

$$\boxed{p = 0 \text{ on } \partial\Omega_t.} \tag{3.11}$$

Finally, we may consider variations $\tilde{v}$ that do not vanish at $t = 1$. However, we require $\tilde{v} \cdot n = 0$ on $\partial\Omega_1$ in this case because the target domain Ω_1 should be fixed. That is, the allowed variations in the flow map X must fix the image at $t = 1$:

$$\Omega_1 = X(\Omega_0, 1). \tag{3.12}$$

The vanishing of the integral term at $t = 1$ in (3.7) then leads to the requirement that v is a gradient at $t = 1$. For $t = 1$ we must have

$$v = \nabla\phi \text{ in } \Omega_t. \tag{3.13}$$

We claim this gradient representation actually must hold for all $t \in [0, 1]$. Let $v = \nabla\phi + w$ be the Helmholtz decomposition of v, and for small ε consider the family of flow maps generated by

$$\dot{X}_\varepsilon(z, t) = (v + \varepsilon w)(X_\varepsilon(z, t), t), \qquad X_\varepsilon(z, 0) = z. \tag{3.14}$$

Corresponding to this family, the action from (3.6) takes the form

$$\mathcal{A}_\varepsilon = \int_0^1 \int_{\Omega_0} |\dot{X}_\varepsilon(z, t)|^2 dz\, dt = \int_0^1 \int_{\Omega_t} |\nabla\phi|^2 + |(1 + \varepsilon)w|^2\, dx\, dt \tag{3.15}$$

Because $w \cdot n = 0$ on $\partial\Omega_t$, the domains Ω_t do not depend on ε, and the same is true of $\nabla\phi$ and w, so this expression is a simple quadratic polynomial in ε. Thus

$$\frac{1}{2} \frac{d\mathcal{A}_\varepsilon}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_0^1 \int_{\Omega_t} |w|^2\, dx\, dt \tag{3.16}$$

and consequently it is necessary that $w = 0$ if $\delta\mathcal{A} = 0$. This proves the claim.

The Euler equation in (3.10) is now a spatial gradient, and one can add to ϕ a function of t alone (possibly different on each component of Q) to ensure that

$$\boxed{\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + p = 0, \quad \Delta\phi = 0 \quad \text{in } \Omega_t.} \tag{3.17}$$

The equations boxed above, including (3.17) together with the zero-pressure boundary condition (3.11) and the kinematic condition that the boundary of Ω_t moves with normal velocity $v \cdot n$ (coming from (3.2)–(3.3)), comprise what we shall call the *Euler droplet* equations, for incompressible, inviscid, potential flow of fluid with zero surface tension and zero pressure at the boundary.

Definition 3.1 A smooth *solution* of the Euler droplet equations is a triple (Q, ϕ, p) such that $\phi, p \colon \bar{Q} \to \mathbb{R}$ are smooth and the Eqs. (3.1), (3.2), (3.3), (3.13), (3.17), (3.11) all hold.

Proposition 3.2 *For smooth flows X that deform Ω_0 as above, that respect the density constraint $\rho = 1$ and fix $\Omega_1 = X(\Omega_0, 1)$, the action $\mathcal{A}$ in (3.6) is critical with respect to smooth variations if and only if X corresponds to a smooth solution of the Euler droplet equations.*

3.2 Weak solutions and Galilean boost

Here we record a couple of simple basic properties of solutions of the Euler droplet equations.

Proposition 3.3 *Let (Q, ϕ, p) be a smooth solution of the Euler droplet equations. Let $\rho = \mathbb{1}_{\bar{Q}}$ and $v = \mathbb{1}_{\bar{Q}}\nabla\phi$, and extend p as zero outside $\bar{Q}$.*
(a) The Euler equations (1.9)–(1.10) hold in the sense of distributions on $\mathbb{R}^d \times [0, 1]$.
(b) The mean velocity

$$\bar{v} = \frac{1}{|\Omega_t|} \int_{\Omega_t} v(x, t)\, dx \tag{3.18}$$

is constant in time, and the action decomposes as

$$\mathcal{A} = \int_0^1 \int_{\Omega_t} |v - \bar{v}|^2 dx\, dt + |\Omega_0||\bar{v}|^2. \tag{3.19}$$

(c) Given any constant vector $b \in \mathbb{R}^d$, another smooth solution $(\widehat{Q}, \hat{\phi}, \hat{p})$ of the Euler droplet equations is given by a Galilean boost, via

$$\widehat{Q} = \bigcup_{t \in [0,1]} (\Omega_t + bt) \times \{t\}, \tag{3.20}$$

$$\hat{\phi}(x + bt, t) = \phi(x, t) + b \cdot x + \frac{1}{2}|b|^2 t, \quad \hat{p}(x + bt, t) = p(x, t). \tag{3.21}$$

Proof To prove (a), what we must show is the following: For any smooth test functions $q \in C_c^\infty(\mathbb{R}^d \times [0, 1], \mathbb{R})$ and $\tilde{v} \in C_c^\infty(\mathbb{R}^d \times [0, 1], \mathbb{R}^d)$,

$$\int_Q (\partial_t q + v \cdot \nabla q) \, dx \, dt = \int_{\Omega_t} q \, dx \Big|_{t=0}^{t=1} \tag{3.22}$$

$$\int_Q v \cdot (\partial_t \tilde{v} + v \cdot \nabla \tilde{v}) + p \nabla \cdot \tilde{v} \, dx \, dt = \int_{\Omega_t} \tilde{v} \cdot v \, dx \Big|_{t=0}^{t=1} \tag{3.23}$$

Changing to Lagrangian variables via $x = X(z, t)$, writing $\hat{q}(z, t) := q(X(z, t), t)$, and using incompressibility, equation (3.22) is equivalent to

$$\int_0^1 \int_{\Omega_0} \frac{d}{dt} \hat{q}(z, t) \, dz \, dt = \int_{\Omega_0} \hat{q}(z, t) \, dz \Big|_{t=0}^{t=1}. \tag{3.24}$$

Evidently this holds. In (3.23), we integrate the pressure term by parts, and treat the rest as in (3.7) to find that (3.23) is equivalent to

$$\int_Q (\partial_t v + v \cdot \nabla v + \nabla p) \cdot \tilde{v} \, dx \, dt = 0. \tag{3.25}$$

Then (a) follows. The proof of parts (b) and (c) is straightforward. □

3.3 Ellipsoidal Euler droplets

The initial-value problem for the Euler droplet equations is a difficult fluid free boundary problem, one that may be treated by the methods developed by Wu [57,58]. For flows with vorticity and smooth enough initial data, smooth solutions for short time have been shown to exist in [14,15,37].

In this section, we describe simple, particular Euler droplet solutions for which the fluid domain Ω_t remains ellipsoidal for all t. As mentioned by Longuet-Higgins [39], equations governing such solutions were known to Dirichlet [19] and are discussed in Lamb's treatise [36]. Our result below associates Dirichlet's ellipsoids with an interesting geodesic interpretation.

Proposition 3.4 *Given a constant $r > 0$, let $a(t) = (a_1(t), \ldots, a_d(t))$ be any constant-speed geodesic on the surface in $\mathbb{R}_+^d$ determined by the relation*

$$a_1 \cdots a_d = r^d. \tag{3.26}$$

Then this determines an Euler droplet solution (Q, ϕ, p) with Ω_t equal to the ellipsoid $E_{a(t)}$ given by

$$E_a = \Big\{ x \in \mathbb{R}^d : \sum_j (x_j/a_j)^2 < 1 \Big\}, \tag{3.27}$$

and potential and pressure given by

$$\phi(x, t) = \frac{1}{2} \sum_j \frac{\dot{a}_j x_j^2}{a_j} - \beta(t), \qquad p(x, t) = \dot{\beta} \left(1 - \sum_j \frac{x_j^2}{a_j^2} \right), \tag{3.28}$$

with

$$\dot{\beta}(t) = \frac{1}{2} \frac{\sum_j \dot{a}_j^2 / a_j^2}{\sum_j 1/a_j^2}. \tag{3.29}$$

Proof The flow X associated with a velocity potential of the form in (3.28) must satisfy

$$\dot{X}_j = \eta_j(t) X_j, \qquad \eta_j = \frac{\dot{a}_j}{a_j}, \quad j = 1, \dots, d. \tag{3.30}$$

Then for all j, $(X_j/a_j)^{\cdot} = 0$ and hence

$$X_j(z, t) = \frac{a_j(t)}{a_j(0)} z_j, \tag{3.31}$$

so the flow is purely dilational along each axis and consequently ellipsoids are deformed to ellipsoids as claimed. Note that incompressibility corresponds to the relation

$$\Delta \phi = \sum_j \eta_j = \sum_j \frac{\dot{a}_j}{a_j} = \frac{d}{dt} \log(a_1 \cdots a_d) = 0.$$

From (3.28) we next compute

$$\partial_t \phi_t + \frac{1}{2} |\nabla \phi|^2 = -\dot{\beta} + \frac{1}{2} \sum_j (\dot{\eta}_j + \eta_j^2) x_j^2 = -\dot{\beta} + \frac{1}{2} \sum_j \frac{\ddot{a}_j x_j^2}{a_j}.$$

This must equal zero on the boundary where $x_j = a_j s_j$ with $s \in S_{d-1}$ arbitrary. We infer that for all j,

$$a_j \ddot{a}_j = 2 \dot{\beta}. \tag{3.32}$$

The expression for p in (3.28) in terms of $\dot{\beta}$ then follows from (3.17), and $p = 0$ on $\partial \Omega_t$.

We recover $\dot{\beta}$ by differentiating the constraint twice in time. We find

$$0 = \sum_j \left(\sum_k a_1 \cdots a_d \frac{\dot{a}_k}{a_k} \frac{\dot{a}_j}{a_j} + a_1 \dots a_d \frac{a_j \ddot{a}_j - \dot{a}_j^2}{a_j^2} \right) = 0 + \sum_j \frac{2 \dot{\beta} - \dot{a}_j^2}{a_j^2}$$

whence (3.29) holds.

To get the first integral that corresponds to kinetic energy, multiply (3.32) by $2 \dot{a}_j / a_j$ and sum to find

$$0 = \sum_j \dot{a}_j \ddot{a}_j, \quad \text{whence} \quad \sum_j \dot{a}_j^2 = c^2$$

and we see that $c = |\dot{a}(t)|$ is the constant speed of motion.

It remains to see that (3.32) are the geodesic equations on the constraint surface. This follows because (3.32) says that $\ddot{a}$ is parallel to the gradient of $F(a) = \sum_j \log a_j$, and the constraint (3.26) corresponds to staying on the level set $F(a) = \log r^d$. This finishes the demonstration of Proposition 3.4. □

Remark 3.5 For later reference, we note that $\ddot{a}_j \geq 0$ for all t, due to (3.32) and (3.29).

Remark 3.6 Given any two points on the surface described by the constraint (3.26), there exists a constant-speed geodesic connecting them. This fact is a straightforward consequence of the Hopf-Rinow theorem on geodesic completeness [34, Theorem 1.7.1], because all closed and bounded subsets on the surface are compact.

Remark 3.7 The Euclidean metric on the hyperboloid-like surface arises, in fact, as the metric induced by the Wasserstein distance [53, Chap. 15], because, given any incompressible path $t \mapsto X(\cdot, t)$ of dilations, satisfying (3.30) for some smooth $\eta(t)$ and with $a_1 \cdots a_d = r^d$,

$$\int_{\Omega_t} |v|^2 \, dx = \int_{\Omega_t} \sum_j \eta_j^2 x_j^2 \, dx = \sum_j \dot{a}_j^2 \int_{|z| \leq 1} z_j^2 \, dz \, r^d = \frac{\omega_d r^d}{d+2} \sum_j \dot{a}_j^2,$$

where $\omega_d = |B(0, 1)|$ is the volume of the unit ball in $\mathbb{R}^d$. For a geodesic, this expression is constant for $t \in [0, 1]$ and equals the action $\mathcal{A}_a$ in (3.6) for the ellipsoidal Euler droplet.

3.4 Ellipsoidal Wasserstein droplets

Let (Q, ϕ, p) be an ellipsoidal Euler droplet solution as given by Proposition 3.4, so that $\Omega_0 = E_{a(0)}$ and $\Omega_1 = E_{a(1)}$ are co-axial ellipsoids. We will call the optimal transport map T between these co-axial ellipsoids an *ellipsoidal Wasserstein droplet*. This is described and related to the Euler droplet as follows.

Given $A \in \mathbb{R}^d$, let $D_A = \text{diag}(A_1, \ldots, A_d)$ denote the diagonal matrix with diagonal A. Then, given $\Omega_0 = E_{a(0)}$, $\Omega_1 = E_{a(1)}$ as above, the particle paths for the Wasserstein geodesic between the corresponding shape densities are given by linear interpolation via

$$T_t(z) = D_{A(t)} D_{A(0)}^{-1} z, \qquad A(t) = (1 - t)a(0) + ta(1). \tag{3.33}$$

Note that a point $z \in E_A$ if and only if $D_A^{-1} z$ lies in the unit ball $B(0, 1)$ in $\mathbb{R}^d$. Thus the Wasserstein geodesic flow takes ellipsoids to ellipsoids:

$$T_t(\Omega_0) = E_{A(t)}, \quad t \in [0, 1].$$

Let $a(t)$, $t \in [0, 1]$, be the geodesic on the hyperboloid-like surface that corresponds to the Euler droplet that we started with. Recall that $\Omega_t = E_{a(t)}$ from Proposition 3.4. Because each component $t \mapsto a_j(t)$ is convex by Remark 3.5, it follows that for each $j = 1, \ldots, d$,

$$a_j(t) \leq A_j(t), \quad t \in [0, 1]. \tag{3.34}$$

Because $E_A = D_A B(0, 1)$, we deduce from this the following important nesting property, which is illustrated in Fig. 2 (where for visibility the ellipses at times $t = \frac{1}{2}$ and $t = 1$ are offset horizontally by $\frac{b}{2}$ and b respectively).

