J Nonlinear Sci (2017) 27:873–926
DOI 10.1007/s00332-016-9354-1

Journal of Nonlinear Science

Continuum Limit of a Mesoscopic Model with Elasticity of Step Motion on Vicinal Surfaces

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Received: 27 June 2016 / Accepted: 15 December 2016 / Published online: 29 December 2016

Abstract This work considers the rigorous derivation of continuum models of step motion starting from a mesoscopic Burton–Cabrera–Frank-type model following the Xiang's work (Xiang in SIAM J Appl Math 63(1):241–258, 2002). We prove that as the lattice parameter goes to zero, for a finite time interval, a modified discrete model converges to the strong solution of the limiting PDE with first-order convergence rate.

Keywords Epitaxial growth · BCF · Hilbert transformation · Convergence rate positivity

Mathematics Subject Classification 35K25 · 35K55 · 74A50

Communicated by Robert V. Kohn.

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1 Introduction

In this work, we revisit the derivation of continuum model for step flow with elasticity on vicinal surfaces. The starting point is the Burton–Cabrera–Frank (BCF)-type models for step flow (Burton et al. 1951); see Duport et al. (1995a, b), Tersoff et al. (1995), Liu et al. (1998) for extensions to include elastic effects. These are mesoscopic models which track the position of each individual step (and hence keep the discrete nature of the step fronts), while adopt a continuum approximation for the interactions of the steps with surrounding atoms of the thin film. The step motion is hence characterized by a system of ODEs. Such models are widely used for crystal growth of thin films on substrates, with many scientific and engineering applications (Pimpinelli and Villain 1998; Weeks and Gilmer 1979; Zangwill 1988). The goal of this work is to rigorously understand the PDE limit of such models.

To avoid unnecessary technical difficulties, we will study a periodic train of steps in this work. Denote the step locations at time t by $x_i(t)$, $i \in \mathbb{Z}$, we assume that

$$x_{i+N}(t) - x_i(t) = L, \qquad \forall i \in \mathbb{Z}, \forall t \geq 0, \tag{1.1}$$

where L is a fixed length of the period. Thus, only the step locations in one period $\{x_i(t),\ i = 1, \ldots, N\}$ are considered as degrees of freedom, see Fig. 1 for example.

We denote the height of each step as $a = \frac{1}{N}$, and thus the total height change across the N steps in the period is given by 1. Corresponding to the step locations, we define the height profile h_N of the steps as

$$h_N(x,t) = \frac{N-i}{N}, \quad \text{for } x \in [x_i(t), x_{i+1}(t)),\ \ i = 1, \ldots, N. \tag{1.2}$$

Moreover, h_N can be further extended, consistent with the periodic assumption (1.1), such that

$$h_N(x+L) - h_N(x) = -1, \qquad \forall x \in \mathbb{R}. \tag{1.3}$$

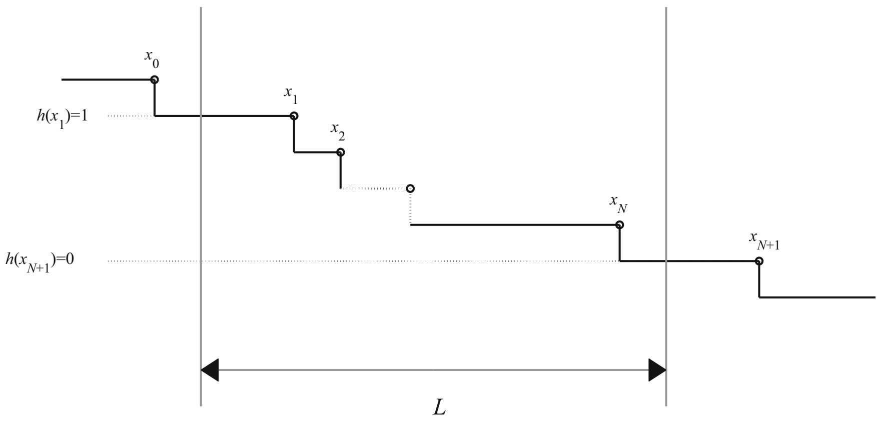

Fig. 1 An example of one periodic steps

For the continuum limit, we consider the step height $a \to 0$ or equivalently, the number of steps in one period $N \to \infty$.

