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Self-similar Spreading in a Merging-Splitting Model of Animal Group Size

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Abstract
In a recent study of certain merging-splitting models of animal-group size (Degond et al. in J Nonlinear Sci 27(2):379–424, 2017), it was shown that an initial size distribution with infinite first moment leads to convergence to zero in weak sense, corresponding to unbounded growth of group size. In the present paper we show that for any such initial distribution with a power-law tail, the solution approaches a self-similar spreading form. A one-parameter family of such self-similar solutions exists, with densities that are completely monotone, having power-law behavior in both small and large size regimes, with different exponents.

Keywords Fish schools · Bernstein functions · Complete monotonicity · Heavy tails · Convergence to equilibrium

Mathematics Subject Classification 45J05 · 70F45 · 92D50 · 37L15 · 44A10 · 35Q99

1 Introduction

Coagulation-fragmentation equations can be used to describe a large variety of merging and splitting processes, including the evolution of animal group sizes [9]. We refer to [3] for an extensive discussion of the relevant literature in this particular application area.

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Here we consider a model with constant coagulation and overall fragmentation rate coefficients that lacks detailed balance and a corresponding H-theorem. This model is motivated by a compelling analysis of fisheries data that was carried out by Niwa in [16], and a first mathematical study of the behavior of its solutions was performed in [3]. As demonstrated in [3], the nature of equilibria of this model as well as their domains of attractions can be rigorously studied using the theory of Bernstein functions. More precisely, it was shown that equilibria can be expressed by a single smooth scaling profile which is not explicit, but it has a convergent power-series representation and its behaviour for small and large cluster sizes can be completely characterized by different power laws with exponential cutoff [3, Eq. (1.5)–(1.7)]. Furthermore, if the initial data have finite first moment, solutions converge to equilibrium in the large time limit.

In addition, it was also shown that if the initial data have infinite first moment, then solutions converge weakly to zero, which means that clusters grow without bound as time goes to infinity. Our goal in the present paper is to investigate whether this growth behaviour is described by self-similar solutions. Indeed, we are going to show that there exists a family of self-similar profiles with completely monotone densities, characterized by different power-law tail behaviours for small and large cluster sizes. Furthermore, if the cumulative mass distribution of the initial data has power law growth for large cluster sizes, the corresponding solution converges to the profile whose mass distribution diverges with the same power-law tail.

Self-similar solutions with fat tails have recently received quite some attention, in particular in the analysis of coagulation equations, starting with work on models with solvable kernels [1,12]. For coagulation equations with non-solvable kernels, existence of self-similar profiles with fat tails has been studied in [2,14,15], but to our knowledge this is the first time that such solutions are found for a class of coagulation-fragmentation equations.

We describe both the discrete- and continuous-size versions of the model in Sect. 2. Our proofs use and extend the methods of complex function theory and in particular Bernstein functions as developed in [3,12,13] and we give a brief overview of the main definitions and results in Sect. 3. Our main results are stated in Sect. 4, while the remaining sections are devoted to their respective proofs.

2 Coagulation-Fragmentation Models D and C

In this section we describe both the discrete coagulation-fragmentation equations under study as well as their continuous-size analogue.

2.1 Discrete-Size Distributions

The number density of clusters of size i at time t is denoted by $f_i(t)$. The size distribution $f(t) = (f_i(t))_{i\in\mathbb{N}}$ evolves according to discrete coagulation-fragmentation equations, written in strong form as follows:

$$\frac{\partial f_i}{\partial t}(t) = Q_a(f)_i(t) + Q_b(f)_i(t), \tag{2.1}$$

$$Q_a(f)_i(t) = \frac{1}{2}\sum_{j=1}^{i-1} a_{j,i-j}\, f_j(t)\, f_{i-j}(t) - \sum_{j=1}^{\infty} a_{i,j}\, f_i(t)\, f_j(t), \tag{2.2}$$

$$Q_b(f)_i(t) = \sum_{j=1}^{\infty} b_{i,j}\, f_{i+j}(t) - \frac{1}{2}\sum_{j=1}^{i-1} b_{j,i-j}\, f_i(t)\,. \tag{2.3}$$

The terms in $Q_a(f)_i(t)$ describe the gain and loss rate of clusters of size i due to aggregation or coagulation, and correspondingly the terms in $Q_b(f)_i(t)$ describe the rate of breakup or fragmentation.

These equations can be written in the following weak form, suitable for comparing to the continuous-size analog: We require that for any bounded test sequence (φ_i),

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^{\infty}\varphi_i\, f_i(t) &= \frac{1}{2}\sum_{i,j=1}^{\infty}\left(\varphi_{i+j}-\varphi_i-\varphi_j\right) a_{i,j}\, f_i(t)\, f_j(t)\\ &\quad -\frac{1}{2}\sum_{i=2}^{\infty}\Big(\sum_{j=1}^{i-1}\left(\varphi_i-\varphi_j-\varphi_{i-j}\right) b_{j,i-j}\Big) f_i(t)\,.\end{aligned} \tag{2.4}$$

The present study deals with the particular case when the rate coefficients take the form

$$a_{i,j} = \alpha\,, \qquad b_{i,j} = \frac{\beta}{i+j+1}\,, \qquad \alpha = \beta = 2. \tag{2.5}$$

We refer to the coagulation-fragmentation equations (2.1)–(2.3) with the coefficients in (2.5) as **Model D** (D for discrete size). By a simple scaling we can achieve any values of α, $\beta > 0$ and so we keep $\alpha = \beta = 2$ for simplicity. As discussed in [3], Model D arises as a modification of the time-discrete model written in [9] which essentially corresponds to the choice of rate coefficients as

$$a_{i,j} = \alpha\,, \qquad b_{i,j} = \frac{\beta}{i+j-1}\,. \tag{2.6}$$

These choices correspond to taking the rate that pairs of individual clusters coalesce, and the rate that individual clusters fragment, to be constants independent of size.

The modification in (2.5), however, permits an analysis in terms of the *Bernstein transform* of the size-distribution measure $\sum_{j=1}^{\infty} f_j(t)\,\delta_j(dx)$. This Bernstein transform is given by

$$\breve{f}(\hat{s},t) = \sum_{j=1}^{\infty}(1-e^{-j\hat{s}}) f_j(t)\,. \tag{2.7}$$

Taking $\varphi_j = 1 - e^{-j\hat{s}}$ in (2.4), it can be shown (see [3, Eq. (10.6)]) that $\breve{f}(\hat{s},t)$ satisfies the integro-differential equation

$$\partial_t \breve{f}(\hat{s},t) = -\breve{f}^2 - \breve{f} + \frac{2}{1-e^{-\hat{s}}}\int_0^{\hat{s}} \breve{f}(r,t)e^{-r}\,dr. \tag{2.8}$$

for $\hat{s}, t > 0$. By the simple change of variables

$$s = 1 - e^{-\hat{s}}\,, \qquad U(s,t) = \breve{f}(\hat{s},t)\,, \tag{2.9}$$

one finds that (2.8) for $\hat{s} \in (0,\infty)$, $t > 0$, is equivalent to

$$\partial_t U(s,t) = -U^2 - U + 2\int_0^1 U(sr,t)\,dr\,, \tag{2.10}$$

for $s \in (0,1)$, $t > 0$. This equation has the same form that arises in the continuous-size case, as we discuss next.