Proposition 3.8 *Given any corresponding ellipsoidal Euler droplet and Wasserstein droplet that deform one ellipsoid $\Omega_0 = E_{a(0)}$ to another $\Omega_1 = E_{a(1)}$, the Euler domains remain nested inside their Wasserstein counterparts, with*

$$X(\Omega_0, t) = \Omega_t \subset T_t(\Omega_0), \quad t \in [0, 1]. \tag{3.35}$$

Fig. 2 Euler droplet (light blue) deforming a circle to an ellipse, nested inside a Wasserstein droplet (dark orange). Tracks of the center and endpoints of vertical major axis are indicated for both droplets (color figure online)
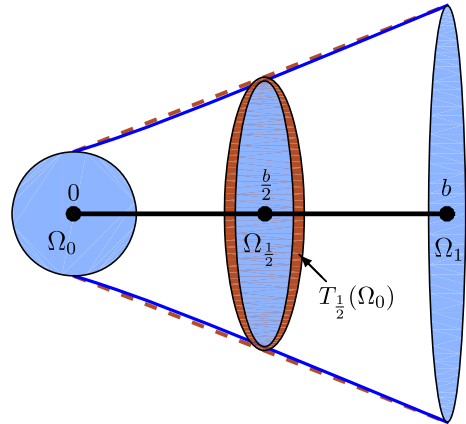
Remark 3.9 The dilational flow X from (3.31) associated with the Euler droplet is given by $X(z, t) = D_{a(t)} D_{a(0)}^{-1} z$ in terms of the notation used in (3.33). Due to (3.34), this flow satisfies, for each $j = 1, \ldots, d$ and $z \in \mathbb{R}^d$,

$$|X_j(z, t)| = \frac{a_j(t)}{a_j(0)} |z_j| \leq \frac{A_j(t)}{A_j(0)} |z_j| = |T_t(z)_j|.$$

For the nesting property $X(\hat{\Omega}, t) \subset T_t(\hat{\Omega})$ to hold, convexity of $\hat{\Omega}$ is not sufficient in general. However, a sufficient condition is that whenever $\eta_j \in [0, 1]$ for $j = 1, \ldots, d$,

$$x = (x_1, \ldots, x_d) \in \hat{\Omega} \quad \text{implies} \quad D_\eta x = (\eta_1 x_1, \ldots, \eta_n x_n) \in \hat{\Omega}.$$

3.5 Action estimate for ellipsoidal Euler droplets

For later use below, we describe how to bound the action for a boosted ellipsoidal Euler droplet in terms of action for the corresponding boosted ellipsoidal Wasserstein droplet, in the case when the source and target domains are respectively a ball and translated ellipse:

Lemma 3.10 *Given $r > 0$, $\hat{a} \in \mathbb{R}_+^d$ with $\hat{a}_1 \cdots \hat{a}_d = r^d$, and $b \in \mathbb{R}^d$, let*

$$\Omega_0 = B(0, r), \qquad \Omega_1 = E_{\hat{a}} + b.$$

Let $a(t)$, $t \in [0, 1]$, be a minimizing geodesic on the surface (3.26) with

$$a(0) = \hat{r} = (r, \ldots, r), \qquad a(1) = \hat{a} = (\hat{a}_1, \ldots, \hat{a}_d).$$

Let (Q, ϕ, p) be the ellipsoidal Euler droplet solution corresponding to the geodesic a, and let $\mathcal{A}_a$ denote the corresponding action. Then

$$d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 \leq \mathcal{A}_a \leq d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \frac{\overline{\lambda}^4}{\underline{\lambda}^2} \omega_d r^{d+2}, \tag{3.36}$$

where

$$\underline{\lambda} = \min \frac{\hat{a}_i}{r}, \qquad \overline{\lambda} = \max \frac{\hat{a}_i}{r}. \tag{3.37}$$

Proof First, consider the transport cost for mapping Ω_0 to Ω_1. The (constant) velocity of particle paths starting at $x \in B(0, r)$ is

$$u(x) = (r^{-1} D_{\hat{a}} - I)x + b,$$

and the squared transport cost or action is (substituting $x = rz$)

$$d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 = \int_{B(0,r)} |u(x)|^2 \, dx = \sum_j \int_{B(0,r)} \left(\frac{\hat{a}_j}{r} - 1\right)^2 z_j^2 + b_j^2 \, dz$$

$$= \omega_d r^d \left(|b|^2 + \frac{|\dot{A}|^2}{d+2}\right), \tag{3.38}$$

where $A(t) = (1 - t)\hat{r} + t\hat{a}$ is the straight-line path from $\hat{r}$ to $\hat{a}$.

The mass density inside the transported ellipsoid $T_t(\Omega_0)$ is constant in space, given by

$$\rho(t) = \det DT_t^{-1} = \prod_i \frac{r}{A_i(t)} = \prod_i \left(1 - t + t\frac{\hat{a}_i}{r}\right)^{-1}.$$

Due to Remark 3.7, the corresponding action for the Euler droplet is bounded by that of the constant-volume path found by dilating the ellipsoidal Wasserstein droplet: Let

$$\gamma_j(t) = \rho(t)^{1/d} A_j(t). \tag{3.39}$$

Then the flow $S_t(z) = r^{-1} D_{\gamma(t)} z$ is dilational and volume-preserving (with $\prod_j \gamma_j(t) \equiv r^d$) and has zero mean velocity. The flow $z \mapsto S_t(z) + tb$ takes Ω_0 to Ω_1, as on Fig. 2, with action

$$\mathcal{A}_\gamma = \int_0^1 \int_{B(0,r)} \sum_j \left(b_j + \frac{\dot{\gamma}_j z_j}{r}\right)^2 \, dz \, dt$$

$$= \omega_d r^d \left(|b|^2 + \frac{1}{d+2} \int_0^1 |\dot{\gamma}|^2 \, dt\right). \tag{3.40}$$

Note that $\sum_j (\dot{\gamma}_j/\gamma_j)^2 \leq \sum_j (\dot{A}_j/A_j)^2$, because

$$\frac{\dot{\gamma}_j}{\gamma_j} = \frac{\dot{A}_j}{A_j} + \frac{\dot{\rho}}{d\rho} = \frac{\dot{A}_j}{A_j} - \frac{1}{d} \sum_i \frac{\dot{A}_i}{A_i}.$$

Because ρ is convex we have $\rho \leq 1$, hence $\gamma_j^2 \leq \max A_i^2$. Thus

$$|\dot{\gamma}|^2 \leq (\max A_i^2) \sum_j \frac{\dot{A}_j^2}{A_j^2} \leq \left(\frac{\max A_i^2}{\min A_i^2}\right) |\dot{A}|^2 \leq \left(\frac{\max \hat{a}_i^2}{\min \hat{a}_i^2}\right) |\hat{a} - \hat{r}|^2. \tag{3.41}$$

Plugging this back into (3.40) and using (3.38), we deduce that

$$\mathcal{A}_\gamma \leq d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \frac{\omega_d r^d}{d+2} \left(\frac{\max \hat{a}_i^2}{\min \hat{a}_i^2}\right) |\hat{a} - \hat{r}|^2. \tag{3.42}$$

With the notation in (3.37), $\underline{\lambda}$ and $\overline{\lambda}$ respectively are the maximum and minimum eigenvalues of DT_t, and because $|1 - \hat{a}_i/r| \leq \max(1, \hat{a}_i/r) \leq \overline{\lambda}$ for all $i = 1, \ldots, d$, this estimate implies

$$\mathcal{A}_a \le \mathcal{A}_\gamma \le d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + \frac{d}{d+2} \frac{\overline{\lambda}^4}{\underline{\lambda}^2} \omega_d r^{d+2}. \tag{3.43}$$

This yields (3.36) and completes the proof.   □

3.6 Velocity and pressure estimates

Lastly in this section we provide bounds on the velocity $v = \nabla\phi$ and pressure p for the ellipsoidal Euler droplet solutions.

Due to the action-minimizing property of the geodesic $a(t)$, and because $1/a_j^2 \le \sum_i (1/a_i^2)$, the pressure in (3.28) is bounded in terms of the dilated path γ from (3.39), by

$$0 \le p \le \dot{\beta} \le \frac{1}{2} \sum_j \dot{a}_j^2 \le \frac{1}{2} \int_0^1 |\dot{\gamma}|^2 \, dt$$

Using (3.41) and the notation in (3.37), it follows

$$0 \le p \le \frac{\overline{\lambda}^4}{\underline{\lambda}^2} r^2 d. \tag{3.44}$$

For the velocity, it suffices to note that in (3.30), $|X_j/a_j| \le 1$ hence $|\dot{X}|^2 \le \sum_j \dot{a}_j^2$. Thus the same bounds as above apply and we find

$$|\nabla\phi|^2 \le \frac{\overline{\lambda}^4}{\underline{\lambda}^2} r^2 d. \tag{3.45}$$

Finally, for a boosted ellipsoidal Euler droplet, with velocity boosted as in (3.21) by a constant vector $b \in \mathbb{R}^d$, the same pressure bound as above in (3.44) applies, and the same bound on velocity becomes

$$|\nabla\hat{\phi} - b|^2 \le \frac{\overline{\lambda}^4}{\underline{\lambda}^2} r^2 d. \tag{3.46}$$

We remark that in the constructions that we make in the next section, for a given distortion ratio $\overline{\lambda}^4/\underline{\lambda}^2$, the bounds in (3.44)–(3.46) can be made arbitrarily small by requiring r^2 is small.

4 Euler sprays

Heuristically, an Euler spray is a countable disjoint superposition of solutions of the Euler droplet equations. Recall that the notation $\sqcup_n Q_n$ means the union of disjoint sets Q_n.

Definition 4.1 An **Euler spray** is a triple (Q, ϕ, p), with Q a bounded open subset of $\mathbb{R}^d \times [0, 1]$ and ϕ, $p : Q \to \mathbb{R}$, such that there is a sequence $\{(Q_n, \phi_n, p_n)\}_{n \in \mathbb{N}}$ of smooth solutions of the Euler droplet equations, such that $Q = \sqcup_{n=1}^\infty Q_n$ is a disjoint union of the sets Q_n, and for each $n \in \mathbb{N}$, $\phi_n = \phi|_{Q_n}$ and $p_n = p|_{Q_n}$.

With each Euler spray that satisfies appropriate bounds we may associate a weak solution (ρ, v, p) of the Euler system (1.9)–(1.10). The following result is a simple consequence of

the weak formulation in (1.11)–(1.12) together with Proposition 3.3(a) and the dominated convergence theorem.

Proposition 4.2 *Suppose (Q, ϕ, p) is an Euler spray such that $|\nabla\phi|^2$ and p are integrable on Q. Then with $\rho = \mathbb{1}_Q$ and $v = \mathbb{1}_Q \nabla\phi$ and with p extended as zero outside Q, the triple (ρ, v, p) satisfies the Euler system (1.9)–(1.10) in the sense of distributions on $\mathbb{R}^d \times [0, 1]$.*

Our main goal in this section is to prove Theorem 1.1. The strategy of the proof is simple to outline: We will approximate the optimal transport map $T: \Omega_0 \to \Omega_1$ for the Monge–Kantorovich distance, up to a null set, by an 'ellipsoidal transport spray' built from a countable collection of ellipsoidal Wasserstein droplets. The spray maps Ω_0 to a target Ω_1^ε whose shape distance from Ω_1 is as small as desired. Then from the corresponding ellipsoidal Euler droplets nested inside the Wasserstein ones, we construct the desired Euler spray (Q, ϕ, p) that connects Ω_0 to Ω_1^ε by a critical path for the action in (1.1).

Remark 4.3 In general, for the Euler sprays that we construct, the domain $Q = \sqcup_{n=1}^\infty Q_n$ has an irregular boundary ∂Q strictly larger than the infinite union $\sqcup_{n=1}^\infty \partial Q_n$ of smooth boundaries of individual ellipsoidal Euler droplets, since ∂Q contains limit points of sequences belonging to infinitely many Q_n.

4.1 Approximating optimal transport by an ellipsoidal transport spray

Heuristically, an ellipsoidal transport spray is a countable disjoint superposition of transport maps on ellipsoids, whose particle trajectories do not intersect.

Definition 4.4 An **ellipsoidal transport spray** on Ω_0 is a map $S: \Omega_0 \to \mathbb{R}^d$, such that

$$\Omega_0 = \bigsqcup_{n \in \mathbb{N}} \Omega_0^n$$

is a disjoint union of ellipsoids, the restriction of S to Ω_0^n is an ellipsoidal Wasserstein droplet, and the linear interpolants S_t defined by

$$S_t(z) = (1 - t)z + t S(z), \qquad z \in \Omega_0,$$

remain injections for each $t \in [0, 1]$.

Proposition 4.5 *Let Ω_0, Ω_1 be a pair of bounded open sets in $\mathbb{R}^d$ of equal volume, and let $T: \Omega_0 \to \Omega_1$ be the optimal transport map for the Monge–Kantorovich distance with quadratic cost. For any $\varepsilon > 0$, there is an ellipsoidal transport spray $S^\varepsilon: \Omega_0^\varepsilon \to \mathbb{R}^d$ such that*

(i) Ω_0^ε is a countable union of balls with $|\Omega_0 \setminus \Omega_0^\varepsilon| = 0$,
(ii) $\sup_{z \in \Omega_0^\varepsilon} |T(z) - S^\varepsilon(z)| < \varepsilon \operatorname{diam} \Omega_1$, and
(iii) the L^∞ transportation distance between the uniform distributions on Ω_1^ε and Ω_1 satisfies $d_\infty(\Omega_1^\varepsilon, \Omega_1) < \varepsilon \operatorname{diam} \Omega_1$.