In the pioneering work (Xiang 2002) (see also Xiang and E 2004), XIANG considered a BCF-type model which incorporates the elastic interaction as[1]

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = a^2 \left(\frac{f_{i+1} - f_i}{x_{i+1} - x_i} - \frac{f_i - f_{i-1}}{x_i - x_{i-1}} \right), \quad i = 1, \ldots, N, \tag{1.4}$$

where f_i's are the local chemical potential given by

$$f_i := \frac{\partial E}{\partial x_i} = -\sum_{j \neq i} \left(\frac{\alpha_1}{x_j - x_i} - \frac{\alpha_2}{(x_j - x_i)^3} \right),$$

with the parameters $\alpha_1 = \frac{4}{\pi} a^4,\ \alpha_2 = \frac{2}{\pi} a^6$ and the energy functional E given by

$$E = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \left(\alpha_1 \ln|x_i - x_j| + \frac{\alpha_2}{2} \frac{1}{(x_i - x_j)^2} \right).$$

For the limit $a \to 0$, Xiang (2002) asymptotically derived the corresponding continuum model

$$h_t = \pi \alpha_1 a^2 \left(-H(h_x) + \frac{1}{2\pi} \frac{a h_{xx}}{h_x} + \frac{\pi}{2} \frac{\alpha_2}{\alpha_1} \frac{h_x h_{xx}}{a} \right)_{xx}. \tag{1.5}$$

Here $H(\cdot)$ is the L-periodic Hilbert transform:

$$(Hu)(x) := \frac{1}{L} \mathrm{PV} \int_0^L u(x - s) \cot\left(\frac{\pi s}{L} \right) \mathrm{d}s. \tag{1.6}$$

Observe that for the particular choice of the parameters α_1 and α_2, (1.5) suggests to rescale t to consider timescale of the order $O(a^{-6})$. Moreover, the coefficients in front of the term $h_x h_{xx}$ and the term $\frac{h_{xx}}{h_x}$ in the bracket scale as a so they become higher-order terms compared with the first one. As argued in Xiang and E (2004), the term $a\frac{h_{xx}}{h_x}$ is the correction to the misfit elastic energy density due to the discrete nature of the stepped surface. Although it is small compared to the leading-order term $H(h_x)$, it is comparable with the term $a h_x h_{xx}$, which comes from the broken bond elastic interaction between steps. When formally ignoring these terms with small a-dependent amplitude, the PDE analysis for $h_t = -H(h_x)_{xx}$ is easy because the operator $H(\cdot)_x$ is a negative operator.

[1] Compared to Xiang (2002), we drop all the physical constants that are mathematically unimportant.

Recently, motivated by the PDE (1.5) proposed by Xiang (2002), Dal Maso et al. (2014) studied the weak solution of [2]

$$h_t = \left(-\frac{2\pi}{L}H(h_x) + \left(3h_x + \frac{1}{h_x}\right)h_{xx}\right)_{xx}, \tag{1.7}$$

in terms of a variational inequality. Note that all the coefficients in this PDE are $O(1)$, unlike the PDE (1.5). They validated (1.7) analytically by verifying the positivity of h_x. Rather remarkably, they found an approximation problem and proved the limit of the solution to the approximation problem also satisfies the weak version of variational inequality, which is satisfied by strong solution. Moreover, Fonseca et al. (2015) obtained the existence and uniqueness of the weak solution. They applied Rothe method and truncation method to carefully deal with the singularity term.

Our goal is to rigorously prove the continuum limit of BCF-type models for step flow. While it would be nice to recover (1.5) using the scaling considered in Xiang (2002), it is quite challenging (if not impossible) since the PDE (1.5) involves two scales, corresponding to the three terms on the right-hand side:

$$O(1): \quad H(h_x); \qquad O(a): \quad h_x h_{xx}; \qquad O(a): \quad \frac{h_{xx}}{h_x}.$$

Instead, we follow the scaling of the PDE (1.7) considered in Dal Maso et al. (2014), Fonseca et al. (2015). We will derive (1.7) as the continuum limit from a slightly modified BCF-type mesoscopic model: We consider the step-flow ODE (1.4) with a rescaled time, i.e.,

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{1}{a}\left(\frac{f_{i+1}-f_i}{x_{i+1}-x_i} - \frac{f_i - f_{i-1}}{x_i - x_{i-1}}\right), \quad i = 1, \ldots, N, \tag{1.8}$$

with a modified chemical potential

$$f_i := -\frac{2}{L}\sum_{j\neq i}\frac{a}{x_j - x_i} + \left(\frac{1}{x_{i+1}-x_i} - \frac{1}{x_i - x_{i-1}}\right) + \left(\frac{a^2}{(x_{i+1}-x_i)^3} - \frac{a^2}{(x_i - x_{i-1})^3}\right); \tag{1.9}$$

see Sect. 4. The first term in f_i comes from the misfit elastic interaction between the steps, which is an attractive interaction. The second and third terms come from the broken bond elastic interaction between steps, which are repulsive terms. Different from XIANG's chemical potential in Xiang (2002), we choose the scaling so that the attractive and repulsive interactions have the same order as $a \to 0$. We add the repulsive term $\frac{1}{x_{i+1}-x_i} - \frac{1}{x_i - x_{i-1}}$ to cancel a singularity from the first term, which seems to be necessary. Moreover, to ease the mathematical derivation, we restrict the repulsive terms to the nearest neighbor, which is the dominant contribution.