2.2 Continuous-Size Distributions

For clusters of any real size $x > 0$, the size distribution at time t is characterized by a measure F_t, whose distribution function we denote using the same symbol:

$$F_t(x) = \int_{(0,x]} F_t(dx).$$

The measure F_t evolves according to the following size-continuous coagulation-fragmentation equation, which we write in weak form. One requires that for any suitable test function $\varphi(x)$,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}_+} \varphi(x)\, F_t(dx) &= \frac{1}{2}\int_{\mathbb{R}_+^2} \big(\varphi(x+y) - \varphi(x) - \varphi(y)\big) a(x,y)\, F_t(dx)\, F_t(dy) \\ &- \frac{1}{2}\int_{\mathbb{R}_+} \Big(\int_0^x \big(\varphi(x) - \varphi(y) - \varphi(x-y)\big)\, b(y, x-y)\, dy\Big) F_t(dx). \end{aligned} \tag{2.11}$$

The specific rate coefficients that we study correspond to constant coagulation rates and constant overall binary fragmentation rates with uniform distribution of fragments:

$$a(x,y) = A\,, \qquad b(x,y) = \frac{B}{x+y}, \qquad A = B = 2. \tag{2.12}$$

(Again, by scaling one can achieve any $A, B > 0$.) We refer to the coagulation-fragmentation equations (2.11) with these coefficients as **Model C** (C for continuous size).

For size distributions with density, written as $F_t(dx) = f(x,t)\,dx$, Model C is written formally in strong form as follows:

$$\partial_t f(x,t) = Q_a(f)(x,t) + Q_b(f)(x,t), \tag{2.13}$$

$$Q_a(f)(x,t) = \int_0^x f(y,t)\, f(x-y,t)\, dy - 2f(x,t)\int_0^\infty f(y,t)\, dy, \tag{2.14}$$

$$Q_b(f)(x,t) = -f(x,t) + 2\int_x^\infty \frac{f(y,t)}{y}\, dy. \tag{2.15}$$

Importantly, Model C has a scaling invariance involving dilation of size. If $F_t(x)$ is any solution and $\lambda > 0$, then

$$\hat{F}_t(x) := F_t(\lambda x) \tag{2.16}$$

is also a solution.

When we take as test function $\varphi(x) = 1 - e^{-sx}$, we find that the Bernstein transform of F_t, defined by

$$U(s,t) = \breve{F}_t(s) = \int_{\mathbb{R}_+} (1 - e^{-sx})\, F_t(dx)\,, \tag{2.17}$$

satisfies

$$\partial_t U(s,t) = -U^2 - U + 2\int_0^1 U(sr,t)\, dr\,. \tag{2.18}$$

This equation has exactly the same form as (2.10).

According to the well-posedness result for Model C established in [3, Thm. 6.1], given any initial $F_0 \in \mathcal{M}_+(0,\infty)$ (the set of nonnegative finite measures on $(0,\infty)$), Model C admits a unique narrowly continuous map $t \mapsto F_t \in \mathcal{M}_+(0,\infty)$ that satisfies (2.11) for all continuous φ on $[0, \infty]$. In particular, (2.18) holds for all $s \in [0, \infty]$. For $s = \infty$ in particular this means that the zeroth moment $m_0(t) = U(\infty, t)$ satisfies the logistic equation

$$\partial_t m_0(t) = -m_0(t)^2 + m_0(t)\,, \tag{2.19}$$

whence $m_0(t) \to 1$ as $t \to \infty$.

3 Preliminaries

All of our main results concern the behavior of solutions of Models C and D having power-law tails and infinite first moment, and the analysis involves the behavior of their Bernstein transforms. Hence, before we state our main results it is useful to recall some basic definitions and results on Bernstein functions and transforms.

A function $g\colon (0, \infty) \to \mathbb{R}$ is *completely monotone* if it is infinitely differentiable and its derivatives satisfy $(-1)^n g^{(n)}(x) \geq 0$ for all real $x > 0$ and integer $n \geq 0$. By Bernstein's theorem, g is completely monotone if and only if it is the Laplace transform of some (Radon) measure on $[0, \infty)$.

Definition 3.1 A function $U\colon (0, \infty) \to \mathbb{R}$ is a *Bernstein function* if it is infinitely differentiable, nonnegative, and its derivative U' is completely monotone.

The main representation theorem for these functions [18, Thm. 3.2] says that a function $U\colon (0, \infty) \to \mathbb{R}$ is a Bernstein function if and only if it has the representation

$$U(s) = a_0 s + a_\infty + \int_{(0,\infty)} (1 - e^{-sx})\, F(dx)\,, \quad s \in (0, \infty), \tag{3.1}$$

where a_0, $a_\infty \geq 0$ and F is a measure satisfying $\int_{(0,\infty)} (x \wedge 1) F(dx) < \infty$. In particular, the triple (a_0, a_∞, F) uniquely determines U and vice versa.

We point out that U determines a_0 and a_∞ via the relations

$$a_0 = \lim_{s\to\infty} \frac{U(s)}{s}\,, \qquad a_\infty = U(0^+) = \lim_{s\to 0} U(s)\,. \tag{3.2}$$

Whenever (3.1) holds, we call U the *Bernstein transform* of the Lévy triple (a_0, a_∞, F). If $a_0 = a_\infty = 0$, we call U the Bernstein transform of the Lévy measure F, and write $U = \breve{F}$, in accordance with the definitions in Sect. 2.

We will also make use of the theory of so-called *complete* Bernstein functions, as developed in [18, Chap. 6]:

Theorem 3.2 *The following are equivalent.*

(i) *The Lévy measure F in* (3.1) *has a completely monotone density g, so that*

$$U(s) = a_0 s + a_\infty + \int_{(0,\infty)} (1 - e^{-sx}) g(x)\, dx\,, \quad s \in (0, \infty). \tag{3.3}$$

(ii) *U is a Bernstein function that admits a holomorphic extension to the cut plane* $\mathbb{C}\backslash(-\infty, 0]$ *satisfying* $(\operatorname{Im} s) \operatorname{Im} U(s) \geq 0$.

In complex function theory, a function holomorphic on the upper half of the complex plane that leaves it invariant is called a *Pick function* (alternatively a *Herglotz* or *Nevalinna* function). Condition (ii) of the theorem above says simply that U is a Pick function analytic and nonnegative on $(0, \infty)$. Such functions are called *complete Bernstein functions* in [18].