The proof of this result will comprise the remainder of this subsection. The strategy is as follows. Due to Alexandrov's theorem on the twice differentiability of convex functions, the Brenier map $T = \nabla\psi$ is differentiable a.e. The set Ω_0^ε will be chosen to be the union of a suitable Vitali covering of Ω_0 a.e. by balls B_i, whose centers are points of differentiability

of T. On each ball B_i we approximate T by an affine map S^ε which takes the ball center x_i to $(1 + \varepsilon)T(x_i)$, taking the form

$$S^\varepsilon(x) = (1 + \varepsilon)T(x_i) + DT(x_i)(x - x_i), \qquad x \in B_i. \tag{4.1}$$

The corresponding displacement interpolation map S_t^ε has three key properties: (i) it is locally affine so maps balls to ellipsoids, (ii) it is volume-preserving, and (iii) it spreads out the ball centers by the dilation factor $1 + \varepsilon$, which ensures that the ball images remain non-overlapping, because they are nested inside corresponding images under a dilated version of the displacement interpolation map T_t.

4.1.1 Nesting by subgradient approximation

It turns out to be quite convenient to construct this dilated version based on the *subgradient* $\partial \psi$ of the Brenier potential ψ, to deal with the problem that the Brenier map T may be discontinuous, perhaps on a complicated set.

We recall that the subgradient of ψ is a set-valued function defined by

$$\partial \psi(x) = \{z \in \mathbb{R}^d : \psi(x + h) \ge \psi(x) + \langle z, h \rangle \ \forall h \in \mathbb{R}^d\}. \tag{4.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^d$. For each $x \in \mathbb{R}^d$, the set $\partial \psi(x)$ is closed, convex, and nonempty. For the convenience of readers, in "Appendix 1" we provide proofs of the few basic facts about subgradients that we will use.

According to Alexandrov's theorem (see [43, Thm. 1.3] or [23]), for almost every $x_0 \in \mathbb{R}^d$ the subgradient $\partial \psi$ admits a local first-order expansion

$$\partial \psi(x) \subset T(x_0) + H(x - x_0) + B(0, \omega(x_0, r)) \qquad \forall x \in B(x_0, r), \tag{4.3}$$

where H is a positive semidefinite matrix and $\omega(x_0, r) = o(r)$ as $r \to 0$. Note we may assume $\omega(x_0, r)/r$ is increasing in r. The quantity $T(x_0) = \nabla \psi(x_0)$ provides the Brenier transport map at x_0, and we let $\mathrm{Hess}\,\psi(x_0)$ denote the matrix H, which is the Hessian of ψ at x_0 in case the gradient $\nabla \psi$ is differentiable at x_0.

Let us say x_0 is an *Alexandrov point* if (4.3) holds. Because $T = \nabla \psi$ pushes forward the Lebesgue measure on Ω_0 to that on Ω_1, it follows that $\det \mathrm{Hess}\,\psi(x) = 1$ for a.e. Alexandrov point x in Ω_0 (see [40, Thm. 4.4] or [52, Thm. 4.8]). Denoting by Ω_A the set of these points, we have $|\Omega_0 \setminus \Omega_A| = 0$, and with $\lambda_1(x), \dots, \lambda_d(x)$ denoting the eigenvalues of $\mathrm{Hess}\,\psi(x)$,

$$\lambda_1(x) \cdots \lambda_d(x) = 1 \qquad \text{for all } x \in \Omega_A. \tag{4.4}$$

Note it follows $\lambda_j(x) > 0$ for all $x \in \Omega_A$, $j = 1, \dots, d$.

Our construction involves an expanded, subgradient extension of the displacement interpolating map T_t. Namely, given $\varepsilon > 0$ and $t \in (0, 1)$, we define

$$\psi_t^\varepsilon(x) = \frac{1}{2}(1 - t)|x|^2 + t(1 + \varepsilon)\psi(x). \tag{4.5}$$

The subgradient of this function is (see Prop. A.1.ii),

$$\partial \psi_t^\varepsilon(x) = (1 - t)x + t(1 + \varepsilon)\partial \psi(x). \tag{4.6}$$

In case $\varepsilon = 0$, this map extends T_t in the sense that $\partial \psi_t^0(x) = \{T_t(x)\}$ for all $x \in \Omega_A$. Further, the range of this subgradient is all of $\mathbb{R}^d$ (by Prop. A.1.iii). Just as in (2.2), due to the monotonicity of the subgradient (Prop. A.1.i) one has

$$|z - \hat{z}| \ge (1 - t)|x - \hat{x}| \quad \text{whenever} \ z \in \partial \psi_t^\varepsilon(x), \ \hat{z} \in \partial \psi_t^\varepsilon(\hat{x}). \tag{4.7}$$

By consequence, the inverse $L_t^\varepsilon := (\partial \psi_t^\varepsilon)^{-1}$ is a single-value Lipschitz map with Lipschitz constant bounded by $(1-t)^{-1}$. Note that for all $z \in \mathbb{R}^d$,

$$z \in \partial \psi_t^\varepsilon(x) \quad \text{if and only if} \quad L_t^\varepsilon(z) = x. \tag{4.8}$$

Lemma 4.6 *Let $\varepsilon > 0$ and let $x_0 \in \Omega_A$. Choose $r_0 > 0$ such that*

$$(1+\varepsilon)\omega(x_0, r_0) < \frac{1}{2}\varepsilon \underline{\lambda}(x_0)r_0, \tag{4.9}$$

where $\underline{\lambda}(x_0) \in (0, 1)$ is the smallest eigenvalue of $H = \text{Hess } \psi(x_0)$. Then, with

$$x_t^\varepsilon = \nabla \psi_t^\varepsilon(x_0) = (1-t)x_0 + t(1+\varepsilon)T(x_0), \qquad H_t = (1-t)I + tH,$$

the ellipsoid

$$E_t^\varepsilon(x_0, r) := x_t^\varepsilon + H_t B(0, r) \subset \partial \psi_t^\varepsilon(x_0 + B(0, r)), \tag{4.10}$$

whenever $0 < r < r_0$ and $0 < t < 1$.

We note that the fact that the term tH in H_t does not contain a factor $1+\varepsilon$ is needed to guarantee the inclusion in (4.10).

Proof Note that $L_t^\varepsilon(x_t^\varepsilon) = x_0$, so by (4.8), the inclusion in (4.10) is equivalent to the statement

$$|L_t^\varepsilon(x_t^\varepsilon + H_t x) - L_t^\varepsilon(x_t^\varepsilon)| < r \quad \text{whenever } |x| < r. \tag{4.11}$$

The proof that this holds whenever $0 < r \le r_0$ and $t \in (0, 1)$ shall be based upon the local expansion of $\partial \psi$ in (4.3). We begin with the following *conditional* estimate of the quantity

$$f_t^\varepsilon(x) := L_t^\varepsilon(x_t^\varepsilon + H_t x) - L_t^\varepsilon(x_t^\varepsilon).$$

Sublemma 4.7 *For each $t \in (0, 1)$ there exists $\theta_t < 1$ such that if $|x| \le r_0$ and we further assume $|f_t^\varepsilon(x)| \le r_0$, then $|f_t^\varepsilon(x)| \le \theta_t|x|$.*

Proof Under the stated assumption, we have $x_t^\varepsilon + H_t x \in \partial \psi_t^\varepsilon(x_0 + y)$ where $y = f_t^\varepsilon(x)$. Due to (4.6) and (4.3), there exists $w \in B(0, \omega(x_0, |y|))$ such that

$$\begin{aligned}
x_t^\varepsilon + H_t x &= (1-t)(x_0 + y) + t(1+\varepsilon)(T(x_0) + Hy + w) \\
&= x_t^\varepsilon + (H_t + \varepsilon t H)y + t(1+\varepsilon)w,
\end{aligned}$$

whence

$$y = (H_t + \varepsilon t H)^{-1}(H_t x - t(1+\varepsilon)w).$$

By diagonalizing H and noting $\lambda_t = 1 - t + t\underline{\lambda}$ is the smallest eigenvalue of H_t, one finds

$$|(H_t + \varepsilon t H)^{-1} H_t x| \le \frac{\lambda_t}{\lambda_t + \varepsilon t \underline{\lambda}}|x|, \qquad |(H_t + \varepsilon t H)^{-1}w| \le \frac{\omega(x_0, |y|)}{\lambda_t + \varepsilon t \underline{\lambda}}.$$

Since we assume $|y| \le r_0$ and this entails $(1+\varepsilon)\omega(x_0, |y|) \le \frac{1}{2}\varepsilon \underline{\lambda}|y|$, the result $|y| \le \theta_t|x|$ follows by taking

$$\theta_t = \frac{\lambda_t}{\lambda_t + \frac{1}{2}\varepsilon t \underline{\lambda}} < 1.$$

$\square$

Now we finish the proof of Lemma 4.6, establishing (4.11) by continuation. Fix $t \in (0, 1)$ and let

$$r_t = \sup \left\{ r \in [0, r_0] \: : \: |f_t^\varepsilon(x)| \leq \theta_t |x| \quad \text{whenever} \quad |x| \leq r \right\} \tag{4.12}$$

(without the extra assumption made in the sublemma). The set in (4.12) is closed and $r_t > 0$, because f_t^ε is continuous and $f_t^\varepsilon(0) = 0$. Note that $|x| \leq r_t$ implies $|f_t^\varepsilon(x)| \leq \theta_t r_t < r_t$. Now it follows $r_t = r_0$, because if $r_t < r_0$, then it follows from continuity that for some $r \in (r_t, r_0]$, $|x| \leq r$ implies $|f_t^\varepsilon(x)| \leq r_t < r_0$, whence $|f_t^\varepsilon(x)| \leq \theta_t |x|$ by the sublemma, contradicting the definition of r_t.   □

4.1.2 Proof of Proposition 4.5

We suppose $0 < \varepsilon < 1$. The first step in the proof is to produce a suitable Vitali covering of Ω_0, up to a null set, by a countable disjoint union of balls. By translating the target Ω_1 so that it contains the origin, we may assume

$$\operatorname*{ess\,sup}_{x \in \Omega_0} |T(x)| = \sup_{y \in \Omega_1} |y| < \operatorname{diam} \Omega_1. \tag{4.13}$$

We may choose $\bar{r}(x, \varepsilon) > 0$ for each $x \in \Omega_A$ and $\varepsilon > 0$ such that whenever $0 < r < \bar{r}(x, \varepsilon)$ we have (see (4.9))

$$(1 + \varepsilon)\omega(x, r) < \frac{1}{2}\varepsilon\underline{\lambda}(x)r, \qquad \frac{\overline{\lambda}(x)}{\underline{\lambda}(x)}r < \varepsilon \operatorname{diam} \Omega_1, \tag{4.14}$$

where $\overline{\lambda}(x)$ is the largest eigenvalue of $\operatorname{Hess}(x)$ and $\underline{\lambda}(x)$ is the smallest. (The second condition on r will be used in the next subsection.) Then $|\Omega_0 \setminus \Omega_A| = 0$, and the family of balls

$$\{B(x, r) : x \in \Omega_A, \: 0 < r < \bar{r}(x, \varepsilon)\}$$

forms a Vitali cover of Ω_A. Therefore, by Vitali's covering theorem [20, Theorem III.12.3], there is a countable family of mutually disjoint balls $B(x_i, r_i)$, with $x_i \in \Omega_A$ and $0 < r_i < \bar{r}(x_i, \varepsilon)$, such that

$$|\Omega_A \setminus \cup_{i \in \mathbb{N}} B(x_i, r_i)| = 0.$$

We let

$$\Omega_0^\varepsilon = \bigsqcup_{i \in \mathbb{N}} B_i, \qquad B_i = B(x_i, r_i). \tag{4.15}$$

Define the map S^ε by (4.1). To show S^ε is an ellipsoidal transport spray on Ω_0^ε, we first prove that the linear interpolants defined by

$$S_t^\varepsilon(z) = (1 - t)z + t S^\varepsilon(z), \qquad z \in \Omega_0^\varepsilon,$$

remain injections for each $t \in [0, 1)$. Clearly the restriction to each B_i is an injection. But by invoking Lemma 4.6 with $H = \operatorname{Hess} \psi(x_i)$, we conclude that the image of B_i under S_t^ε satisfies

$$S_t^\varepsilon(B_i) = E_t^\varepsilon(x_i, r_i) \subset \partial \psi_t^\varepsilon(B_i). \tag{4.16}$$

Recalling that the inverse $(\partial \psi_t^\varepsilon)^{-1}$ is a single-value Lipschitz map by (4.7), this implies that the images $S_t^\varepsilon(B_i)$ are pairwise disjoint.

Now, for $t = 1$ we necessarily have S^ε is injective, for if not then for some $i \neq j$, the open set $S_t^\varepsilon(B_i) \cap S_t^\varepsilon(B_j)$ is nonempty for $t = 1$ and hence for t near 1, contradiction. This proves that S^ε is an ellipsoidal transport spray on the set Ω_0^ε in (4.15), so that property (i) holds.