[2] For the convenience of calculation, we set the coefficients slightly different from Dal Maso et al. (2014). Moreover, instead of taking h to be increasing as in Dal Maso et al. (2014), we take h to be decreasing corresponding to physical interpretation of h being the height of the vicinal surface, which is the same convention as Xiang (2002), Xiang and E (2004).

Our modified ODE system, from both the view of chemical potential and free energy, is balanced in order. Therefore unlike the original ODE systems which (at least heuristically) lead to a PDE with multiple scales, our system converges to PDE (1.7) in the limit. We are also able to obtain the convergence rate of order a for local strong solution of the continuum PDE.

For the study of the PDE (1.7), we discover four variational structures with four corresponding energy functionals, in terms of step height h, step location ϕ, step density ρ and anti-derivative of h, denoted as u. Those four kinds of descriptions are equivalent rigorously for strong local solution, but it is convenient to use different one when studying different aspects of our problem. The height h is the original variable indicating the evolution of surface height, while it is a better idea to use ρ and u to study the strong local solution of continuum model (1.7) due to its concise variational structure. In the proof of convergence rate in Sects. 4, 5 and 6, since the original discrete model is described by each step location x_i, it is more natural to use the variational structure of step location ϕ, which is the inverse function of step height h, i.e.,

$$\alpha = h(\phi(\alpha, t), t), \quad \forall \alpha. \tag{1.10}$$

For the properties of local strong solution of continuum PDE (1.7), we used the variational structures for u and ρ to establish some a priori estimates and then obtain the existence and uniqueness for local strong solution to the continuum PDE; see Sect. 3. We state the main result of Sect. 3 below, with the notations $I := [0, L]$,

$$W^{k,p}_{\mathrm{per}^\star}(I) := \left\{u(x) \in W^{k,p}_{\mathrm{loc}}(\mathbb{R});\ u(x+L) - u(x) = -1\right\}, \tag{1.11}$$

and

$$W^{k,p}_{\mathrm{per}_0}(I) := \{u \in W^{k,p}(I);\ u \text{ is } L\text{-periodic and mean value zero in one period}\}. \tag{1.12}$$

Standard notations for Sobolev spaces are assumed above.

Theorem 1.1 *Assume $h^0 \in W^{m,2}_{per^\star}(I)$, $h^0_x \le \beta$, for some constant $\beta < 0$, $m \in \mathbb{Z}$, $m \ge 6$. Then there exists time $T_m > 0$ depending on β, $\|h^0\|_{W^{m,2}_{per^\star}}$ such that*

$$h \in L^\infty([0, T_m]; W^{m,2}_{per^\star}(I)) \cap L^2([0, T_m]; W^{m+2,2}_{per^\star}(I)) \cap C([0, T_m]; W^{m-4,2}_{per^\star}(I)),$$
$$h_t \in L^\infty([0, T_m]; W^{m-4,2}_{per}(I))$$

is the unique strong solution of (1.7) with initial data h^0, and h satisfies

$$h_x \le \frac{\beta}{2}, \quad a.e.\ t \in [0, T_m],\ x \in [0, L]. \tag{1.13}$$

Moreover, we also study the stability of the linearized ϕ-PDE. This is important in the construction of approximate solutions to the PDE with high-order consistency, which is crucial in the proof of convergence.

For the convergence result of mesoscopic model, we first testify our modified ODE system has a global-in-time solution; see Proposition 4.1. More explicitly, we prove that the steps and terraces will keep monotone if we have monotone initial data. This is consistent with the positivity of step density ρ of the PDE. Then we calculate the consistency of the step location continuum equation and ODE system till order a; see Theorem 5.1. However, due to the nonlinearity and fourth-order derivative in our problem, we need to utilize a priori assumption method and construct an auxiliary solution with high-order consistency. By establishing the stability of the linearized ODE system and carefully calculating the Hessian of coefficient matrix of ODE system, which is a third-order tensor, we finally get the convergence rate $O(a)$ of modified ODE system to its continuum PDE limit.