The power-law tail behavior of size distributions is related to power-law behavior of Bernstein transforms near the origin through use of Karamata's Tauberian theorem [4, Thm. III.5.2] and Lemma 3.3 of [12]. To explain, suppose a measure F on $(0, \infty)$ has a density f satisfying

$$f(x) \sim Ax^{-\alpha-1}, \quad x \to \infty. \tag{3.4}$$

Necessarily $\alpha \in (0, 1]$ if F is finite with infinite first moment. The derivative $\partial_s \breve{F}$ of the Bernstein transform of F is the Laplace transform of the measure with distribution function

$$\int_0^x y\, F(dy) \sim \frac{Ax^{1-\alpha}}{1-\alpha} \tag{3.5}$$

for $\alpha \in (0, 1)$. By Karamata's theorem, this is equivalent to

$$\partial_s \breve{F}(s) \sim \frac{A\Gamma(2-\alpha)}{1-\alpha} s^{\alpha-1}, \quad s \to 0. \tag{3.6}$$

Then by Lemma 3.3 of [12] this is equivalent to

$$\breve{F}(s) \sim \frac{A\Gamma(2-\alpha)}{\alpha(1-\alpha)} s^{\alpha}, \quad s \to 0. \tag{3.7}$$

4 Main Results

The choice of coefficients in the asymptotic expressions below is made to simplify Bernstein transform calculations in the sequel. In the following we denote by

$$F_t(x) := \int_{(0,x]} F_t(dx)$$

the cumulative distribution function.

Theorem 4.1 (Self-similar solutions for Model C) *For each* $\alpha \in (0, 1)$ *and* $\lambda > 0$*, Model C admits a unique self-similar solution having the form*

$$F_t(x) = F_{\star\alpha}(\lambda x e^{-\beta t}), \tag{4.1}$$

where $F_{\star\alpha}$ *is a probability measure having the tail behavior*

$$\int_0^x y F_{\star\alpha}(dy) \sim \frac{\alpha}{\Gamma(2-\alpha)} x^{1-\alpha}, \quad x \to \infty. \tag{4.2}$$

For this solution,

$$\beta = \frac{1-\alpha}{\alpha(1+\alpha)}, \tag{4.3}$$

and $F_{\star\alpha}$ *has a completely monotone density* $f_{\star\alpha}$ *having the following asymptotics:*

$$f_{\star\alpha}(x) \sim \begin{cases} \dfrac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1} & x \to \infty, \\ \dfrac{\hat{c}}{\Gamma(-\hat{\alpha})} x^{\hat{\alpha}-1} & x \to 0^+, \end{cases} \tag{4.4}$$

where the constants $\hat{\alpha} \in (0, 1)$*,* $\hat{c} > 0$ *are as described in Lemma 5.1.*

Theorem 4.2 (Large-time behavior for Model C with algebraic tails) *Suppose that the initial data for Model C satisfies*

$$\int_0^x yF_0(dy) \sim \int_0^x yF_{\star\alpha}(\lambda\, dy) \sim \frac{\alpha\lambda^{-\alpha}}{\Gamma(2-\alpha)} x^{1-\alpha}, \quad x \to \infty, \tag{4.5}$$

where $\alpha \in (0, 1)$, $\lambda > 0$. *Then for every* $x \in [0, \infty]$ *we have*

$$F_t(xe^{\beta t}) \to F_{\star\alpha}(\lambda x), \quad t \to \infty. \tag{4.6}$$

Theorem 4.3 (Large-time behavior for Model D with algebraic tails) *Suppose that the initial data for Model D satisfies*

$$\sum_{1\le k\le x} kf_k(0) \sim \int_0^x yF_{\star\alpha}(\lambda\, dy) \sim \frac{\alpha\lambda^{-\alpha}}{\Gamma(2-\alpha)} x^{1-\alpha}, \quad x \to \infty, \tag{4.7}$$

where $\alpha \in (0, 1)$, $\lambda > 0$. *Then for every* $x \in [0, \infty]$ *we have*

$$\sum_{1\le k\le xe^{\beta t}} f_k(t) \to F_{\star\alpha}(\lambda x), \quad t \to \infty. \tag{4.8}$$

These convergence results relate to the notion of weak convergence of measures on $(0, \infty)$ sometimes known as narrow convergence. Let $\mathcal{M}_+(0, \infty)$ be the space of nonnegative finite (Radon) measures on $(0, \infty)$. Given F, $F_n \in \mathcal{M}_+(0, \infty)$ for $n \in \mathbb{N}$, we say F_n converges to F *narrowly* and write $F_n \xrightarrow{n} F$ if

$$\int_{(0,\infty)} g(x)\, F_n(dx) \to \int_{(0,\infty)} g(x)\, F(dx)$$

for all functions $g \in C_b(0, \infty)$, the space of bounded continuous functions on $(0, \infty)$. The convergence statements (4.6) and (4.8) correspond to the statement that

$$\hat{F}_t(dx) \xrightarrow{n} F_{\star\alpha}(\lambda\, dx), \quad t \to \infty$$

where, respectively,

$$\hat{F}_t(dx) = \begin{cases} F_t(e^{\beta t} dx) & \text{for Model C,} \\ \sum_k f_k(t)\delta_{ke^{-\beta t}}(dx) & \text{for Model D.} \end{cases} \tag{4.9}$$

The proofs of (4.6) and (4.8) make use of the following result from [3] (cf. [3, Proposition 3.6]) that characterizes narrow convergence in terms of the Bernstein transform.

Proposition 4.4 *Assume* F, $F_n \in \mathcal{M}_+(0, \infty)$ *for* $n \in \mathbb{N}$. *Then the following are equivalent as* $n \to \infty$.

(i) F_n *converges narrowly to* F, *i.e.,* $F_n \xrightarrow{n} F$.
(ii) *The Bernstein transforms* $\breve{F}_n(s) \to \breve{F}(s)$, *for each* $s \in [0, \infty]$.
(iii) *The Bernstein transforms* $\breve{F}_n(s) \to \breve{F}(s)$, *uniformly for* $s \in (0, \infty)$.

The proofs of our main results will proceed in stages as follows. In Sect. 5 we identify the family of relevant self-similar solutions of Eq. (2.18). The argument involves a phase plane analysis that does not yet establish that the profile function is actually a Bernstein function. In Sect. 6 we prove a comparison principle for the nonlocal evolution equation (2.18), then use this in Sect. 7 to show that solutions of (2.18) with initial data $U_0(s) \sim s^\alpha$ approach the corresponding self-similar form found in Sect. 5. From this we deduce the self-similar profiles are limits of complete Bernstein functions, hence they are Bernstein transforms themselves of measures $F_{\star\alpha}$ having completely monotone densities, and the results of Theorems 4.2 and 4.3 follow. The remaining properties of the profiles stated in Theorem 4.1, including complete monotonicity of densities and asymptotics for small and large size, are established in Sects. 8 and 9.

The results of Theorems 4.2 and 4.3 show that the long-time behavior of solutions with algebraic tails depends upon the algebraic rate of decay. We recall that for the pure coagulation equation with constant rate kernel (corresponding to Model C without fragmentation), all domains of attraction for self-similiar solutions with algebraic tails were characterized in [12] by the condition that initial data are regularly varying. Here in Theorem 4.2, for example, this would correspond to the condition that the initial data satisfy

$$\int_0^x yF_0(dy) \sim x^{1-\alpha}L(x)$$

where L is slowly varying at ∞. In the present context, however, we do not know whether this more general condition is either sufficient or necessary for convergence to self-similar form.

5 Self-similar Scaling: Necessary Conditions

We begin our analysis by finding the necessary forms for any self-similar solution to Eq. (2.18) that governs the Bernstein transform of solutions to Model C.