Next we prove property (ii). Using (4.14), for each $x \in B_i$ we have, since $T(x) \in \partial \psi(x)$,

$$|T(x) - S^\varepsilon(x)| \leq |T(x) - T(x_i) - DT(x_i)(x - x_i)| + \varepsilon |T(x_i)|$$
$$\leq \omega(x_i, r_i) + \varepsilon |T(x_i)|$$
$$\leq \frac{1}{2}\varepsilon\underline{\lambda}(x_i)r_i + \varepsilon \, \mathrm{diam}\,\Omega_1 \leq \frac{3}{2}\varepsilon \, \mathrm{diam}\,\Omega_1. \tag{4.17}$$

This shows (ii) after replacing ε by $\varepsilon/2$. For part (iii), we note that the set $\Omega_0^\varepsilon = (S^\varepsilon)^{-1}(\Omega_1^\varepsilon)$ has full measure in Ω_0, and the map T pushes forward Lebesgue measure on Ω_0 to Lebesgue measure on Ω_1. It follows that the map $T \circ (S^\varepsilon)^{-1}: \Omega_1^\varepsilon \to \Omega_1$ pushes forward uniform measure to uniform measure and satisfies

$$\sup_{x \in \Omega_1^\varepsilon} |T \circ (S^\varepsilon)^{-1}(x) - x| < \varepsilon \, \mathrm{diam}\,\Omega_1.$$

The result claimed in part (iii) follows, due to (2.7). This finishes the proof of Proposition 4.5.

4.2 Action estimate for Euler spray

Each of the ellipsoidal Wasserstein droplets that make up the ellipsoidal transport spray S^ε is associated with a boosted ellipsoidal Euler droplet nested inside, due to the nesting property in Proposition 3.8. The disjoint superposition of these Euler droplets make up an Euler spray that deforms Ω_0^ε to the same set Ω_1^ε.

In order to complete the proof of Theorem 1.1, it remains to bound the action of this Euler spray in terms of the Wasserstein distance between the uniform measures on Ω_0 and Ω_1. Toward this goal, we first note that because the maps T and S^ε are volume-preserving, due to the estimate in part (ii) of Proposition 4.5 and (2.7) we have

$$d_W(T(B_i), S^\varepsilon(B_i))^2 \leq (\varepsilon K_1)^2 |B_i|, \qquad K_1 = \mathrm{diam}\,\Omega_1.$$

Now by the triangle inequality,

$$d_W(B_i, S^\varepsilon(B_i))^2 \leq \left(d_W(B_i, T(B_i)) + \varepsilon K_1 |B_i|^{1/2}\right)^2$$
$$\leq d_W(B_i, T(B_i))^2(1 + \varepsilon) + (\varepsilon + \varepsilon^2)K_1^2|B_i| \tag{4.18}$$

Recall that by inequality (3.36) of Lemma 3.10, the action of the i-th ellipsoidal Euler droplet, denoted by $\mathcal{A}_i$, satisfies

$$\mathcal{A}_i \leq d_W(B_i, S^\varepsilon(B_i))^2 + \frac{\overline{\lambda}(x_i)^4}{\underline{\lambda}(x_i)^2}r_i^2|B_i|$$
$$\leq d_W(B_i, T(B_i))^2(1 + \varepsilon) + 3\varepsilon K_1^2|B_i|, \tag{4.19}$$

where we make use of the second constraint in (4.14).

By summing over all i, we obtain the required bound,

$$\mathcal{A}^\varepsilon = \sum_i \mathcal{A}_i \leq d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + K\varepsilon$$

where

$$K = d_W(\mathbb{1}_{\Omega_0}, \mathbb{1}_{\Omega_1})^2 + 4|\Omega_0|(\operatorname{diam} \Omega_1)^2.$$

This concludes the proof of Theorem 1.1.

5 Shape distance equals Wasserstein distance

Our main goal in this section is to prove Theorem 1.3, which establishes the existence of paths of shape densities (as countable concatenations of Euler sprays) that exactly connect any two compactly supported measures having densities with values in [0, 1] and have action as close as desired to the Wasserstein distance squared between the measures. Theorem 1.2 follows as an immediate corollary, showing that shape distance between arbitrary bounded measurable sets with positive, equal volume is the Wasserstein distance between the corresponding characteristic functions.

   Theorem 1.3 will be deduced from Theorem 1.1 by essentially 'soft' arguments. Theorem 1.1 shows that the relaxation of shape distance, in the sense of lower-semicontinuous envelope with respect to the topology of weak-⋆ convergence of characteristic functions, is Wasserstein distance. Essentially, here we use this result to compute the completion of the shape distance in the space of bounded measurable sets.

Lemma 5.1 *Let $\rho\colon \mathbb{R}^d \to [0, 1]$ be a measurable function of compact support. Then for any $\varepsilon > 0$ there is an open set Ω such that its volume is the total mass of ρ and the L^∞ transport distance from ρ to its characteristic function is less than ε:*

$$|\Omega| = \int_{\mathbb{R}^d} \rho \, dx \quad and \quad d_\infty(\rho, \mathbb{1}_\Omega) < \varepsilon.$$

Proof We recall that weak-⋆ convergence of probability measures supported in a fixed compact set is equivalent to convergence in (either L^2 or L^∞) Wasserstein distance. Given $k \in \mathbb{N}$, cover the support of ρ a.e. by a grid of disjoint open rectangles of diameter less than $\varepsilon_k = 1/k$. For each rectangle R in the grid, shrink the rectangle homothetically from any point inside to obtain a sub-rectangle $\hat{R} \subset R$ such that $|\hat{R}| = \int_R \rho \, dx$. Let Ω_k be the disjoint union of the non-empty rectangles $\hat{R}$ so obtained. Then the sequence of characteristic functions $\mathbb{1}_{\Omega_k}$ evidently converges weak-⋆ to ρ in the space of fixed-mass measures on a fixed compact set: for any continuous test function f on $\mathbb{R}^d$, as $k \to \infty$ we have

$$\int_{\Omega_k} f(x) \, dx \to \int_{\mathbb{R}^d} f(x)\rho(x) \, dx.$$

Choosing $\Omega = \Omega_k$ for some sufficiently large k yields the desired result.   □

Proof of Theorem 1.3 part (a) Let ρ_0, ρ_1 have the properties stated, and suppose $D := d_W(\rho_0, \rho_1) > 0$. (The other case is trivial.) Let $\varepsilon > 0$. By Lemma 5.1 we may choose open sets Ω_0 and $\hat{\Omega}_1$ whose volume is $\int_{\mathbb{R}^d} \rho_0$ and such that

$$d_\infty(\rho_0, \mathbb{1}_{\Omega_0}) + d_\infty(\rho_1, \mathbb{1}_{\hat{\Omega}_1}) < \frac{\varepsilon}{2}, \qquad d_W(\Omega_0, \hat{\Omega}_1)^2 \le d_W(\rho_0, \rho_1)^2 + \frac{\varepsilon}{2}. \qquad (5.1)$$

Then we can apply Theorem 1.1 to find an Euler spray that connects Ω_0 to a set $\hat{\Omega}_1^\varepsilon =: \Omega_1$ close to $\hat{\Omega}_1$ with the properties

$$d_\infty(\Omega_1, \hat{\Omega}_1) < \frac{\varepsilon}{3}, \qquad \mathcal{A}^\varepsilon \le d_W(\Omega_0, \hat{\Omega}_1)^2 + \frac{\varepsilon}{3}, \qquad (5.2)$$

where $\mathcal{A}^\varepsilon$ is the action of this Euler spray. By combining the inequalities in (5.1) and (5.2) we find that the sets Ω_0, Ω_1 have the properties required. $\square$

Before we establish part (b), we separately discuss the concatenation of transport paths. Let $\rho^k = (\rho_t^k)_{t \in [0,1]}$ be a path of shape densities for each $k = 1, 2, \ldots, K$, with associated transport velocity field $v^k \in L^2(\rho^k \, dx \, dt)$ and action

$$\mathcal{A}_k = \int_0^1 \int_{\mathbb{R}^d} \rho_t^k(x) |v^k(x, t)|^2 \, dx \, dt.$$

We say this set of paths forms a *chain* if $\rho_1^k = \rho_0^{k+1}$ for $k = 1, \ldots, K - 1$. Given such a chain, and numbers $\tau_k > 0$ such that $\sum_{k=1}^K \tau_k = 1$, we define the *concatenation of the chain of paths* ρ^k compressed by τ_k to be the path $\rho = (\rho_t)_{t \in [0,1]}$ given by

$$\rho_t = \rho_s^k \quad \text{for} \quad t = \tau_k s + \sum_{j < k} \tau_k, \quad s \in [0, 1]. \tag{5.3}$$

The transport velocity associated with the concatenation is

$$v(\cdot, t) = \tau_k^{-1} v^k(\cdot, s) \quad \text{for} \quad t = \tau_k s + \sum_{j < k} \tau_k, \quad s \in [0, 1], \tag{5.4}$$

and the action is

$$\mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho_t |v|^2 \, dx \, dt = \sum_{k=1}^K \tau_k^{-1} \int_0^1 \int_{\mathbb{R}^d} \rho_s^k(x) |v(x, s)|^2 \, dx \, ds = \sum_{k=1}^K \tau_k^{-1} \mathcal{A}_k. \tag{5.5}$$

Remark 5.2 We mention here how the triangle inequality for the shape distance defined in (1.5) follows directly from this concatenation procedure. Given the chain ρ^k as above with actions $\mathcal{A}_k$, let $\delta_k = \sqrt{\mathcal{A}_k}$ and set

$$\tau_k = \frac{\delta_k}{\sum_j \delta_j}, \quad k = 1, \ldots, K.$$

Let $\mathcal{A}$ be the action of the concatenation of paths ρ^k compressed by τ_k, and let $\delta = \sqrt{\mathcal{A}}$. Then

$$\mathcal{A} = \delta^2 = \sum_k \tau_k^{-1} \delta_k^2 = \left(\sum_k \delta_k \right)^2.$$

From this the triangle inequality follows.

Proof of Theorem 1.3 part (b) Next we establish part (b). The idea is to construct a path of shape densities $\rho = (\rho_t)_{t \in [0,1]}$ connecting ρ_0 to ρ_1 by concatenating the Euler spray from part (a) together with two paths of small action that themselves are concatenated chains of Euler sprays that respectively connect Ω_0 to sets that approximate ρ_0, and connect Ω_1 to sets that approximate ρ_1.

Let $\varepsilon > 0$, and let ρ^ε be a shape density determined by an Euler spray as from part (a) that connects bounded open sets Ω_0 and Ω_1 of volume $\int_{\mathbb{R}^d} \rho_0$, but with the (perhaps tighter) conditions

$$dw(\mathbb{1}_{\Omega_0}, \rho_0) + dw(\mathbb{1}_{\Omega_1}, \rho_1) < \frac{1}{4}\varepsilon 2^{-1}, \qquad \mathcal{A}^\varepsilon < dw(\rho_0, \rho_1)^2 + \varepsilon,$$

where $\mathcal{A}^\varepsilon$ is the action of this spray.

Next we construct a chain of Euler sprays with shape densities ρ^k, $k = 1, 2, \ldots$, with action $\mathcal{A}_k$ that connect Ω_1 with a chain of sets Ω_k such that $\mathbb{1}_{\Omega_k} \xrightarrow{\star} \rho_1$ as $k \to \infty$ and

$$dw(\mathbb{1}_{\Omega_k}, \rho_1) < \frac{1}{4}\varepsilon 2^{-k}, \qquad \mathcal{A}_k < (\varepsilon 2^{-k})^2. \tag{5.6}$$

We proceed by recursion by applying Theorem 1.1 like in the proof of part (a). Given $k \geq 1$, suppose Ω_k is defined and ρ^j are defined for $j < k$. Using Lemma 5.1 we can find a bounded open set $\hat{\Omega}_{k+1}$ such that

$$|\hat{\Omega}_{k+1}| = \int_{\mathbb{R}^d} \rho_0 \quad \text{and} \quad dw(\mathbb{1}_{\hat{\Omega}_{k+1}}, \rho_1) < \frac{1}{8}\varepsilon 2^{-k-1}.$$

Then by invoking Theorem 1.1 and the triangle inequality for dw, we obtain an Euler spray with action $\mathcal{A}_k$ that connects Ω_k to a bounded open set Ω_{k+1}, such that

$$dw(\Omega_{k+1}, \hat{\Omega}_{k+1}) < \frac{1}{8}\varepsilon 2^{-k-1} \quad \text{and} \quad \mathcal{A}_k < dw(\Omega_k, \hat{\Omega}_{k+1})^2 + \frac{1}{2}(\varepsilon 2^{-k})^2 < (\varepsilon 2^{-k})^2.$$

We let $\rho^k = (\rho_t^k)_{t \in [0,1]}$ be the path of shape densities for this spray, so that $\rho_0^k = \mathbb{1}_{\Omega_k}$ and $\rho_1^k = \mathbb{1}_{\Omega_{k+1}}$. This completes the construction of the chain of paths ρ^k satisfying (5.6).

It is straightforward to see that $dw(\rho_t^k, \rho_1) \to 0$ as $k \to \infty$ uniformly for $t \in [0, 1]$. Now we let $\rho^+ = (\rho_t^+)_{t \in [0,1]}$ be the countable concatenation of this chain of paths ρ^k compressed by $\tau_k = 2^{-k}$ according to the formulas (5.3)–(5.5) above taken with $K \to \infty$, and with $\rho_1^+ = \rho_1$. The action $\mathcal{A}^+$ of this concatenation then satisfies

$$\mathcal{A}^+ = \sum_{k=1}^{\infty} 2^k \mathcal{A}_k < \varepsilon^2. \tag{5.7}$$

In exactly analogous fashion we can construct a countable concatenation $\hat{\rho}^-$ of a chain of paths coming from Euler sprays, that connects $\hat{\rho}_0^- = \mathbb{1}_{\Omega_0}$ with $\hat{\rho}_1^- = \rho_0$ and having action $\mathcal{A}^- < \varepsilon^2$. Then define ρ^- be the *reversal* of $\hat{\rho}^-$, given by

$$\rho_t^- = \hat{\rho}_{1-t}^-.$$

This path ρ^- has the same action $\mathcal{A}^-$.