Recall the definition (1.2) and (1.10). Denote

$$\alpha_i = h(x_i(0), 0) = \frac{N-i}{N},$$

and

$$\phi_i(t) = \phi(\alpha_i, t).$$

We state the main convergence result in this work as follows:

Theorem 1.2 *Let the step height be $a = \frac{1}{N}$. Assume for some constant $\beta < 0$, some $m \in \mathbb{N}$ large enough, the initial datum $h(0) \in W^{m,2}_{per^\star}(I)$ satisfies*

$$h_x(0) \leq \beta < 0. \tag{1.14}$$

Let $h(x,t)$ be the exact solution of (1.7) on $[0, T_m]$, where T_m is the maximal existence time for strong solution defined in Theorem 1.1. Let $\phi(\alpha, t)$ be the inverse function of $h(x,t)$ defined in (1.10), whose nodal values are denoted as $\phi_N(t) := \{\phi(\alpha_i, t),\ i = 1, \ldots, N\}$. Let $x(t) = (x_1(t), \ldots, x_N(t))$ be the solution to ODE (1.8) with f_i defined in (1.9) and initial data $x(0) = \phi_N(0)$. Then there exists N_0 large enough such that for $N > N_0$, we have $x(t)$ converges to $\phi(\alpha, t)$ with convergence rate a, in the sense of

$$\|x(t) - \phi_N(t)\|_{\ell^2} \leq C\left(\beta, \|h^0\|_{W^{m,2}_{per^\star}}\right) a, \quad \text{for } t \in [0, T_m], \tag{1.15}$$

where $C(\beta, \|h^0\|_{W^{m,2}_{per^\star}})$ is a constant depending only on β and $\|h^0\|_{W^{m,2}_{per^\star}}$.

Several remarks of the main result are in order.

Remark 1 In fact, we can achieve a better convergence rate $O(a^2)$, if f_i is modified to be

$$\begin{aligned}\widetilde{f}_i := & -\frac{2}{L}\sum_{j\neq i}\frac{a}{x_j - x_i} + \left(1 - \frac{a}{2}\right)\left(\frac{1}{x_{i+1} - x_i} - \frac{1}{x_i - x_{i-1}}\right) \\ & + \left(\frac{a^2}{(x_{i+1} - x_i)^3} - \frac{a^2}{(x_i - x_{i-1})^3}\right).\end{aligned}$$

Compared with (1.9), the coefficient of the second term is changed from 1 to $1 - \frac{a}{2}$. This is done to better correct the error from the discretization of the Hilbert transform as $a \to 0$ (recall the second term in (1.9) is introduced to correct the singularity from the first term). In fact, by Lemma 5.2, we know the leading error $\frac{a}{2}\frac{\phi_{\alpha\alpha}}{\phi_\alpha^2}$ in Lemma 5.3 can be removed by such a correction term. Hence we can get $O(a^2)$ consistency in Sect. 5, and consequently, the convergence rate can be improved to $O(a^2)$ in Theorem 1.2 for the modified microscopic model.

Remark 2 Theorem 1.2 is a result of local convergence to strong solutions to the PDE. The global convergence of the ODE system to the (weak) global-in-time solution to the PDE (1.7) is more challenging and will be left for the future. We hope the additional understanding of the variational structures of the PDE (1.7) provided in this work would help the future investigation on global convergence.

Remark 3 To avoid unnecessary technical complications and to make the presentation of the convergence result clear, in this work we do not try to optimize the initial regularity that is needed in the Theorem 1.2. We just set m to be large enough, so that we may assume sufficient regularity of the solution.

While a comprehensive review of the vast literature of crystal growth is beyond the scope of this work, let us review here some related works mostly in the mathematical literature. Besides the work of Xiang (2002), the derivation of the continuum limit of BCF models has also been considered in other works, see, e.g., Tang (1997), E and Yip (2001), Shenoy and Freund (2002), Margetis and Nakamura (2011). However, as far as we know, the derivation has not been done on the rigorous level, and moreover, the convergence rate is provided here, which seems to be missing before in the literature. The idea using step location for formal asymptotic analysis was inspired by Xiang (2002). In order to get the convergence rate rigorously, we find it is better to first study the continuum PDE for the inverse function ϕ, instead of the height h. Recently, in the attachment–detachment-limited (ADL) regime, Al Hajj Shehadeh et al. (2011) studied the continuum limit of self-similar solution and obtained the convergence rate. Related to the stability analysis, the linear stability of thin film (known as the ATG instability) has been analyzed in previous works, see, e.g., Xiang and E (2004), Grinfeld (1986), Srolovitz (1989). While we consider here the one spatial-dimensional models, the asymptotic derivation of two-dimensional continuum models has been considered in Margetis and Kohn (2006) and Xu and Xiang (2009), and the rigorous aspects of these results will be interesting future research directions.