We look for self-similar solutions to (2.18) of the form

$$U(s,t) = u(sX(t)),$$

where $X(\cdot)$ is smooth with $X(t) \to \infty$ as $t \to \infty$. Because in general $U(\infty, t) = m_0(t) \to 1$ as $t \to \infty$, we require $u(\infty) = 1$. After substituting into (2.18), we find that for nontrivial solutions we must have

$$\beta := X'(t)/X(t)$$

to be a positive constant independent of t, and $u(z)$ must satisfy

$$\beta z \partial_z u + u^2 + u = 2\int_0^1 u(zr)\,dr. \tag{5.1}$$

With

$$v(z) = \int_0^1 u(zr)\,dr = \frac{1}{z}\int_0^z u(r)\,dr, \tag{5.2}$$

the variables $(v(z), u(z))$ satisfy the ODE system

$$\beta z \partial_z u = -u - u^2 + 2v\,, \tag{5.3}$$
$$z\partial_z v = u - v\,. \tag{5.4}$$

Under the change of variables $\tau = \log z$ we have $\partial_\tau = z\partial_z$ and this system becomes autonomous. We seek a solution defined for $\tau \in \mathbb{R}$ satisfying

$$(u,v) \to \begin{cases} (0,0) & \tau \to -\infty, \\ (1,1) & \tau \to +\infty, \end{cases}$$

with both components increasing in τ. What is rather straightforward to check, is that the origin $(0, 0)$ is a saddle point in the (v, u) phase plane, and the region

$$R = \left\{(u, v) \mid 0 < \frac{1}{2}(u + u^2) < v < u\right\}$$

is positively invariant and contained in the unit square $[0, 1]^2$. Inside this region both u and v increase with τ. The unstable manifold at $(0, 0)$ enters this region and must approach the stable node $(1, 1)$ as $\tau \to \infty$, satisfying $1 \le dv/du \le \frac{3}{2}$ asymptotically since the trajectory approaches from inside R.

This trajectory provides the following result.

Lemma 5.1 *Let $\beta > 0$. Then, up to a dilation in z, there is a unique solution of* (5.1) *which is positive and increasing for $z \in (0, \infty)$ with $u(0) = 0$ and $u(\infty) = 1$, satisfying*

$$u(z) \sim z^{\alpha} \qquad \text{as } z \to 0^+,$$
$$1 - u(z) \sim \hat{c} z^{-\hat{\alpha}} \quad \text{as } z \to \infty,$$

where $\alpha \in (0, 1)$, $\hat{\alpha} \in (0, \frac{1}{3})$ are determined by the relations

$$\beta = \frac{1 - \alpha}{\alpha(1 + \alpha)} = \frac{1 - 3\hat{\alpha}}{\hat{\alpha}(1 - \hat{\alpha})} \tag{5.5}$$

We note that the relations (5.5) arise from the eigenvalue equations

$$\begin{vmatrix} -1 - \beta\alpha & 2 \\ 1 & -1 - \alpha \end{vmatrix} = 0, \qquad \begin{vmatrix} -3 + \beta\hat{\alpha} & 2 \\ 1 & -1 + \hat{\alpha} \end{vmatrix} = 0\,. \tag{5.6}$$

In what follows we let u_α denote the solution described by this lemma, noting that the relation between β and α is monotone and given by (4.3). The phase-plane argument above does not show that u_α is a Bernstein function, however. Our plan is to show that in fact u_α is a complete Bernstein function (a Pick function), by showing that it arises as the pointwise limit of rescaled solutions of (2.18) which are complete Bernstein functions. Thus, our proof of the existence theorem 4.1 will depend upon a proof of stability.

6 Comparison Principle

Our next goal is to study the long-time dynamics of solutions of (2.18) with appropriate initial data. For this purpose we develop a comparison principle showing that solutions of (2.18) preserve the ordering of the initial data on any interval of the form $[0, S]$.

Given $S > 0$ and $u \in C([0, S])$, define an averaging operator $\mathcal{A}$ by

$$(\mathcal{A}u)(s) = \int_0^1 u(sr)\, dr\,, \quad s \in [0, S]. \tag{6.1}$$

Then clearly $\mathcal{A}$ is a linear contraction on $C([0, S])$, with

$$(\mathcal{A}u)(0) = u(0). \tag{6.2}$$

We recall that by Hardy's inequality,

$$\left(\int_0^S |(\mathcal{A}u)(s)|^2\, ds\right)^{1/2} \le 2\left(\int_0^S |u(s)|^2\, ds\right)^{1/2}. \tag{6.3}$$

Indeed, due to Minkowski's inequality in integral form we have

$$\left(\int_0^S \left|\int_0^1 u(xr)\,dr\right|^2 dx\right)^{1/2} \le \int_0^1 \left(\int_0^S |u(sr)|^2\,ds\right)^{1/2} dr$$

and thus

$$\begin{aligned}\left(\int_0^S |(\mathcal{A}u)(s)|^2\,ds\right)^{1/2} &\le \int_0^1 \left(\int_0^S |u(sr)|^2\,ds\right)^{1/2} dr\\ &\le \int_0^1 \frac{dr}{r^{1/2}} \left[\int_0^S |u(s)|^2\,ds\right]^{1/2}.\end{aligned}$$

Proposition 6.1 *Given $S, T > 0$ suppose that $U, V \in C^1([0,T], C([0,S])$ have the following properties:*

(i) $U(s,0) \ge V(s,0)$ *for all* $s \in [0,S]$,
(ii) *for all* $(s,t) \in [0,S] \times [0,T]$ *the equations*

$$\partial_t U + U^2 + U(s,t) = 2\mathcal{A}U + F\,, \tag{6.4}$$
$$\partial_t V + V^2 + V(s,t) = 2\mathcal{A}V + G\,, \tag{6.5}$$

hold, where $F \ge G$.

Then $U \ge V$ *everywhere in* $[0,S] \times [0,T]$.

Proof We write

$$w = U - V = w_+ - w_- \quad \text{where } w_+, w_- \ge 0.$$

Let $M \ge \max|U+V|$. Subtracting (6.5) from (6.4) we find

$$\partial_t w + M|w| + w \ge 2\mathcal{A}w + F - G\,.$$

Because $w_\pm$ is Lipschitz in t, $w_+ w_- = 0$, and $\mathcal{A}w_\pm \ge 0$, we can multiply by $-2w_- \le 0$ and invoke [5, Lemma 7.6] to infer that the weak derivative

$$\partial_t(w_-^2) - 2Mw_-^2 \le 4w_-\mathcal{A}w_-\,.$$

Integrating over $s \in [0,S]$ and using Hardy's inequality we find

$$\partial_t \int_0^S w_-(s)^2\,ds \le (8+2M)\int_0^S w_-(s)^2\,ds.$$

Because $w_-(s,0) = 0$, integrating in t and using Gronwall's lemma concludes the proof that $U \ge V$ in $[0,S] \times [0,T]$. □

7 Convergence to Equilibrium for Initial Data with Power-Law Tails

We begin with a result for solutions of (2.18) that is suitable for use in treating both Model C and Model D.