Finally, define the path ρ by concatenating ρ^-, ρ^ε, ρ^+ compressed by ε, $1 - 2\varepsilon$ and ε respectively. This path satisfies the desired endpoint conditions and has action $\mathcal{A}$ that satisfies

$$\mathcal{A} = \varepsilon^{-1}\mathcal{A}^- + (1 - 2\varepsilon)^{-1}\mathcal{A}^\varepsilon + \varepsilon^{-1}\mathcal{A}^+ < dw(\rho_0, \rho_1)^2 + K\varepsilon,$$

for some constant K independent of ε small. The result of part (b) follows. $\square$

Remark 5.3 Our construction here of a sequence of action-infimizing paths involves connecting geodesics given by Euler sprays only for simplicity. A more general approach to constructing non-geodesic near-optimal incompressible paths can be taken that exploits the convexity of density along Wasserstein transport paths. Such an approach was implemented in an earlier preprint version of this article [38].

5.1 Rigidity of minimizing incompressible paths

The result of Theorem 1.4, providing a sharp criterion for the existence of a minimizer for the shape distance in (1.5), follows by combining the uniqueness property of Wasserstein geodesics with the result of Theorem 1.2.

Proof of Theorem 1.4 Let $\rho = (\rho_t)_{t \in [0,1]}$ be the density along the Wasserstein geodesic path that connects $\mathbb{1}_{\Omega_0}$ and $\mathbb{1}_{\Omega_1}$, where Ω_0, Ω_1 are bounded open sets in $\mathbb{R}^d$ with equal volume. Clearly, if ρ is a characteristic function, then the Wasserstein geodesic provides a minimizing path for (1.5). On the other hand, if a minimizer for (1.5) exists, it must have constant speed by a standard reparametrization argument. Then by Theorem 1.2 it provides a constant-speed minimizing path for Wasserstein distance as well, hence corresponds to the unique Wasserstein geodesic. Thus the Wasserstein geodesic density ρ is a characteristic function. □

A consequence of Theorem 1.2 is that existence of a minimizer among incompressible transport paths in (1.5) imposes a rigid structure on the optimal transport map T. We describe this below in Corollary 5.8. We begin by examining the density along the Wasserstein geodesic path that connects $\mathbb{1}_{\Omega_0}$ and $\mathbb{1}_{\Omega_1}$. This density is the pushforward of $\rho_0 = \mathbb{1}_{\Omega_0}$ under the transport map T_t from (2.1). Invoking a result of McCann [40, Proposition 4.2], we know

$$\rho(T_t(z), t)^{-1} = \det \frac{\partial T_t}{\partial z} = \det((1-t)I + t \, \text{Hess} \, \psi(z)) = \prod_{j=1}^{d} (1 - t + t\lambda_j(z)), \quad (5.8)$$

for each z in the set Ω_A of full measure in Ω_0, which appears in (4.4) and consists of the Alexandrov points z of ψ at which $\det \text{Hess} \, \psi(z) = 1$.

Lemma 5.4 *Along the particle paths $t \mapsto T_t(z)$ of displacement interpolation between the measures with respective densities $\mathbb{1}_{\Omega_0}$ and $\mathbb{1}_{\Omega_1}$ as above, the density is log-convex, that is $t \mapsto \log \rho(T_t(z), t)$ is convex, for each $z \in \Omega_A$. Moreover, this function is constant if and only if $\text{Hess} \, \psi(z) = I$.*

Proof We compute

$$\frac{d^2}{dt^2} \log \rho = -\frac{d}{dt} \sum_{j=1}^{d} \frac{\lambda_j - 1}{1 - t + t\lambda_j} = \sum_{j=1}^{d} \left(\frac{\lambda_j - 1}{1 - t + t\lambda_j} \right)^2 \geq 0, \quad (5.9)$$

and this vanishes if and only if $\lambda_j = 1$ for all j. □

Remark 5.5 A related fact stated in [40, Lemma 2.1] implies the (stronger) property that $\rho^{-1/d}$ is concave along particle paths and is connected to a well-known proof of the Brunn-Minkowski inequality by Hadwiger and Ohmann. The proof of Lemma 5.4 can be easily modified to show ρ^{-q} is concave along particle paths for any $q \in (0, 1/d]$. Indeed, due to the Cauchy–Schwartz (or Jensen's) inequality, for $g(t) = \rho(T_t(z), t)^{-1/d}$, one has

$$\frac{g''}{g} = \left(\frac{1}{d} \sum_{j=1}^{d} \frac{\lambda_j - 1}{1 - t + t\lambda_j} \right)^2 - \frac{1}{d} \sum_{j=1}^{d} \left(\frac{\lambda_j - 1}{1 - t + t\lambda_j} \right)^2 \leq 0. \quad (5.10)$$

Remark 5.6 We note that Lemma 5.4 implies that $\rho_t \leq 1$ a.e., for all $t \in [0, 1]$, with a simple proof relying only on basic tools of optimal transport.

If a minimizer for the shape distance in (1.5) exists, the result of Lemma 5.4 shows that Hess $\psi(z) = 1$ for every $z \in \Omega_A$ and hence

$$\nabla T_t(z) = 1 \quad \text{for every } z \in \Omega_A \text{ and } t \in [0, 1]. \tag{5.11}$$

To discuss further the consequences of these tight restrictions, it is convenient to invoke the regularity theory of Caffarelli [13], Figalli [24] and Figalli and Kim [25]. These authors have shown (see Theorem 3.4 in [17] and also [18]) that, due to the fact that the characteristic functions are smooth inside Ω_0 and Ω_1, the optimal transportation potential ψ is smooth away from a set of measure zero. More precisely, there exist relatively closed sets of measure zero, $\Sigma_i \subset \Omega_i$ for $i = 0, 1$ such that $T : \Omega_0 \backslash \Sigma_0 \to \Omega_1 \backslash \Sigma_1$ is a smooth diffeomorphism between two open sets.

Remark 5.7 In a previous draft of this paper, this regularity theory was used to prove Theorem 1.1 through a Vitali covering argument. The present approach to the proof in Sect. 4 exploits the simpler property that the subgradient maps $\partial\psi_t$ have single-valued inverses.

As a consequence of this regularity theory, the rigidity of gradients in (5.11) implies the following rigiditiy for transport maps.

Corollary 5.8 *The Wasserstein geodesic density ρ is a characteristic function if and only if the displacement interpolant is piecewise a rigid translation:*

$$T_t(z) = z + tb(z),$$

where $b(\cdot)$ is constant on each component of the open set $\Omega_0 \setminus \Sigma_0$.

In case the result of this Corollary applies, the target $\Omega_1 = T(\Omega_0)$ represents some kind of decomposition of the source Ω_0 by fracturing into pieces that can separate without overlapping.

Remark 5.9 In the case of one dimension ($d = 1$) it is always the case that the Wasserstein geodesic density $\rho(T_t(z), t) \equiv 1$ for all z in the non-singular set. This is so because the diffeomorphism $T : \Omega_0 \backslash \Sigma_0 \to \Omega_1 \backslash \Sigma_1$ must always be a rigid translation on each component, as it pushes forward Lebesgue measure to Lebesgue measure.

As a nontrivial example, let $\mathcal{C} \subset [0, 1]$ be the standard Cantor set, and let $\Omega_0 = (0, 1)$. Define the Brenier map $T(z) = z + c(z)$ with c given by the Cantor function, increasing and continuous on $[0, 1]$ with $c(0) = 0$, $c(1) = 1$ and $c' = 0$ on $(0, 1) \setminus \mathcal{C}$. Then $T(\Omega_0) = (0, 2)$, but the pushforward of uniform measure on Ω_0 is the uniform measure on the set $\Omega_1 = T(\Omega_0 \setminus \mathcal{C})$, which has countably many components, and total length $|\Omega_1| = 1$.

6 Displacement interpolants as weak limits

This section focuses on the proofs of Theorems 1.5 and 1.6 on the convergence of Euler sprays towards Wasserstein geodesics. These results establish convergence of mass densities, momenta, and momentum flux tensors, first in the standard weak-$\star$ sense in L^∞, then in the more precise TL^p sense. While both theorems deal with general densities, in each case we first treat the case that ρ_0 is a characteristic function (Propositions 6.1 and 6.5). To extend to the general case we use results on stability of Wesserstein geodesics (Propositions 6.3 and 6.6).

We conclude this section in Sect. 6.3 with a result on convergence of Lagrangian flow maps, in the TL^2 sense.

6.1 Proof of Theorem 1.5

We will proceed in two steps, first dealing with the case that the endpoint densities ρ_0, ρ_1 are characteristic functions of bounded open sets. To extend this result to the general case of bounded densities, we will make use of fundamental results on stability of optimal transport plans from [2] and [53].

Proposition 6.1 *Let Ω_0, Ω_1 be bounded open sets of equal volume. Let (ρ, v) be the density and transport velocity determined by the unique Wasserstein geodesic (displacement interpolant) that connects the uniform measures on Ω_0 and Ω_1 as described in Sect. 2.*

Then, as $\varepsilon \to 0$, the weak solutions $(\rho^\varepsilon, v^\varepsilon, p^\varepsilon)$ associated to the Euler sprays of Theorem 1.1 by Proposition 4.2 converge to $(\rho, v, 0)$, and (ρ, v) is a weak solution to the pressureless Euler system (1.13)–(1.14). The convergence holds in the following sense: $p^\varepsilon \to 0$ uniformly, and

$$\rho^\varepsilon \overset{\star}{\rightharpoonup} \rho, \qquad \rho^\varepsilon v^\varepsilon \overset{\star}{\rightharpoonup} \rho v, \qquad \rho^\varepsilon v^\varepsilon \otimes v^\varepsilon \overset{\star}{\rightharpoonup} \rho v \otimes v, \tag{6.1}$$

weak-$\star$ in L^∞ on $\mathbb{R}^d \times [0, 1]$.

Toward proving this result, we describe the bounds on pressure and velocity that come from the construction of the Euler sprays constructed above, for any given $\varepsilon \in (0, 1)$.

Lemma 6.2 *Let $(Q^\varepsilon, \phi^\varepsilon, p^\varepsilon)$, $0 < \varepsilon < 1$, denote the Euler sprays constructed in the proof of Theorem 1.1, and let $X^\varepsilon : \Omega_0^\varepsilon \times [0, 1] \to \mathbb{R}^d$ denote the associated flow maps, which satisfy*

$$\dot{X}^\varepsilon(z, t) = \nabla \phi^\varepsilon(X^\varepsilon(z, t), t), \quad (z, t) \in \Omega_0^\varepsilon \times [0, 1],$$

with $X^\varepsilon(z, 0) = z$. Then for some $\hat{K} > 0$ independent of ε, we have

$$0 \le p^\varepsilon(x, t) \le \hat{K}\varepsilon \tag{6.2}$$

for all $(x, t) \in Q^\varepsilon$, and

$$|X^\varepsilon(z, t) - T_t(z)| + |\dot{X}^\varepsilon(z, t) - \dot{T}_t(z)| \le \hat{K}\varepsilon \tag{6.3}$$

for all $(z, t) \in \Omega_0^\varepsilon \times [0, 1]$, where $(z, t) \mapsto T_t(z)$ is the flow map from (2.1) for the Wasserstein geodesic.

Proof By the pressure bound for individual droplets in (3.44) together with the second condition in (4.14), we have the pointwise bound

$$0 \le p^\varepsilon \le K_0\varepsilon, \qquad K_0 = K_1^2 d, \quad K_1 = \operatorname{diam} \Omega_1. \tag{6.4}$$

Next consider the velocity. The boosted ellipsoidal Euler droplet that transports B_i to $S^\varepsilon(B_i)$ is translated by x_i, and boosted by the vector

$$b_i := (1 + \varepsilon)T(x_i) - x_i = \dot{T}_t(x_i) + \varepsilon T(x_i). \tag{6.5}$$

In this "i-th droplet," the velocity satisfies, by the estimate (3.46),

$$|\nabla \phi^\varepsilon - b_i| = |v^\varepsilon - b_i| \le K_0\varepsilon. \tag{6.6}$$

Now the particle velocity for the Euler spray compares to that of the Wasserstein geodesic according to

$$
\begin{aligned}
|\dot{X}^{\varepsilon}(z,t) - \dot{T}_t(z)| &\leq |\dot{X}^{\varepsilon} - b_i| + |b_i - \dot{T}_t(z)| \\
&\leq K_0\varepsilon + \varepsilon|T(x_i)| + |T(z) - z - (T(x_i) - x_i)| \\
&\leq K_0\varepsilon + \varepsilon|T(x_i)| + r_i \max_j |\lambda_j - 1| + \omega(x_i, r_i) \\
&\leq K_0\varepsilon + 3K_1\varepsilon.
\end{aligned}
\tag{6.7}
$$

(Here λ_j are the eigenvalues of Hess $\psi(x_i)$, and we use (4.3) with the fact that $|\lambda_j - 1|r_i \leq \bar{\lambda}(x_i)r_i < K_1\varepsilon$ and $\omega(x_i, r_i) < \varepsilon r_i$ by (4.14).) Integrating we get both bounds in (6.3). □

Proof of Proposition 6.1 Now, let (ρ, v) be the density and velocity of the particle paths for the Wasserstein geodesic, from (5.8) and (2.4). Recall that the density $\rho^{\varepsilon} = \mathbb{1}_{Q^{\varepsilon}}$ and velocity $v^{\varepsilon} = \mathbb{1}_{Q^{\varepsilon}}\nabla\phi^{\varepsilon}$ associated with the Euler sprays are uniformly bounded. To prove $\rho^{\varepsilon} \overset{\star}{\rightharpoonup} \rho$ weak-$\star$ in L^{∞}, it suffices to show that as $\varepsilon \to 0$,

$$
\int_0^1 \int_{\mathbb{R}^d} (\rho^{\varepsilon} - \rho) q \, dx \, dt \to 0,
\tag{6.8}
$$

for every smooth test function $q \in C_c^{\infty}(\mathbb{R}^d \times [0,1], \mathbb{R})$. Changing to Lagrangian variables using X^{ε} for the term with $\rho^{\varepsilon} = \mathbb{1}_{Q^{\varepsilon}}$ and T_t for the term with ρ, the left-hand side becomes

$$
\int_0^1 \int_{\Omega_0} (q(X^{\varepsilon}(z,t),t) - q(T_t(z),t)) \, dz \, dt.
\tag{6.9}
$$

Evidently this does approach zero as $\varepsilon \to 0$, due to (6.3).