For the discrete BCF model considered in Xiang (2002), very recently, Luo et al. (2016) rigorously proved the step bunch phenomenon, which characterized the limiting behavior of the system as $t \to \infty$. They have also connected the step bunching with continuum models through a Γ-convergence argument [16]. These works motivate further study of the continuum limit of mesoscopic models of crystal growth.

Let us also mention that while our starting point is step-flow models, the derivation of the continuum limit can be also considered starting from a more atomistic description, such as a kinetic Monte Carlo-type model. See the works Guo et al. (1988), Yau (1991), Funaki and Spohn (1997), Nishikawa (2002) and more recently Marzuola and

Weare (2013). See also a recent work that aims to derive BCF-type models from a kinetic Monte Carlo lattice model (Lu et al. 2015).

The rest of this paper is organized as follows: In Sect. 2, after setting up some notations, we introduce four equivalent forms of continuum PDE (1.7) and their variational structures. Section 3 is devoted to establishing the existence, uniqueness and stability for local strong solution of the PDE. We then introduce the modified step-flow ODE in Sect. 4 and state the global existence result for the modified ODE system. Section 5 is devoted to proving the consistency result for ODE system and its continuum limit PDE. Finally, by constructing an auxiliary solution with high-order consistency, we obtain the convergence rate of the modified ODE to its continuum PDE limit in Sect. 6, which completes the proof of our main result Theorem 1.2.

2 The Continuum Model

In this section, we discuss the properties of the continuum model. Besides using the height profile h, it would be useful to rewrite the dynamics in a few equivalent ways. Let us introduce the following definitions

- Step location $\phi(\alpha, t)$, the inverse function of h:

$$\alpha = h(\phi(\alpha, t), t), \quad \forall \alpha;$$

- Step density $\rho(x, t)$, the (negative) gradient of h:

$$\rho(x, t) = -h_x(x, t); \tag{2.1}$$

- $u(x, t)$, the (negative) anti-derivative of h:

$$h(x, t) = -u_x(x, t) - bx - k_0, \tag{2.2}$$

where $b,\ k_0$ are constants chosen to guarantee the periodicity of u_x.

Now we establish the variational structures for $h,\ u,\ \rho,\ \phi$. In Sect. 3, it will be convenient to use ρ-equation and u-equation, while it will be proper to use ϕ-equation when studying the continuum limit in Sects. 4, 5, 6.

2.1 Equation for Height Profile h

Let us consider the PDE for the height profile

$$h_t = \left(-\frac{2\pi}{L}H(h_x) + \left(3h_x + \frac{1}{h_x}\right)h_{xx}\right)_{xx}.$$

As mentioned in Introduction, the coefficients here are independent of a. In Sect. 5, we will show that this continuum PDE can be derived as the limit of a BCF-type discrete atomistic model.

First we observe that the evolution equation (1.7) has a variational structure. Define the total energy E_h as a functional of h:

$$E_h(h) := \int_0^L \left(\frac{1}{L} \int_0^L \ln \left| \sin\left(\frac{\pi}{L}(x-y)\right)\right| h_x h_y \,\mathrm{d}y - h_x \ln(-h_x) - \frac{h_x^3}{2} \right) \mathrm{d}x. \tag{2.3}$$

Then we have

$$h_t = \mu_{xx} = \left(\frac{\delta E_h}{\delta h}\right)_{xx}, \tag{2.4}$$

where the chemical potential μ is given by

$$\mu := \frac{\delta E_h}{\delta h} = -\,\mathrm{PV} \int_0^L \frac{2\pi}{L^2} \cot \frac{\pi(x-y)}{L} h_y(y)\,\mathrm{d}y + \frac{h_{xx}}{h_x} + 3 h_x h_{xx}. \tag{2.5}$$

To see this, let us calculate in Lemma 2.1 the functional derivative $\frac{\delta E_h^0}{\delta h}$ for

$$E_h^0(h) := \int_0^L \int_0^L \ln \left| \sin \frac{\pi(x-y)}{L} \right| h_x h_y \,\mathrm{d}x\,\mathrm{d}y. \tag{2.6}$$

The derivative of the other two terms in E_h is straightforward.