Proposition 7.1 *Suppose $U(s,t)$ is any C^1 solution of (2.18) for $s \in [0,\bar{s})$, $t \in [0,\infty)$, and assume that its initial data satisfies*

$$U_0(s) \sim s^\alpha \quad \text{as } s \to 0^+, \tag{7.1}$$

where $\alpha \in (0,1)$. Then with β given by (4.3), we have

$$U(se^{-\beta t},t) \to u_\alpha(s) \quad \text{as } t \to \infty, \text{ for all } s \in (0,\infty), \tag{7.2}$$

with uniform convergence for s in any bounded subset of $(0,\infty)$, where u_α is the self-similar profile u described in Lemma 5.1.

The proof is rather different from the proof of convergence to equilibrium for initial data with finite first moment, in Section 7 of [3]. In the present case, the behavior of $U(s,t)$ globally in t is determined by the local behavior of the initial data U_0 near $s=0$.

Proof First, let u_α be given by Lemma 5.1, and note that for any $c>0$ the function given by

$$V(s,t) = u_\alpha(cse^{\beta t})$$

is a solution of (6.5) with $G=0$. Second, it is not difficult to prove that

$$u_\alpha(cz) \to u_\alpha(z) \quad \text{as } c \to 1, \text{ uniformly for } z \in (0,\infty). \tag{7.3}$$

Now, let $S>0$ and let $\varepsilon>0$. Choose $c<1<C$ such that

$$u_\alpha(cz) < u_\alpha(z) < u_\alpha(Cz) < u_\alpha(cz)+\varepsilon \quad \text{for all } z \in (0,\infty). \tag{7.4}$$

Due to the hypothesis (7.1), there exists $S_0 = S_0(c,C) > 0$ such that

$$u_\alpha(cs) \le U(s,0) \le u_\alpha(Cs) \qquad \text{for all } s \in [0,S_0]. \tag{7.5}$$

Invoking the comparison principle in Proposition 6.1 we infer that

$$u_\alpha(cse^{\beta t}) \le U(s,t) \le u_\alpha(Cse^{\beta t}) \quad \text{for all } s \in [0,S_0], t>0. \tag{7.6}$$

Replacing $s \in [0,S_0]$ by $se^{-\beta t}$ with $s \in [0,S_0e^{\beta t}]$, this gives

$$u_\alpha(cs) \le U(se^{-\beta t},t) \le u_\alpha(Cs) \quad \text{for all } s \in [0,S_0e^{\beta t}], t>0. \tag{7.7}$$

By consequence, whenever $S_0e^{\beta t} > S$ it follows that

$$|U(se^{-\beta t},t) - u_\alpha(s)| < \varepsilon \quad \text{for all } s \in [0,S].$$

This finishes the proof. □

Proof of Theorem 4.2 Because of the dilation invariance of Model C, we may assume the initial data satisfies (4.5) with $\lambda=1$. By the discussion of (3.5)–(3.7) we infer that

$$U_0(s) = \int_0^\infty (1-e^{-sx})F_0(dx) \sim s^\alpha, \quad s \to 0. \tag{7.8}$$

Next, we invoke Proposition 7.1 to deduce that

$$U(se^{-\beta t},t) = \int_0^\infty (1-e^{-sx})F_t(e^{\beta t}\,dx) \to u_\alpha(s) \tag{7.9}$$

for all $s \in [0,\infty)$. The limit also holds for $s=\infty$ as a consequence of the logistic equation (2.19) for $m_0(t) = U(\infty,t)$. At this point we use the fact that the pointwise limit $u_\alpha(s)$ of the Bernstein functions $s \mapsto U(se^{-\beta t}, t)$ is necessarily Bernstein [18, Cor. 3.7, p. 20] and the facts that

$$\lim_{s\to 0} u_\alpha(s) = 0, \qquad \lim_{s\to\infty} u_\alpha(s) = 1\,,$$

to infer the following (cf. [3, Eq. (3.3)]).

Lemma 7.2 *For any $\alpha \in (0,1)$, the function u_α described in Lemma 5.1 is the Bernstein transform of a probability measure $F_{\star\alpha}$ on $(0,\infty)$, satisfying*

$$u_\alpha(s) = \int_0^\infty (1 - e^{-sx}) F_{\star\alpha}(dx)\,, \quad s \in [0,\infty].$$

Finally, we use Proposition 4.4 to infer the narrow convergence result

$$F_t(e^{\beta t}\,dx) \xrightarrow{n} F_{\star\alpha}(dx)\,, \quad t \to \infty, \tag{7.10}$$

to conclude the proof of Theorem 4.2, □

Proof of Theorem 4.3 For Model D, the discussion of (3.5)–(3.7) implies that the hypothesis (4.7) on initial data is equivalent to the condition

$$\breve{f}(\hat{s}, 0) \sim \lambda^{-\alpha}\hat{s}^\alpha\,, \quad \hat{s} \to 0, \tag{7.11}$$

on the Bernstein transform of the initial data. Under the change of variables $s = 1 - e^{-\hat{s}}$ in (2.9) this is evidently equivalent to

$$U(s, 0) \sim \lambda^{-\alpha}s^\alpha\,, \quad s \to 0. \tag{7.12}$$

As $U(s,t)$ is a solution of the dilation-invariant equation (2.10), so is the function $\hat{U}(s,t) = U(\lambda s, t)$ which satisfies $\hat{U}(s,0) \sim s^\alpha$, $s \to 0$. Invoking Proposition 7.1, we deduce that for all $s \in [0,\infty)$,

$$U(se^{-\beta t}, t) \to u_\alpha(s/\lambda) \qquad \text{as } t \to \infty. \tag{7.13}$$

Note that the left-hand side is well-defined only for $e^{\beta t} > s$.

We can now write

$$\breve{f}(\hat{s}e^{-\beta t}, t) = U(\bar{s}(\hat{s}, t)e^{-\beta t}, t), \tag{7.14}$$

where $\bar{s}(\hat{s}, t)e^{-\beta t} = 1 - \exp(-\hat{s}e^{-\beta t})$. Then for any fixed $\hat{s} \in (0,\infty)$,

$$\bar{s}(\hat{s}, t) = \hat{s} + O(e^{-\beta t}) \quad \text{as } t \to \infty. \tag{7.15}$$

Because the convergence in (7.13) is uniform for s in bounded sets by Proposition 7.1, it follows that for each $\hat{s} \in [0,\infty)$,

$$\breve{f}(\hat{s}e^{-\beta t}, t) \to u_\alpha(\hat{s}/\lambda). \tag{7.16}$$

Next we establish (7.16) for $\hat{s} = \infty$, recalling $\breve{f}(\infty, t) = m_0(f(t))$. In the present case of Model D, the evolution equation for $m_0(f(t))$ is not closed, and we formulate our result as follows.