Next, we claim $\rho^{\varepsilon} v^{\varepsilon} \overset{\star}{\rightharpoonup} \rho v$ weak-$\star$ in L^{∞}. Because these quantities are uniformly bounded, it suffices to show that as $\varepsilon \to 0$,

$$
\int_0^1 \int_{\mathbb{R}^d} (\rho^{\varepsilon} v^{\varepsilon} - \rho v) \cdot \tilde{v} \, dx \, dt \to 0
\tag{6.10}
$$

for each $\tilde{v} \in C_c^{\infty}(\mathbb{R}^d \times [0,1], \mathbb{R}^d)$. Changing variables in the same way, the left-hand side becomes

$$
\int_0^1 \int_{\Omega_0} \left(\dot{X}^{\varepsilon}(z,t) \cdot \tilde{v}(X^{\varepsilon}(z,t),t) - \dot{T}_t(z) \cdot \tilde{v}(T_t(z),t) \right) dz \, dt.
\tag{6.11}
$$

But because $\tilde{v}$ is smooth and due to the bounds in (6.3), this also tends to zero as $\varepsilon \to 0$.

It remains to prove $\rho^{\varepsilon} v^{\varepsilon} \otimes v^{\varepsilon} \overset{\star}{\rightharpoonup} \rho v \otimes v$ weak-$\star$ in L^{∞}. Considering the terms componentwise, the proof is extremely similar to the previous steps. This finishes the proof of Theorem 1.5. □

To generalize Proposition 6.1 to handle general densities $\rho_0, \rho_1 \colon \mathbb{R}^d \to [0,1]$, we will use a double approximation argument, comparing Euler sprays to optimal Wasserstein geodesics for open sets whose characteristic functions approximate ρ_0, ρ_1 in the sense of Lemma 5.1, then comparing these to the Wasserstein geodesic that connects ρ_0 to ρ_1. We prove weak-star convergence for the second comparison by extending the results from [2] and [53] on weak-$\star$ stability of transport plans to establish weak-$\star$ stability of Wasserstein geodesic flows (in the Eulerian framework).

Proposition 6.3 *Let (ρ, v) be the density and transport velocity determined by the Wasserstein geodesic that connects the measures with given densities $\rho_0, \rho_1 \colon \mathbb{R}^d \to [0,1]$, measurable with compact support such that*

$$\int_{\mathbb{R}^d} \rho_0 = \int_{\mathbb{R}^d} \rho_1.$$

Let $\Omega_0^k, \Omega_1^k, k = 1, 2, \ldots$, be bounded open sets such that $|\Omega_0^k| = |\Omega_1^k| = \int_{\mathbb{R}^d} \rho_0$ and

$$d_\infty(\rho_0, \mathbb{1}_{\Omega_0^k}) + d_\infty(\rho_1, \mathbb{1}_{\Omega_1^k}) \to 0 \quad as \ k \to \infty,$$

and let $(\bar{\rho}^k, \bar{v}^k)$ be the density and transport velocity determined by the Wasserstein geodesic that connects the measures with densities $\bar{\rho}_0^k = \mathbb{1}_{\Omega_0^k}$ and $\bar{\rho}_1^k = \mathbb{1}_{\Omega_1^k}$. Then

$$\bar{\rho}^k \overset{\star}{\rightharpoonup} \rho, \qquad \bar{\rho}^k \bar{v}^k \overset{\star}{\rightharpoonup} \rho v, \qquad \bar{\rho}^k \bar{v}^k \otimes \bar{v}^k \overset{\star}{\rightharpoonup} \rho v \otimes v, \tag{6.12}$$

weak-$\star$ in L^∞ on $\mathbb{R}^d \times [0, 1]$. Consequently $0 \le \rho \le 1$ a.e. in $\mathbb{R}^d \times [0, 1]$.

Proof Let π (resp. π^k) be the optimal transport plan connecting ρ_0 to ρ_1 (resp. $\mathbb{1}_{\Omega_0^k}$ to $\mathbb{1}_{\Omega_1^k}$). These plans take the form $\pi = (\mathrm{id} \times T)_\sharp \rho_0$ (resp. $\pi^k = (\mathrm{id} \times T^k)_\sharp \mathbb{1}_{\Omega_0^k}$) where T (resp. T^k) is the Brenier map. Then by [53, Theorem 5.20] or [2, Proposition 7.1.3], we know that π^k converges weak-$\star$ to π in the space of Radon measures on $\mathbb{R}^d \times \mathbb{R}^d$.

The densities $\bar{\rho}^k$ are uniformly bounded above a.e. by 1, due to Remark 5.6. Then the momenta $\bar{\rho}^k \bar{v}^k$ are uniformly bounded, since the velocities $\bar{v}^k$ are bounded (cf. (2.3)). We will prove that $\bar{\rho}^k \bar{v}^k \overset{\star}{\rightharpoonup} \rho v$; it will be clear that the remaining results in (6.12) are similar. Let $\varphi \colon \mathbb{R}^d \times [0, 1] \to \mathbb{R}^d$ be smooth with compact support. We claim that

$$\int_0^1 \int_{\mathbb{R}^d} \bar{\rho}^k \bar{v}^k \varphi(x, t) \, dx \, dt \to \int_0^1 \int_{\mathbb{R}^d} \rho v \varphi(x, t) \, dx \, dt. \tag{6.13}$$

Recall from (2.3) that the geodesic velocities $\bar{v}^k(x, t)$ satisfy

$$\bar{v}^k((1 - t)z + tT^k(z), t) = T^k(z) - z.$$

Hence the left-hand side of (6.13) can be written in the form

$$\int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} (y - z)\varphi((1 - t)z + ty, t) \, d\pi^k(z, y) \, dt = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(z, y) \, d\pi^k(z, y),$$

where

$$\psi(z, y) = \int_0^1 (y - z)\varphi((1 - t)z + ty, t) \, dt.$$

Due to the fact that $\pi^k \overset{\star}{\rightharpoonup} \pi$ and all these measures are supported in a fixed compact set, as $k \to \infty$ we obtain the limit

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(z, y) \, d\pi(z, y) = \int_0^1 \int_{\mathbb{R}^d \times \mathbb{R}^d} (y - z)\varphi((1 - t)z + ty, t) \, d\pi(z, y) \, dt$$
$$= \int_0^1 \int_{\mathbb{R}^d} (T(z) - z)\varphi(T_t(z), t) \, \rho_0(z) \, dz \, dt, \tag{6.14}$$

where $T_t(z) = (1 - t)z + tT(z)$. To conclude the proof, we need to recall how ρ and v are determined by displacement interpolation, in a precise technical sense for the present case when ρ_0 and ρ_1 lack smoothness. Indeed, from the results in Lemma 5.29 and Proposition 5.30 of [47] (also see Proposition 8.1.8 of [2]), we find that with the notation

$$x_t(z, y) = (1 - t)z + ty,$$

the measure μ_t with density ρ_t is given by the pushforward

$$\mu_t = (x_t)_\sharp \pi = (x_t)_\sharp (\mathrm{id} \times T)_\sharp \mu_0 = (T_t)_\sharp (\rho_0 \, dz), \tag{6.15}$$

and the transport velocity is given by

$$v(x, t) = (T - \mathrm{id}) \circ (T_t)^{-1}(x). \tag{6.16}$$

Thus we may use T_t to push forward the measure $\rho_0(z) \, dz = d\mu_0(z)$ in (6.14) to write, for each $t \in [0, 1]$,

$$\int_{\mathbb{R}^d} (T(z) - z)\varphi(T_t(z), t) \, \rho_0(z) \, dz = \int_{\mathbb{R}^d} v(x, t)\varphi(x, t) \, \rho_t(x) \, dx. \tag{6.17}$$

It then follows that (6.13) holds, as desired. $\qquad \square$

Remark 6.4 The validity of the continuity equation (1.13) for (ρ, v) is well known and established in several sources, e.g., see [52, Theorem 5.51] or [47, Chapter 5]. The step above going from (6.14) to (6.17) provides an answer to the related exercise 5.52 in [52]. We are not aware, however, of any source where the momentum equation (1.14) for (ρ, v) is explicitly and rigorously justified.

Proof of Theorem 1.5. Let us now finish the proof of Theorem 1.5. Any ball in $L^\infty(\mathbb{R}^d \times [0, 1])$ is metrizable, by [20, Theorem V.5.1], hence we may fix a metric d in a large enough ball, and select $\varepsilon_k > 0$ for each $k \in \mathbb{N}$ such that for the quantities $(\rho^k, v^k, p^k) := (\rho^{\varepsilon_k}, v^{\varepsilon_k}, p^{\varepsilon_k})$ coming from the Euler sprays of Proposition 6.1, the components of ρ^k, $\rho^k v^k$ and $\rho^k v^k \otimes v^k$ approximate the corresponding quantities $\bar{\rho}^k$, $\bar{\rho}^k \bar{v}^k$ and $\bar{\rho}^k \bar{v}^k \otimes \bar{v}^k$ that appear in Proposition 6.3, within distance $1/k$. That is,

$$\max \left(d(\rho^k, \bar{\rho}^k), \ d(\rho^k v_i^k, \bar{\rho}^k \bar{v}_i^k), \ d(\rho^k v_i^k v_j^k, \bar{\rho}^k \bar{v}_i^k \bar{v}_j^k) \right) < \frac{1}{k}.$$

Then the convergences asserted in (1.15) evidently hold. $\qquad \square$

6.2 Convergence in the stronger TL^p sense

Here we prove Theorem 1.6. It establishes that the convergences described in Propositions 6.1 and 6.3 and Theorem 1.5 actually hold in a stronger sense related to the TL^p metric that was introduced in [28] to measure differences between functions defined with respect to different measures. For the convenience of readers, in "Appendix 1" we recall the definition and characterization of the TL^p metric from [28], and establish a needed TL^p stability property for transport maps and Wasserstein geodesics.

Our first result here strengthens the conclusions drawn in Proposition 6.1.

Proposition 6.5 *Under the same hypotheses as Proposition 6.1 and Lemma 6.2, the map that associates $T_t(x)$ with $X_t^\varepsilon(x) = X^\varepsilon(x, t)$, defined by $Y_t^\varepsilon = X_t^\varepsilon \circ T_t^{-1}$, pushes forward ρ_t to ρ_t^ε and we have the estimate*

$$|x - Y_t^\varepsilon(x)| + |v_t(x) - v_t^\varepsilon(Y_t^\varepsilon(x))| \leq \hat{K}\varepsilon \tag{6.18}$$

for all $t \in [0, 1]$ and ρ_t-a.e. x. By consequence, for all $t \in [0, 1]$ we have

$$d_{TL^\infty}((\rho_t, v_t), (\rho_t^\varepsilon, v_t^\varepsilon)) \leq \hat{K}\varepsilon.$$

This result follows immediately from estimate (6.3) of Lemma 6.2. Expressed in terms of couplings, using the transport plan that associates $X^\varepsilon(z, t)$ with $T_t(z)$ given by the pushforward

$$\pi^\varepsilon = (X^\varepsilon(\cdot, t) \times T_t)_\sharp \rho_0,$$

the estimate (6.3) implies that for π^ε-a.e. (x, y), for all $t \in [0, 1]$ we have

$$|x - y| + |v_\varepsilon(x, t) - v(y, t)| \leq \hat{K}\varepsilon.$$

Next we improve the conclusions of Proposition 6.3, on stability of Wasserstein geodesics, by invoking the results of Corollary B.5 in the "Appendix 1".

Proposition 6.6 *Under the assumptions of Proposition 6.3, there exist transport maps $\bar{S}^k$ that push forward ρ_0 to $\bar{\rho}_0^k = \mathbb{1}_{\Omega_0^k}$, such that*

$$\| \mathrm{id} - \bar{S}^k \|_{L^\infty(\rho_0\, dx)} \to 0 \quad as\ k \to \infty, \tag{6.19}$$

and for any such sequence of transport maps, the maps given by

$$\bar{S}_t^k = T_t^k \circ \bar{S}^k \circ T_t^{-1}$$

push forward ρ_t to $\bar{\rho}_t^k$ and satisfy, as $k \to \infty$,

$$\sup_{t\in[0,1]} \int |x - \bar{S}_t^k(x)|^2 \rho_t(x)\, dx \to 0, \tag{6.20}$$

$$\sup_{t\in[0,1]} \int |v_t(x) - \bar{v}_t^k(\bar{S}_t^k(x))|^2 \rho_t(x)\, dx \to 0. \tag{6.21}$$

Proof The existence of the maps $\bar{S}^k$ follow from the fact that $d_\infty(\rho_0, \bar{\rho}_0^k) \to 0$ as $k \to \infty$, and existence of optimal transport maps for these distances, see Theorem 3.24 of [47]. The remaining statements follow from Corollary B.5 in the "Appendix". □

By combining the last two results, we can prove Theorem 1.6.