Lemma 2.1 *Assume $h(x) \in C^2([0, L])$.*
We have

$$\frac{\delta E_h^0}{\delta h} = -\,\mathrm{PV} \int_0^L \frac{2\pi}{L} \cot \frac{\pi(x-y)}{L} h_y(y)\,\mathrm{d}y.$$

Proof First denote

$$E_h^\delta(h) := \int_0^L \left(\int_0^{x-\delta} + \int_{x+\delta}^L \right) \ln \left| \sin \frac{\pi(x-y)}{L} \right| h_x h_y \,\mathrm{d}y\,\mathrm{d}x.$$

By the definition of the principal value integral, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E_h^0(h+\varepsilon\tilde{h}) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \lim_{\delta\to 0^+} E_h^\delta(h+\varepsilon\tilde{h}),$$

and since $\ln|\sin x|$ is even, we have

$$\lim_{\delta\to 0^+} \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E_h^\delta(h+\varepsilon\tilde{h}) = \lim_{\delta\to 0^+} \int_0^L \left(\int_0^{x-\delta} + \int_{x+\delta}^L \right) \frac{-2\pi}{L} \cot \frac{\pi(x-y)}{L} h_y(y)\tilde{h}(x)\,\mathrm{d}y\,\mathrm{d}x. \tag{2.7}$$

Now we claim

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \lim_{\delta\to 0^+} E_h^\delta(h+\varepsilon\tilde{h}) = \lim_{\delta\to 0^+} \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E_h^\delta(h+\varepsilon\tilde{h}).$$

Obviously, $E_h^\delta(h+\varepsilon\tilde{h})$ is continuous with respect to δ. It suffices to show that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}E_h^\delta(h+\varepsilon\tilde{h})$ is also continuous respect to δ. Hence from (2.7), it suffices to prove

$$\lim_{\delta\to 0^+}\int_0^L\int_{x-\delta}^{x+\delta}\frac{\pi}{L}\cot\frac{\pi(x-y)}{L}h_y(y)\tilde{h}(x)\,\mathrm{d}y\,\mathrm{d}x=0.$$

Indeed

$$\begin{aligned}&\int_0^L\int_{x-\delta}^{x+\delta}\frac{\pi}{L}\cot\frac{\pi(x-y)}{L}h_y(y)\tilde{h}(x)\,\mathrm{d}y\,\mathrm{d}x\\&=\int_0^L-\ln\left|\sin\frac{\pi(x-y)}{L}\right|h_y(y)\bigg|_{y=x-\delta}^{x+\delta}\tilde{h}(x)\,\mathrm{d}x\\&\quad+\int_0^L\int_{x-\delta}^{x+\delta}\ln\left|\sin\frac{\pi(x-y)}{L}\right|h_{yy}(y)\tilde{h}(x)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Notice that $h(x)\in C^2([0,L])$. Let $\delta\to 0$. The first term tends to zero by Taylor expansion, and the second term tends to zero as the integrand is integrable. □

Note that the energy E_h we use here has a slightly different form compared to the one in Xiang (2002), denoted by $\bar{E}_h(h)$, which reads in the periodic setting as

$$\bar{E}_h(h)=\int_0^L\left(-\frac{\pi}{L}\left(h+\frac{x}{L}\right)H(h_x)-h_x\ln(-h_x)-\frac{h_x^3}{2}\right)\mathrm{d}x. \tag{2.8}$$

In fact, the two energy functionals only differ by a null Lagrangian, as we show below, so we prefer the more symmetric expression E_h.

Lemma 2.2 *Let*

$$W(h):=\frac{1}{L^2}\int_0^L\int_0^L\ln\left|\sin\frac{\pi(x-y)}{L}\right|h_y\,\mathrm{d}x\,\mathrm{d}y. \tag{2.9}$$

Then we have

$$E_h(h)=\bar{E}_h(h)+W(h),$$

and

$$\frac{\delta E_h}{\delta h}=\frac{\delta\bar{E}_h}{\delta h}.$$

Proof First by the definition of the periodic Hilbert transform,

$$\bar{E}_h(h)=\int_0^L\left(-\frac{\pi}{L^2}\left(h+\frac{x}{L}\right)\mathrm{PV}\int_0^L\cot\frac{\pi(x-y)}{L}h_y\,\mathrm{d}y-h_x\ln(-h_x)-\frac{h_x^3}{2}\right)\mathrm{d}x.$$