Lemma 7.3 *For any solution of Model D, $m_0(f(t)) \to 1$ as $t \to \infty$.*

Proof 1. According to [3, Thm. 12.1], the zeroth moment $m_0(f(t)) = \breve{f}(\infty, t)$ is a smooth function of $t \in [0, \infty)$ that satisfies the inequality

$$\partial_t m_0(f(t)) \le -m_0(f(t))^2 + m_0(f(t)), \quad t \ge 0. \tag{7.17}$$

We infer that for all $t \ge 0$,

$$m_0(f(t)) \le \frac{1}{1 - e^{-t}}, \tag{7.18}$$

as the right-hand size solves the logistic equation $y' = -y^2 + y$ on $(0, \infty)$. Thus we infer

$$\limsup_{t\to\infty} m_0(f(t)) \le 1. \tag{7.19}$$

2. We claim $\liminf_{t\to\infty} m_0(f(t)) \ge 1$. For this we use the result of Proposition 7.1, with $U(s, t)$ for $0 < s < 1$ determined from $\breve{f}(\hat{s}, t)$ by (2.9). Choose $S > 0$ such that $u_\alpha(S) > 1 - \varepsilon$. Then for t sufficiently large we have

$$m_0(f(t)) \ge U(Se^{-\beta t}, t) > 1 - \varepsilon.$$

Hence $\liminf_{t\to\infty} m_0(f(t)) \ge 1$. This finishes the proof of the Lemma.

□

Now, because (7.16) holds for all $s \in [0, \infty]$, the desired conclusion of narrow convergence in Theorem 4.3 follows by using Proposition 4.4. □

8 Pick Properties of Self-similar Profiles

Lemma 8.1 *For any $\alpha \in (0, 1)$ the measure $F_{\star\alpha}$ of Lemma 7.2 has a completely monotone density $f_{\star\alpha}$, whose Bernstein transform is the function u_α described in Lemma 5.1, i.e.,*

$$u_\alpha(s) = \int_0^\infty (1 - e^{-sx}) f_\alpha(x)\, dx, \quad s \in [0, \infty].$$

Proof By Theorem 6.1(ii) of [3], if the initial data F_0 for Model C has a completely monotone density, then the solution F_t has a completely monotone density for every $t \ge 0$, with $F_t(dx) = f_t(x)\, dx$ where f_t is completely monotone. By the representation theorem for complete Bernstein functions, this property is equivalent to saying that the Bernstein transform $U(\cdot, t) = \breve{F}_t$ is a Pick function.

As dilates and pointwise limits of complete Bernstein functions are complete Bernstein functions [18, Cor. 7.6], we infer directly from our Theorem 4.2 that for any $\alpha \in (0, 1)$, the self-similar profile u_α is a complete Bernstein function. Therefore, its Lévy measure $F_{\star\alpha}$ has a completely monotone density f_α. □

Remark 8.1 An example of Pick-function initial data which satisfy the hypotheses of the convergence theorem is the following:

$$U_0(s) = s^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-sx}) x^{-1-\alpha}\, dx \tag{8.1}$$

Remark 8.2 We have no argument establishing the monotonicity of densities for model C that avoids use of the representation theorem for complete Bernstein functions. It would be interesting to have such an argument.

Decomposition. A point which is interesting, but not essential to the main thrust of our analysis, is that we can sometimes 'decompose' the Bernstein transforms $U(s,t) = \breve{F}_t(s)$ of solutions of Model C, writing

$$U(s,t) = V(s^\alpha, t), \tag{8.2}$$

where $V(\cdot, t)$ itself is a complete Bernstein function. By considering limits as $t \to \infty$, this can be used to say something more about the self-similar profiles u_α.

Proposition 8.2 (a) *Suppose $\alpha \in (0,1)$ and $U_0(s) = V_0(s^\alpha)$ where V_0 is completely Bernstein. Then for all $t \geq 0$, (8.2) holds for the solution of (2.18) with initial data U_0, where $V(\cdot, t)$ is completely Bernstein.*

(b) *For each $\alpha \in (0,1)$, the Bernstein transform u_α of the self-similar profiles of Lemma 5.1 have the form*

$$u_\alpha(s) = V_\alpha(s^\alpha) \tag{8.3}$$

where V_α is completely Bernstein, having the representation

$$V_\alpha(s) = \int_0^\infty (1 - e^{-sx}) g_{\star\alpha}(x)\, dx \tag{8.4}$$

for some completely monotone function $g_{\star\alpha}$.

Proof To prove part (a), we define $V(\cdot, t)$ by (8.2) and compute that

$$\partial_t V(s,t) + V^2 + V = 2A_\alpha V(s,t), \tag{8.5}$$

$$A_\alpha V(s,t) = \int_0^1 V(sr, t)\, d(r^{1/\alpha}). \tag{8.6}$$

The implicit-explicit difference scheme used in [3, Sect. 6] to solve (2.18) corresponds precisely here to the difference scheme

$$\hat{V}_n(s) = V_n(s) + 2\Delta t\, A_\alpha V_n(s)\,, \tag{8.7}$$

$$(1 + \Delta t) V_{n+1}(s) + \Delta t\, V_{n+1}(s)^2 = \hat{V}_n(s) \tag{8.8}$$

under the correspondence

$$U_n(s) = V_n(s^\alpha). \tag{8.9}$$

Exactly as argued at the end of [3, Sect. 6], if V_n is completely Bernstein then so is $\hat{V}_n$ since complete Bernstein functions form a convex cone closed under dilations and taking pointwise limits. Then V_{n+1} is completely Bernstein due to [3, Prop. 3.4] (i.e., for the same reason U_{n+1} is). Because of the fact that $U_n(s) \to U(s,t)$ as $\Delta t \to 0$ with $n\Delta t \to t$. which was shown in [3], we infer that similarly $V_n(s) \to V(s,t)$, and hence $V(\cdot, t)$ is completely Bernstein.

Next we prove part (b). From the convergence result of Proposition 7.1 it follows that if $V_0(s) = U_0(s^{1/\alpha}) \sim s$ as $s \to 0$, then for all $s > 0$,

$$V(se^{-\alpha\beta t}, t) = U(s^{1/\alpha} e^{-\beta t}, t) \to V_\alpha(s) \quad \text{as } t \to \infty, \tag{8.10}$$

where V_α is defined by (8.3). By taking V_0 to be completely Bernstein and applying part (a), we conclude V_α is completely Bernstein through taking the pointwise limit. □

Remark 8.3 Formulae such as (8.3), involving the composition of two Bernstein functions, are associated with the notion of subordination of probability measures, as is discussed by Feller [4, XIII.7]. See Sect. 11 below for further information.