Proof *(Proof of Theorem 1.6)* Let $\bar{\rho}_0^k = \mathbb{1}_{\Omega_0^k}$ be as in Proposition 6.6, and let (ρ^k, v^k, p^k) be corresponding solutions of the Euler system (1.13)–(1.14) coming from the Euler sprays of Theorem 1.1, chosen as in the proof of Theorem 1.5. With $\bar{S}^k$ as in Proposition 6.6, let

$$S_t^k = X_t^k \circ \bar{S}^k \circ T_t^{-1},$$

Then $(S_t^k)_\sharp(\rho_t\, dx) = \rho_t^k\, dx$. To prove (1.16), by Proposition B.4 it suffices to show that

$$\sup_{t\in[0,1]} \int |x - S_t^k(x)|^2 \rho_t(x)\, dx \to 0,$$

$$\sup_{t\in[0,1]} \int |v_t(x) - v_t^k(S_t^k(x))|^2 \rho_t(x)\, dx \to 0,$$

as $k \to \infty$. Using Proposition 6.5, we can deduce that when $k \to \infty$ then

$$\sup_{t\in[0,1]} \int |S_t^k(x) - \bar{S}_t^k(x)|^2 \rho_t(x)\, dx \to 0, \tag{6.22}$$

$$\sup_{t\in[0,1]} \int |v_t^k(S_t^k(x)) - \bar{v}_t^k(\bar{S}_t^k(x))|^2 \, \rho_t(x) \, dx \to 0. \tag{6.23}$$

Combining these with the results of Proposition 6.6 finishes the proof of (1.16). Then (1.17) follows from the definition of the TL^1 metric using the Cauchy–Schwartz inequality and the boundedness of the velocities and the domain. $\qquad\square$

6.3 Convergence of Lagrangian flow maps

From weak-$\star$ convergence of momenta (or velocities), one usually cannot conclude much about the motion of particle trajectories. Here, though, we can establish the following convergence result for Lagrangian flow maps.

Corollary 6.7 *Under the assumptions of Proposition 6.3, the flow maps X_t^k for the Euler sprays of Theorem 1.1 that connect $\bar{\rho}_0^k = \mathbb{1}_{\Omega_0^k}$ and $\bar{\rho}_1^k = \mathbb{1}_{\Omega_1^k}$ satisfy*

$$\sup_{t\in[0,1]} d_{TL^2}((\rho_0, T_t), (\bar{\rho}_0^k, X_t^k)) \to 0 \quad as \ k \to \infty.$$

Proof Note $\bar{S}^k$ from Proposition 6.6 is a transport map from ρ_0 to $\bar{\rho}_0^k$. Then for all $t \in [0, 1]$,

$$d_{TL^2}^2((\rho_0, T_t), (\bar{\rho}_0^k, X_t^k)) \le \int |\bar{S}^k(z) - z|^2 \rho_0(z) \, dz + \int |X_t^k(\bar{S}^k(z)) - T_t(z)|^2 \rho_0(z) \, dz.$$

The first term converges to zero as $k \to \infty$ by (6.19), while the second term can be estimated using the change of variables $z = T_t^{-1}(x)$ as

$$\int |X_t^k(\bar{S}^k(z)) - T_t(z)|^2 \rho_0(z) \, dz$$
$$\le \int \left(|X_t^k(\bar{S}^k(z)) - T_t^k(\bar{S}^k(z))|^2 + |T_t^k(\bar{S}^k(z)) - T_t(z)|^2 \right) \rho_0(z) \, dz$$
$$= \int \left(|S_t^k(x) - \bar{S}_t^k(x)|^2 + |\bar{S}_t^k(x) - x|^2 \right) \rho_t(x) \, dx.$$

These terms converge to zero as $k \to \infty$, by (6.22) and (6.20), respectively. $\qquad\square$

7 A Schmitzer–Schnörr-type shape distance without volume constraint

Theorem 1.2 establishes that restricting the Wasserstein metric to paths of shapes of fixed volume does not provide a new notion of distance on the space of such shapes. Namely it shows that for paths in the space of shapes of fixed volume, the infimum of the length of paths between two given shapes is the Wasserstein distance.

Volume change. It is of interest to consider a more general space of shapes in order to compare shapes of different volumes. In particular, Schmitzer and Schnörr [48] considered a space that corresponds to the set of bounded, simply connected domains in $\mathbb{R}^2$ with smooth boundary and arbitrary positive area. To each such shape Ω one associates as its corresponding *shape measure* the probability measure having uniform density on Ω, denoted by

$$\mathcal{U}_\Omega = \frac{1}{|\Omega|} \mathbb{1}_\Omega. \tag{7.1}$$

We consider here this same association between sets and shape measures, but allow for more general shapes. Namely for fixed dimension d, let us consider shapes as bounded measurable subsets of $\mathbb{R}^d$ with positive volume. Let $\mathcal{C}$ be the set of all shape measures corresponding to such shapes. Thus $\mathcal{C}$ is the set of all uniform probability distributions of bounded support.

One can formally consider $\mathcal{C}$ as a submanifold of the space of probability measures of finite second moment, endowed with Wasserstein distance. Then we define a distance between shapes as we did in (1.5), requiring

$$d_{\mathcal{C}}(\Omega_0, \Omega_1)^2 = \inf \mathcal{A}, \qquad \mathcal{A} = \int_0^1 \int_{\mathbb{R}^d} \rho |v|^2 \, dx \, dt, \qquad (7.2)$$

where $\rho = (\rho_t)$ is now required to be a path of shape measures in $\mathcal{C}$, with endpoints

$$\rho_0 = \mathcal{U}_{\Omega_0}, \qquad \rho_1 = \mathcal{U}_{\Omega_1}, \qquad (7.3)$$

and transported according to the continuity equation (1.2) with a velocity field $v \in L^2(\rho \, dx \, dt)$.

Because the characteristic-function restriction (1.3) is replaced by the weaker requirement that ρ_t has a uniform density, for any two shapes of equal volume scaled to unity for convenience, it is clear that

$$d_s(\Omega_0, \Omega_1) \geq d_{\mathcal{C}}(\Omega_0, \Omega_1) \geq d_W(\Omega_0, \Omega_1). \qquad (7.4)$$

Then as a direct consequence of Theorem 1.2, we have

$$d_{\mathcal{C}}(\Omega_0, \Omega_1) = d_W(\Omega_0, \Omega_1). \qquad (7.5)$$

By a minor modification of the arguments of Sect. 5, in general we have the following.

Theorem 7.1 *Let Ω_0 and Ω_1 be any two shapes of positive volume. Then*

$$d_C(\Omega_0, \Omega_1) = d_W(\mathcal{U}_{\Omega_0}, \mathcal{U}_{\Omega_1}).$$

Proof By a simple scaling argument, we may assume $\min\{|\Omega_0|, |\Omega_1|\} \geq 1$ without loss of generality, so that both $\rho_0, \rho_1 \leq 1$. Then the concatenated Euler sprays provided by Theorem 1.3(b) supply a path of shape measures in $\mathcal{C}$ (actually shape densities), with action converging to $d_W(\mathcal{U}_{\Omega_0}, \mathcal{U}_{\Omega_1})^2$. □

Smoothness. For dimension $d = 2$, Theorem 7.1 does not apply to describe distance in the space of shapes considered by Schmitzer and Schnörr in [48], however, for as we have mentioned, they consider shapes to be bounded simply connected domains with smooth boundary.

One point of view on this issue is that it is nowadays reasonable for many purposes to consider 'pixelated' images and shapes, made up of fine-grained discrete elements, to be valid approximations to smooth ones. Thus the microdroplet constructions considered in this paper, which fit with the mathematically natural regularity conditions inherent in the definition of Wasserstein distance, need not be thought unnatural from the point of view of applications.

Nevertheless one may ask whether the infimum of path length in the space of smooth simply connected shapes is still the Wasserstein distance, as in Theorem 7.1. Our proof of Theorem 1.2 in Sect. 5 does not provide paths in this space because the union of droplets is disconnected. However, the main mechanism by which we efficiently transport mass, namely by "dividing" the domain into small pieces (droplets) which almost follow the Wasserstein

geodesics, is still available. In particular, by creating many deep creases in the domain it might be effectively 'divided' into such pieces while still remaining connected and smooth. Thus we conjecture that even in the class of smooth sets considered in [48], the geodesic distance is the Wasserstein distance between uniform distributions as in Theorem 7.1.

Geodesic equations. It is also interesting to compare our Euler droplet equations from Sect. 3.1 with the formal geodesic equations for smooth critical paths of the action $\mathcal{A}$ in the space $\mathcal{C}$ of uniform distributions. These equations correspond to equation (4.12) of Schmitzer and Schnörr in [48].

These geodesic equations may be derived in a manner almost identical to the treatment in Sect. 3.1 above. The principal difference is that due to (3.4), the divergence of the Eulerian velocity may be a nonzero function of time, constant in space:

$$\nabla \cdot v = c(t),$$

and the same is true of virtual displacements $\tilde{v}$. The variation of action now satisfies

$$\frac{\delta \mathcal{A}}{2} = \int_{\Omega_t} v \cdot \tilde{v} \, \rho \, dx \bigg|_{t=1} - \int_0^1 \int_{\Omega_t} (\partial_t v + v \cdot \nabla v) \cdot \tilde{v} \, \rho \, dx \, dt. \tag{7.6}$$

Now, the space of vector fields orthogonal to all constant-divergence fields on Ω_t is the space of gradients ∇p such that p vanishes on the boundary and has average zero in Ω_t, satisfying

$$p = 0 \quad \text{on } \partial \Omega_t, \qquad \int_{\Omega_t} p \, dx = 0. \tag{7.7}$$

Because ρ is spatially constant and $\tilde{v}$ can be (locally in time) arbitrary with spatially constant divergence, necessarily $u = -(\partial_t v + v \cdot \nabla v)$ is such a gradient. The remaining considerations in Sect. 3.1 apply almost without change, and we conclude that $v = \nabla \phi$ where

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + p = 0, \qquad \Delta \phi = c(t), \tag{7.8}$$

where $c(t)$ is spatially constant in Ω_t.

These fluid equations differ from those in Sect. 3.1 in that ϕ gains one degree of freedom (a multiple of the solution of $\Delta \phi = 1$ in Ω_t with Dirichlet boundary condition) while the pressure p loses one degree of freedom (as its integral is constrained).

They have ellipsoidal droplet solutions given by displacement interpolation of ellipsoidal Wasserstein droplets as in Sect. 3.4, because pressure vanishes and density is indeed spatially constant for these interpolants. Because they are Wasserstein geodesics, these particular solutions are also length-minimizing geodesics in the shape space $\mathcal{C}$.

We remark that unlike in the case of Euler sprays, disjoint superposition will not yield a geodesic in general. This is because the requirement of spatially uniform density leads to a global coupling between all shape components. It seems likely that length-minimizing paths in $\mathcal{C}$ will not generally exist even locally, but we have no proof at present.

Appendix A: Some basic facts about subgradients

For the convenience of readers, we include here proofs of a few facts about subgradients that we use in Sect. 4 for the proof of Theorem 1.1. The proofs are standard and simple but seem not to be easy to extract from monographs on the subject, e.g., see [4,10,32].

Proposition A.1 *Let H be a Hilbert space, and let $\varphi \colon H \to (-\infty, \infty]$ be convex, lower semi-continuous, and proper (i.e., somewhere finite). Let $S(x) = \frac{1}{2}\|x\|^2 + \varphi(x)$. Then:*

  i. *The subgradient $\partial \varphi$ is a monotone operator.*
 ii. *$\partial S(x) = x + \partial \varphi(x)$ for all $x \in H$.*
iii. *The range of ∂S is all of H. I.e., for all $y \in H$ there exists $x \in H$ and $z \in \partial \varphi(x)$ such that $y = x + z$.*

Proof i. Given any $x, \hat{x} \in H$, $z \in \partial \varphi(x)$, $\hat{z} \in \partial \varphi(\hat{x})$, by the definition of $\partial \varphi(x)$ and $\partial \varphi(\hat{x})$ respectively we have $\varphi(\hat{x}) - \varphi(x) \geq \langle z, \hat{x} - x \rangle$ and $\varphi(x) - \varphi(\hat{x}) \geq \langle \hat{z}, x - \hat{x} \rangle$, whence $0 \leq \langle z - \hat{z}, x - \hat{x} \rangle$. This proves $\partial \varphi$ is monotone.

ii. 1. Let $z \in \partial \varphi(x)$. We claim $x + z \in \partial S(x)$. Indeed, for all $h \in H$,

$$\frac{1}{2}\|x + h\|^2 + \varphi(x + h) \geq \frac{1}{2}\|x\|^2 + \varphi(x) + \langle z + x, h \rangle.$$

2. Suppose $z \notin \partial \varphi(x)$. We claim $z + x \notin \partial S(x)$. We know there exists $h \in H$ such that

$$t^{-1}(\varphi(x + th) - \varphi(x)) - \langle z, h \rangle < 0$$

for $t = 1$, hence for all $t \in (0, 1]$ by convexity. Then for sufficiently small $t > 0$ we can add $\frac{1}{2}t\|h\|^2$ to the left-hand side and conclude that for small $t > 0$,

$$\frac{1}{2}\|th\|^2 + \varphi(x + th) < \varphi(x) + \langle z, th \rangle,$$

whence $z + x \notin \partial S(x)$, since

$$\frac{1}{2}\|x + th\|^2 + \varphi(x + th) < \frac{1}{2}\|x\|^2 + \varphi(x) + \langle z + x, th \rangle.$$

iii. Let $y \in H$ and define $\hat{S}(x) = S(x) - \langle y, x \rangle = \frac{1}{2}\|x\|^2 + \varphi(x) - \langle y, x \rangle$. Due to our hypotheses, $\hat{S}$ has a minimum at some $x \in H$. This implies that for all $h \in H$,

$$\frac{1}{2}\|x + h\|^2 + \varphi(x + h) \geq \frac{1}{2}\|x\|^2 + \varphi(x) - \langle y, h \rangle,$$

which means that $y \in \partial S(x) = x + \partial \varphi(x)$.   □

Appendix B: TL^p stability of Wasserstein geodesics

Here we recall the notion of TL^p convergence as introduced in [28], which provides a more precise way to compare Wasserstein geodesics than the notion of weak convergence does alone. We establish the TL^p stability of optimal transport maps in Theorem B.1 and TL^p stability of the Wasserstein geodesics in Corollary B.2. In Proposition B.4 we recall a basic property of TL^p convergence and use it to show that the stability in Corollary B.2 holds even if the maps used to couple the relevant measures are not optimal. This technical result is needed in the proofs in Sect. 6.2.