Notice that

$$
\begin{aligned}
&\int_0^L \left(-\frac{\pi}{L^2}\left(h+\frac{x}{L}\right)\mathrm{PV}\int_0^L \cot\frac{\pi(x-y)}{L} h_y\,\mathrm{d}y\right)\mathrm{d}x \\
&\quad = -\frac{1}{L}\int_0^L \left(\left(h+\frac{x}{L}\right)\ln\left|\sin\frac{\pi(x-y)}{L}\right|\Bigg|_0^L\right. \\
&\qquad \left. -PV\int_0^L \left(h_x+\frac{1}{L}\right)\ln\left|\sin\frac{\pi(x-y)}{L}\right|\,\mathrm{d}x\right) h_y\,\mathrm{d}y \\
&\quad = \frac{1}{L}\int_0^L\int_0^L \ln\left|\sin\frac{\pi(x-y)}{L}\right| h_x h_y\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \frac{1}{L^2}\int_0^L\int_0^L \ln\left|\sin\frac{\pi(x-y)}{L}\right| h_y\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

where we have used that $h+\frac{x}{L}$ is L-periodic function. Therefore for W defined in (2.9), we get

$$E_h(h) = \bar{E}_h(h) + W(h).$$

Similar to the proof of Lemma 2.1, we can see

$$
\begin{aligned}
\left\langle \frac{\delta W}{\delta h}, \tilde{h}\right\rangle &= \frac{1}{L^2}\int_0^L\int_0^L \ln\left|\sin\frac{\pi(x-y)}{L}\right|\,\mathrm{d}x\tilde{h}_y\,\mathrm{d}y, \\
&= \frac{1}{L^2}\int_0^L \tilde{h}\ln\left|\sin\frac{\pi(x-y)}{L}\right|\Bigg|_0^L\,\mathrm{d}x \\
&\quad - \int_0^L \mathrm{PV}\int_0^L \frac{2\pi}{L^2}\cot\frac{\pi(x-y)}{L}\,\mathrm{d}x\tilde{h}(y)\,\mathrm{d}y \\
&= 0.
\end{aligned}
$$

Hence $W(h)$ is a null Lagrangian. □

2.2 Equation for Step Location Function ϕ

Consider the step location function ϕ, which is defined in (1.10) as the inverse function of h. From the definition, we have

$$\phi_t = -\frac{h_t}{h_x}, \qquad 1 = h_x\phi_\alpha, \qquad h_{xx} = -\frac{\phi_{\alpha\alpha}}{\phi_\alpha^3}. \tag{2.10}$$

Then changing variable from h to ϕ in (2.4), we have

$$\phi_t = -\phi_\alpha\mu_{xx} = -\partial_\alpha\left(\frac{1}{\phi_\alpha}\mu_\alpha\right), \tag{2.11}$$

due to (2.10) and the chain rule $\mu_x = \mu_\alpha \frac{1}{\phi_\alpha}$. Note that this immediately implies that $\int_0^1 \phi \, \mathrm{d}\alpha$ is a constant of motion.

The equation of ϕ (2.11) also has a variational structure. To this end, let us rewrite the energy in terms of ϕ such that $E_\phi(\phi) = E_h(h)$:

$$E_\phi(\phi) = \int_0^1 \left(\frac{1}{L} \int_0^1 \ln \left| \sin \frac{\pi(\phi(\alpha) - \phi(\beta))}{L} \right| \mathrm{d}\beta - \ln(-\phi_\alpha) + \frac{1}{2\phi_\alpha^2} \right) \mathrm{d}\alpha. \quad (2.12)$$

We will show that

$$\phi_t = -\phi_\alpha \mu_{xx} = -\partial_\alpha \left(\frac{1}{\phi_\alpha} \left(\frac{\delta E_\phi}{\delta \phi} \right)_\alpha \right). \quad (2.13)$$

Similar to the proof of Lemma 2.1, let us first calculate $\frac{\delta E_\phi^0}{\delta \phi}$, where

$$E_\phi^0(\phi) := \int_0^1 \int_0^1 \ln \left| \sin \frac{\pi(\phi(\alpha) - \phi(\beta))}{L} \right| \mathrm{d}\alpha \, \mathrm{d}\beta.$$

Lemma 2.3 *Assume $h(x) \in C^2([0, L])$, and there exists a constant $C > 0$ such that $|h_x| \geq C$. We have*

$$\frac{\delta E_\phi^0}{\delta \phi} = \mathrm{PV} \int_0^1 \frac{2\pi}{L} \cot \frac{\pi(\phi(\alpha) - \phi(\beta))}{L} \, \mathrm{d}\beta.$$