Remark 8.4 Equation (8.5) satisfied by $V(s,t)$ is close to one satisfied by the Bernstein transform of the solution of a system modeling coagulation with multiple-fragmentation [7,10,11]. This system takes the following strong form analogous to (2.13)–(2.15):

$$\partial_t f(x,t) = Q_a(f)(x,t) + Q_b(f)(x,t), \tag{8.11}$$

$$Q_a(f)(x,t) = \int_0^x f(y,t)\, f(x-y,t)\, dy - 2f(x,t)\int_0^\infty f(y,t)\, dy, \tag{8.12}$$

$$Q_b(f)(x,t) = -f(x,t) + \int_x^\infty b(x|y) f(y,t)\, dy, \tag{8.13}$$

where

$$b(x|y) = (\gamma+2)\frac{x^\gamma}{y^{1+\gamma}}\,. \qquad \gamma = \frac{1-\alpha}{\alpha}\,. \tag{8.14}$$

The coefficient $\gamma+2$ is determined by the requirement that mass is conserved:

$$1 = \frac{1}{y}\int_0^y x b(x|y)\, dx = (\gamma+2)\int_0^1 r^{\gamma+1}\, dr\,.$$

A key calculation is that with $\varphi_s(x) = 1 - e^{-sx}$,

$$\begin{aligned}\int_0^y \varphi_s(x) b(x|y)\, dx &= \int_0^y \varphi_s(x)(\gamma+2)\left(\frac{x}{y}\right)^\gamma \frac{dx}{y} \\ &= \frac{\gamma+2}{\gamma+1}\int_0^1 \varphi_s(ry)\, d(r^{\gamma+1}) \\ &= (\alpha+1)\int_0^1 \varphi_s(ry)\, d(r^{1/\alpha})\end{aligned}$$

As a consequence, the Bernstein transform of a solution of (8.11)–(8.13) should satisfy

$$\partial_t V(s,t) + V^2 + V = (1+\alpha)A_\alpha V(s,t)\,. \tag{8.15}$$

The coefficient $(1+\alpha)$ here differs from the factor 2 in (8.5), and we see no way to scale the V in (8.2) to get exactly this coagulation–multiple-fragmentation model.

A last note is that the 'number of clusters' produced from a cluster of size y by this fragmentation mechanism is calculated to be

$$n(y) = \int_0^y b(x|y)\, dx = \frac{\gamma+2}{\gamma+1} = \alpha + 1.$$

9 Asymptotics of Self-similar Profiles

Here we complete the proof of Theorem 4.1, characterizing self-similar solutions of Model C, by describing the asymptotic behavior of the self-similar size-distribution profiles $f_{\star\alpha}$ in the limits of large and small size. This involves a Tauberian analysis based on the behavior of the Bernstein transform u_α as described in Lemma 5.1.

Proof of Theorem 4.1 Given $\alpha \in (0,1)$, recall we know that for any self-similar solution of Model C as in (4.1), the measure $F_{\star\alpha}(dx)$ must have Bernstein transform $u_\alpha(s)$ as described by Lemma 5.1. That indeed the function u_α is the Bernstein transform of a probability measure $F_{\star\alpha}$ follows from Lemma 7.2, and the fact that $F_{\alpha\star}$ has a completely monotone density $f_{\star\alpha}$ was shown in Lemma 8.1. It remains only to establish that $f_{\alpha\star}$ enjoys the asymptotic properties stated in (4.4).

From Lemma 5.1 we infer that as $z \to \infty$,

$$1 - u_\alpha(z) = \int_0^\infty e^{-zx} f_{\star\alpha}(x)\,dx \sim \hat{c} z^{-\hat{\alpha}} \quad \text{as } z \to \infty.$$

Recalling $\hat{\alpha} \in (0, \frac{1}{3})$, invoking the Tauberian theorem [4, Thm. XIII.5.3] and the fact that $f_{\star\alpha}$ is monotone, from [4, Thm. XIII.5.4] we infer

$$f_{\star\alpha}(x) \sim \frac{\hat{c}}{\Gamma(\hat{\alpha})} x^{\hat{\alpha}-1} \quad \text{as } x \to 0. \tag{9.1}$$

Next, from Lemma 5.1, (5.3) and (4.3) we infer that

$$\partial_z u_\alpha(z) = \int_0^\infty e^{-zx} x f_{\star\alpha}(x)\,dx \sim \alpha z^{\alpha-1} \quad \text{as } z \to 0.$$

By Karamata's Tauberian theorem [4, Thm. XIII.5.2] we deduce

$$\int_0^x y f_{\star\alpha}(y)\,dy \sim \frac{\alpha}{\Gamma(2-\alpha)} x^{1-\alpha} \quad \text{as } x \to \infty.$$

Although we do not know $y \mapsto y f_\alpha(y)$ is eventually monotone, the selection argument used in the proof of [4, Thm. XIII.5.4] works without change, allowing us to infer that

$$x f_{\star\alpha}(x) \sim \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha} \quad \text{as } x \to \infty. \tag{9.2}$$

This completes the proof of Theorem 4.1. □

Remark 9.1 We note that in the limit $\alpha \to 1$ we have $\beta \to 0$ and $\hat{\alpha} \to \frac{1}{3}$, and the power-law exponent $\hat{\alpha} - 1 \to -\frac{2}{3}$. This recovers the exponent governing the small-size behavior of the equilibrium distribution analyzed previously in [3, Eq. (1.6)].

Remark 9.2 By (8.3),

$$1 - V_\alpha(z) = \int_0^\infty e^{-zx} g_{\star\alpha}(x)\,dx \sim \hat{c} z^{-\hat{\alpha}/\alpha},$$

hence by the same argument as that leading to (9.1) we find

$$g_{\star\alpha}(x) \sim \frac{\hat{c}}{\Gamma(\hat{\alpha}/\alpha)} x^{-1+\hat{\alpha}/\alpha} \quad \text{as } x \to 0. \tag{9.3}$$

We note that $\hat{\alpha}/\alpha < 1$ for all $\alpha \in (0, 1)$, because the assumption $\hat{\alpha} = \alpha$ together with the relations (5.5) lead to a contradiction.

10 Series in Fractional Powers

In this section we show that the self-similar profile in Lemma 5.1 is expressed, for small $z > 0$, in the form

$$u_\alpha(z) = \sum_{n=1}^\infty (-1)^{n-1} c_n z^{\alpha n}, \tag{10.1}$$

where the series converges for $z^\alpha \in (0, R_\alpha)$ for some positive but finite number R_α, and the coefficient sequence $\{c_n\}$ is positive with a rather nice structure.

By substituting the series expansion (10.1) into (5.1) we find that $c_1 = 1$, and c_n is necessarily determined recursively for $n \geq 2$ by

$$c_n = \frac{1}{a_n} \sum_{k=1}^{n-1} c_k c_{n-k} \,, \tag{10.2}$$

$$a_n = \beta \alpha n + 1 - \frac{2}{\alpha n + 1} = \frac{1-\alpha}{1+\alpha} n + \frac{\alpha n - 1}{\alpha n + 1}. \tag{10.3}$$

Because the relation (4.3) implies that indeed

$$\beta \alpha + 1 = \frac{2}{1+\alpha}, \tag{10.4}$$

plainly $a_1 = 0$ and a_n increases with n, with $a_n > 0$ for $n > 1$.

Recall that we know from Proposition 8.2 that $u_\alpha(s) = V_\alpha(s^\alpha)$ where V_α is completely Bernstein.

Proposition 10.1 *For each $\alpha \in (0, 1)$, V_α is analytic in a neighborhood of $s = 0$, given by the power series*

$$V_\alpha(s) = \sum_{n=1}^{\infty} (-1)^{n-1} c_n s^n \,.$$

This series has a positive radius of convergence R_α satisfying

$$\frac{1-\alpha}{1+\alpha} \leq R_\alpha \leq a_2 < 1, \tag{10.5}$$

and coefficients that take the form

$$c_n = \gamma^\star_{n-1} R_\alpha^{1-n}, \tag{10.6}$$

where $(\gamma^\star_n)_{n\geq 0}$ is a completely monotone sequence with $\gamma^\star_0 = 1$.