The TL^p metric provides a natural setting for comparing optimal transport maps between different probability measures. Let $\mathcal{P}_p(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ with finite p-th moments. On the space $TL^p(\mathbb{R}^d)$, consisting of all ordered pairs (μ, g) where $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ and $g \in L^p(\mu)$, the metric is given as follows: For $1 \le p < \infty$,

$$d_{TL^p}((\mu_0, g_0), (\mu_1, g_1)) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} \left(\int |x - y|^p + |g_0(x) - g_1(y)|^p \, d\pi(x, y) \right)^{1/p},$$

$$d_{TL^\infty}((\mu_0, g_0), (\mu_1, g_1)) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} \operatorname{ess\,sup}_{\pi}(|x - y| + |g_0(x) - g_0(y)|),$$

where $\Pi(\mu_0, \mu_1)$ is the set of transportation plans (couplings) between μ_0 and μ_1.

The following result establishes a (new) TL^p stability property for optimal transport maps, as a consequence of a known general stability property for optimal plans.

Theorem B.1 (TL^p stability of transport maps) *Let $\mu, \mu_k \in \mathcal{P}_p(\mathbb{R}^d)$ be probability measures absolutely continuous with respect to Lebesgue measure, and let $\nu, \nu_k \in \mathcal{P}_p(\mathbb{R}^d)$, for each $k \in \mathbb{N}$. Assume that*

$$d_p(\mu_k, \mu) \to 0 \quad and \quad d_p(\nu_k, \nu) \to 0 \quad as \ k \to \infty.$$

Let T_k and T be the optimal transportation maps between μ_k and ν_k, and μ and ν, respectively. Then

$$(\mu_k, T_k) \xrightarrow{TL^p} (\mu, T) \ as \ k \to \infty.$$

Proof The measures $\pi_k = (\mathrm{id} \times T_k)_\sharp \mu_k$ and $\pi = (\mathrm{id} \times T)_\sharp \mu$ are the optimal transportation plans between μ_k and ν_k, and μ and ν, respectively. By stability of optimal transport plans (Proposition 7.1.3 of [2] or Theorem 5.20 in [53]) the sequence π_k is precompact with respect to weak convergence and each of its subsequential limits is an optimal transport plan between μ and ν. Since π is the unique optimal transportation plan between μ and ν the sequence π_k converges to π. Furthermore, by Theorem 5.11 of [47] or Remark 7.1.11 of [2],

$$\lim_{k\to\infty} \int |x|^p + |y|^p \, d\pi_k(x, y) = \lim_{k\to\infty} \int |x|^p \, d\mu_k + \int |y|^p \, d\nu_k$$
$$= \int |x|^p \, d\mu + \int |y|^p \, d\nu = \int |x|^p + |y|^p \, d\pi(x, y).$$

By Lemma 5.1.7 of [2], it follows the π_k have uniformly integrable p-th moments, therefore

$$d_p(\pi_k, \pi) \to 0 \quad as \ k \to \infty,$$

by Proposition 7.1.5 in [2]. Hence there exists (optimal) $\gamma_k \in \Pi(\pi, \pi_k)$ such that

$$\int |x - \tilde{x}|^p + |y - \tilde{y}|^p \, d\gamma_k(x, y, \tilde{x}, \tilde{y}) \to 0 \quad as \ k \to \infty. \tag{B.1}$$

Since π-almost everywhere $y = T(x)$ and π_k-almost everywhere $\tilde{y} = T_k(\tilde{x})$ and the support $\mathrm{supp}\,\gamma_k$ of γ_k is contained in $\mathrm{supp}\,\pi \times \mathrm{supp}\,\pi_k$, we conclude that γ_k-almost everywhere $(x, y, \tilde{x}, \tilde{y}) = (x, T(x), \tilde{x}, T_k(\tilde{x}))$. Therefore

$$\int |x - \tilde{x}|^p + |T(x) - T_k(\tilde{x})|^p \, d\gamma_k(x, y, \tilde{x}, \tilde{y}) \to 0 \quad as \ k \to \infty.$$

Finally let θ_k be the projection of γ_k to $(x, \tilde{x})$ variables. Since $\theta_k \in \Pi(\mu, \mu_k)$, by above

$$\int |x - \tilde{x}|^p + |T(x) - T_k(\tilde{x})|^p \, d\theta_k(x, \tilde{x}) \to 0 \quad as \ k \to \infty. \tag{B.2}$$

Thus $(\mu_k, T_k) \xrightarrow{TL^p} (\mu, T)$. □

We now consider the convergence of Wasserstein geodesics between measures μ_k and ν_k as in the Lemma B.1, treating only the case $p = 2$. We recall that particle paths along these geodesics are given by

$$T_{k,t}(x) = (1 - t)x + tT_k(x).$$

The displacement interpolation between μ_k and ν_k, and particle velocities (in Eulerian variables) along the geodesics, are given by (cf. (6.15)–(6.16))

$$\mu_{k,t} = T_{k,t\sharp}\mu_k, \quad v_{k,t} = (T_k - \mathrm{id}) \circ T_{k,t}^{-1}, \quad t \in [0, 1).$$

If ν_k is absolutely continuous with respect to Lebesgue measure, then $t = 1$ is allowed. We also recall that

$$\int |v_{k,t}(z)|^2 d\mu_{k,t}(z) = \int |v_{k,0}(x)|^2 d\mu_k(x) = d_2^2(\mu_k, \nu_k).$$

Furthermore it is straightforward to check that $t \mapsto (\mu_{k,t}, v_{k,t})$ is Lipschitz continuous into $TL^2(\mathbb{R}^d)$, satisfying for $0 \le s < t < 1$

$$(t - s)d_2(\mu_k, \nu_k) = d_2(\mu_{k,t}, \mu_{k,s}) \le d_{TL^2}((\mu_{k,t}, v_{k,t}), (\mu_{k,s}, v_{k,s})) \le (t - s)d_2(\mu_k, \nu_k). \tag{B.3}$$

Corollary B.2 (TL^2 stability for displacement interpolants) *Under the assumptions of Theorem B.1 for the case $p = 2$, as $k \to \infty$ we have*

$$\sup_{t \in [0,1]} d_2(\mu_{k,t}, \mu_t) \to 0 \quad and \quad \sup_{t \in [0,1)} d_{TL^2}((\mu_{k,t}, v_{k,t}), (\mu_t, v_t)) \to 0. \tag{B.4}$$

If the measures ν_k and ν are absolutely continuous with respect to Lebesgue measure then the convergence in (B.4) also holds for $t \in [0, 1]$.

Proof Let $\pi \in \Pi(\mu, \nu)$, $\pi_k \in \Pi(\mu_k, \nu_k)$, and $\gamma_k \in \Pi(\pi, \pi_k)$ be as in the proof of Theorem B.1. Similarly to θ_k, we define $\theta_{k,t} = (z_t \times z_t)_\sharp \gamma_k$ where

$$z_t(x, y) = (1 - t)x + ty \quad and \quad (z_t \times z_t)(x, y, \tilde{x}, \tilde{y}) = (z_t(x, y), z_t(\tilde{x}, \tilde{y})).$$

We note that $\theta_{k,t} \in \Pi(\mu_t, \mu_{k,t})$ and hence, for all $t \in [0, 1]$,

$$d_2(\mu_t, \mu_{k,t})^2 \le \int |z - \tilde{z}|^2 d\theta_{k,t}(z, \tilde{z})$$

$$= \int |(1 - t)(x - \tilde{x}) + t(y - \tilde{y})|^2 d\gamma_k(x, y, \tilde{x}, \tilde{y})$$

$$\le 2 \int |x - \tilde{x}|^2 + |y - \tilde{y}|^2 d\gamma_k(x, y, \tilde{x}, \tilde{y}),$$

which by (B.1) converges to 0 as $k \to \infty$.

We use the same coupling $\theta_{k,t}$ to compare the velocities. Using that γ_k-almost everywhere $(x, y, \tilde{x}, \tilde{y}) = (x, T(x), \tilde{x}, T_k(\tilde{x}))$, for any $t \in [0, 1)$ we obtain

$$\int |v_t(z) - v_{k,t}(\tilde{z})|^2 d\theta_{k,t}(z, \tilde{z})$$

$$= \int |v_t((1 - t)x + ty) - v_{k,t}((1 - t)\tilde{x} + t\tilde{y})|^2 d\gamma_k(x, y, \tilde{x}, \tilde{y})$$

$$= \int |v_t(T_t(x)) - v_{k,t}(T_{k,t}(\tilde{x}))|^2 \, d\theta_k(x, \tilde{x})$$

$$= \int |v_0(x) - v_{k,0}(\tilde{x})|^2 \, d\theta_k(x, \tilde{x})$$

$$\leq 2 \int |x - \tilde{x}|^2 + |T(x) - T_k(\tilde{x})|^2 \, d\theta_k(x, \tilde{x}),$$

which converges to 0 as $k \to \infty$, as in (B.2).   □

Remark B.3 If the target measure v_k is not absolutely continuous with respect to Lebesgue measure, then T_k may fail to be invertible on the support of v_k and $(\mu_{k,t}, v_{k,t})$ may fail to converge as $t \to 1$ to some point in $TL^2(\mathbb{R}^d)$ due to oscillations in velocity. However, if v_k and v are absolutely continuous with respect to Lebesgue measure, then the curves $t \mapsto (\mu_{k,t}, v_{k,t}), t \mapsto (\mu_t, v_t)$ extend as continuous maps into TL^2 for all $t \in [0, 1]$, and the uniform convergences in (B.4) holds on [0, 1].

A number of properties of the TL^p metric are established in Section 3 of [28] for measures supported in a fixed bounded set. One useful characterization of TL^p-convergence in this case is stated in Proposition 3.12 of [28], which implies the following.

Proposition B.4 (A characterization of TL^p convergence on bounded domains) *Let $D \subset \mathbb{R}^d$ be open and bounded, and let μ and μ_k ($k \in \mathbb{N}$) be probability measures on D, and suppose μ is absolutely continuous with respect to Lebesgue measure. Call a sequence of transport maps (S_k) that push forward μ to μ_k (satisfying $S_{k\sharp}\mu = \mu_k$) stagnating if*

$$\lim_{n \to \infty} \int_D |x - S_k(x)| \, d\mu(x) = 0. \tag{B.5}$$

Then the following are equivalent, for $1 \leq p < \infty$.

(i) $(\mu_k, f_k) \xrightarrow{TL^p} (\mu, f)$ *as $k \to \infty$.*
(ii) *μ_k converges weakly to μ and there exists a stagnating sequence (S_k) such that*

$$\int_D |f(x) - f_k(S_k(x))|^p \, d\mu(x) \to 0 \quad as \; k \to \infty. \tag{B.6}$$

(iii) *μ_k converges weakly to μ and for every stagnating sequence (S_k) the equality (B.6) holds.*

This result together with Proposition B.2 yields the following.

Corollary B.5 (A characterization of TL^p convergence for displacement interpolants) *Make the same assumptions as in Corollary B.2, and assume all measures μ_k, μ, v_k, v are absolutely continuous with respect to Lebesgue measure and have support in a bounded open set D. Then for any stagnating sequence of transport maps (S_k) that push forward μ to μ_k, with the notation*

$$S_{k,t} = T_{k,t} \circ S_k \circ T_t^{-1}$$

the sequence $(S_{k,t})$ pushes forward μ_t to $\mu_{k,t}$ and is stagnating, and as $k \to \infty$,

$$\sup_{t \in [0,1]} \int |x - S_{k,t}(x)|^2 \, d\mu_t(x) \to 0, \tag{B.7}$$

$$\sup_{t \in [0,1]} \int |v_t(x) - v_{k,t}(S_{k,t}(x))|^2 \, d\mu_t(x) \to 0, \tag{B.8}$$

Proof First we note that indeed

$$\mu_{k,t} = (T_{k,t})_\sharp \mu_k = (T_{k,t} \circ S_k)_\sharp \mu = (S_{k,t})_\sharp \mu_t.$$

Next, fix any $t \in [0, 1]$. Because $d_2(\mu_{k,t}, \mu_t) \to 0$ by (B.4) and $T_{k,t}$ is the optimal transport map pushing forward μ_k to $\mu_{k,t}$, by Theorem B.1 we have $d_2((\mu_k, T_{k,t}), (\mu, T_t)) \to 0$. Now by Proposition B.4, because $(T_t)_\sharp \mu = \mu_t$ we have

$$\int |x - S_{k,t}(x)|^2 \, d\mu_t(x) = \int |T_t(z) - T_{k,t}(S_k(z))|^2 \, d\mu(z) \to 0. \qquad (B.9)$$

We infer that $(S_{k,t})$ is stagnating and the convergence in (B.7) holds pointwise in t. But now, the middle quantity in (B.9) is a quadratic function of t, so the uniform convergence in (B.7) holds.

Next, we note that the quantity in (B.8) is actually independent of t. We have

$$\int |v_t(x) - v_{k,t}(S_{k,t}(x))|^2 \, d\mu_t(x) = \int |v_0(z) - v_{k,0}(S_k(z))|^2 \, d\mu(z) \to 0,$$

due to Proposition B.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

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