Proof It suffices to prove the bounds on the radius of convergence and the representation formula (10.6), as the validity of Eq. (5.1) then follows by substitution. By induction we will establish bounds on the radius of convergence of the power series

$$v_\star(z) = \sum_{n=1}^{\infty} c_n z^n \,, \tag{10.7}$$

which is evidently related to V_α by $V_\alpha(z) = -v_\star(-z)$. Observe that the inequality $c_k \leq m/r^k$ for $1 \leq k < n$ implies

$$c_n \leq \frac{n-1}{a_n} \frac{m^2}{r^n} \leq \frac{m}{r^n} \,,$$

provided that

$$m \leq \frac{a_n}{n-1} = \frac{1-\alpha}{1+\alpha} + \frac{2\alpha}{(1+\alpha)(1+\alpha n)} \,.$$

By choosing

$$m = r = \frac{1-\alpha}{1+\alpha} = \beta\alpha\,,$$

we ensure $c_1 = m/r$ and therefore $c_n \le r^{1-n}$ for all $n \ge 1$, i.e.,

$$c_n \le \left(\frac{1+\alpha}{1-\alpha}\right)^{n-1}, \quad n = 1, 2, \ldots, \tag{10.8}$$

whence

$$v_\star(z) \le \frac{rz}{r-z} < \infty \quad \text{for } 0 < z < r.$$

In a similar way, the choice

$$M = R = a_2 \ge \frac{a_n}{n-1}$$

for all $n \ge 2$ ensures $c_n \ge M/R^n$ for all $n \ge 2$, whence

$$v_\star(z) \ge \frac{Rz}{R-z} \quad \text{for } 0 < z < R$$

By consequence we infer the bounds in (10.5) hold.

Now, because V_α is completely Bernstein, it is a Pick function analytic on the positive half-line. Hence, from what we have shown, the function $v_\star$ is a Pick function analytic on $(-\infty, R_\star)$. From this and Corollary 1 of [8], it follows directly that the coefficients c_n may be represented in the form (10.6) where $\{\gamma_n\}_{n\ge 0}$ is a completely monotone sequence with $\gamma_0^\star = 1$. □

Remark 10.1 In the limiting case $\beta = 0$, $\alpha = 1$, the coefficients c_n reduce to the explicit form appearing in eq. (5.19) of [3]. Namely,

$$c_n = A_n(3,1) = \frac{1}{3n+1}\binom{3n+1}{n}$$

in terms of the Fuss–Catalan numbers defined by

$$A_n(p,r) = \frac{1}{pn+r}\binom{pn+r}{n}.$$

This can be verified directly from the recursion formulae in (10.2) by using a known identity for the Fuss–Catalan numbers [17, p. 148].

Remark 10.2 We are not aware of any combinatorial representation or interpretation of the coefficients $c_n(\alpha)$ for $\alpha \in (0,1)$, however.

Remark 10.3 *(Nature of the singularity at R_α)* Numerical evidence suggests that for $0 < \alpha < 1$, the singularity at R_α is a simple pole. If true, this should imply that as $n \to \infty$, the coefficients $\gamma_n^\star \to \gamma_\infty^\star > 0$, and the completely monotone Lévy density $g_{\star\alpha}(x)$ for the complete Bernstein function $V_\alpha(z)$ has exponential decay at ∞, with

$$g_{\star\alpha}(x) \sim C_\star e^{-R_\alpha x} \quad \text{as } x \to \infty,$$

where $C_\star > 0$.

11 A Subordination Formula

Here we use the subordination formulae from [4, XIII.7(e)] as linearized in [6, Remark 3.10], to describe a relation between the completely monotone Lévy densities for the Bernstein functions u_α and V_α. Recall we have shown

$$u_\alpha(z) = \int_0^\infty (1 - e^{-zx}) f_{\star\alpha}(x)\, dx = V_\alpha(z^\alpha),$$

where V_α is a complete Bernstein function, with

$$V_\alpha(z) = \int_0^\infty (1 - e^{-zx}) g_{\star\alpha}(x)\, dx\,,$$

for some completely monotone function $g_{\star\alpha}$. The complete Bernstein function z^α has power-law Lévy measure

$$\nu_0(dx) = c_\alpha x^{-1-\alpha} dx\,, \qquad c_\alpha = \frac{\alpha}{\Gamma(1-\alpha)}.$$

This is the jump measure for an α-stable Lévy process $\{Y_\tau\}_{\tau \geq 0}$ (increasing in τ) whose time-τ transition kernel $Q_\tau(dy)$ has the Laplace transform

$$\mathbb{E}(e^{-qY_\tau}) = \int_0^\infty e^{-qy} Q_\tau(dy) = e^{-\tau q^\alpha}\,.$$

Recalling the subordination formula in the linearized form (3.20) from [6], we infer that the self-similar profile $f_{\star\alpha}$ may be expressed as

$$f_{\star\alpha}(x) = \int_0^\infty Q_\tau(dx) g_{\star\alpha}(\tau)\, d\tau\,.$$

We know that $Q_1(dy) = p_\alpha(y)\, dy$ where p_α is the maximally skewed Lévy-stable density from [4, XVII.7] given by

$$p_\alpha(x) = p(x; \alpha, -\alpha) = \frac{-1}{\pi x} \sum_{k=1}^\infty \frac{\Gamma(k\alpha + 1)}{k!} (-x^{-\alpha})^k \sin k\pi\alpha\,. \tag{11.1}$$

Then by scaling dual to $\exp(-\tau q^\alpha) = \exp(-(\tau^{1/\alpha} q)^\alpha)$, we find

$$Q_\tau(dy) = p_\alpha\left(\frac{y}{\tau^{1/\alpha}}\right) \frac{dy}{\tau^{1/\alpha}}\,,$$

and obtain the following.

Proposition 11.1 *The self-similar profile $f_{\star\alpha}$ is related to the completely monotone Lévy densitiy $g_{\star\alpha}$ of V_α by*

$$f_{\star\alpha}(x) = \int_0^\infty g_{\star\alpha}(\tau) p_\alpha\left(\frac{x}{\tau^{1/\alpha}}\right) \frac{d\tau}{\tau^{1/\alpha}}$$

We note that in the limit $\alpha \to 1$ one has $p_\alpha(y)\, dy \to \delta_1$, the delta mass at 1, consistent with $g_{\star\alpha} \to f_{\star\alpha}$. Moreover, note that from (11.1) the large-x behavior of the α-stable density p_α is

$$p_\alpha(x) \sim \Gamma(1+\alpha) \frac{\sin \pi\alpha}{\pi} x^{-\alpha-1} \sim f_{\star\alpha}(x)\,, \quad x \to \infty\,, \tag{11.2}$$

due to Euler's reflection formula for the Γ-function. This is consistent with the fact that the Bernstein transform of p_α is

$$\int_0^\infty (1-e^{-sx})p_\alpha(x)\,dx = 1-e^{-s^\alpha} \sim s^\alpha \sim u_\alpha(s)\,, \quad s\to 0.$$

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