Proof First denote

$$E_\phi^\delta(\phi) := \int_0^1 \left(\int_0^{\beta-\delta} + \int_{\beta+\delta}^1 \right) \ln \left| \sin \frac{\pi(\phi(\alpha) - \phi(\beta))}{L} \right| \mathrm{d}\alpha \, \mathrm{d}\beta.$$

It is obvious to see that

$$\left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} E_\phi^0(\phi + \varepsilon\tilde{\phi}) = \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} \lim_{\delta \to 0} E_\phi^\delta(\phi + \varepsilon\tilde{\phi}),$$

and

$$\left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} E_\phi^\delta(\phi + \varepsilon\tilde{\phi}) = \int_0^1 \left(\int_0^{\beta-\delta} + \int_{\beta+\delta}^1 \right) \frac{\pi}{L} \cot \frac{\pi(\phi(\alpha) - \phi(\beta))}{L} (\tilde{\phi}(\alpha) - \tilde{\phi}(\beta)) \, \mathrm{d}\alpha \, \mathrm{d}\beta.$$

Now we claim

$$\left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} \lim_{\delta \to 0^+} E_\phi^\delta(\phi + \varepsilon\tilde{\phi}) = \lim_{\delta \to 0^+} \left. \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \right|_{\varepsilon=0} E_\phi^\delta(\phi + \varepsilon\tilde{\phi}).$$

Obviously, $E^\delta_\phi(\phi+\varepsilon\tilde{\phi})$ is continuous with respect to δ. It is sufficient to proof $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}E^\delta_\phi(\phi+\varepsilon\tilde{\phi})$ is also continuous with respect to δ. In fact, since $\cot x$ is odd,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}E^\delta_\phi(\phi+\varepsilon\tilde{\phi}) = 2\int_0^1\left(\int_0^{\beta-\delta}+\int_{\beta+\delta}^1\right)\frac{\pi}{L}\cot\frac{\pi(\phi(\alpha)-\phi(\beta))}{L}\tilde{\phi}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

Hence it is sufficient to proof

$$\lim_{\delta\to0^+}\int_0^1\int_{\beta-\delta}^{\beta+\delta}\frac{\pi}{L}\cot\frac{\pi(\phi(\alpha)-\phi(\beta))}{L}\tilde{\phi}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta=0.$$

In fact,

$$\begin{aligned}&\int_0^1\int_{\beta-\delta}^{\beta+\delta}\frac{\pi}{L}\cot\frac{\pi(\phi(\alpha)-\phi(\beta))}{L}\tilde{\phi}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta\\ &=\int_0^1\frac{\tilde{\phi}(\alpha)}{\phi_\alpha(\alpha)}\ln\left|\sin\frac{\pi(\phi(\alpha)-\phi(\beta))}{L}\right|\Bigg|_{\alpha=\beta-\delta}^{\beta+\delta}\mathrm{d}\beta\\ &\quad-\int_0^1\int_{\beta-\delta}^{\beta+\delta}\ln\left|\sin\frac{\pi(\phi(\alpha)-\phi(\beta))}{L}\right|\left(\frac{\tilde{\phi}(\alpha)}{\phi_\alpha(\alpha)}\right)_\alpha\mathrm{d}\alpha\,\mathrm{d}\beta.\end{aligned}$$

As $\delta\to0$, the first term tends to zero by Taylor expansion. $\left|\left(\frac{\tilde{\phi}(\alpha)}{\phi_\alpha(\alpha)}\right)_\alpha\right|$ is bounded since $h(x)\in C^2([0,L])$ and $|h_x|\geq C>0$, so the second term tends to zero as the integrand is integrable. □

Hence we have

$$\frac{\delta E_\phi}{\delta\phi}=\frac{2\pi}{L^2}\,\mathrm{PV}\int_0^1\cot\frac{\pi(\phi(\alpha)-\phi(\beta))}{L}\,\mathrm{d}\beta-\frac{\phi_{\alpha\alpha}}{\phi_\alpha^2}-3\frac{\phi_{\alpha\alpha}}{\phi_\alpha^4}. \tag{2.14}$$

It remains to show that $\mu=\frac{\delta E_\phi}{\delta\phi}$, *i.e.*, $\frac{\delta E_\phi}{\delta\phi}=\frac{\delta E_h}{\delta h}$. For $\tilde{\phi}$, $\tilde{h}$ satisfying

$$\alpha=(h+\varepsilon\tilde{h})\circ(\phi+\varepsilon\tilde{\phi}),$$

Taylor expansion shows that

$$0=h_x\tilde{\phi}+\tilde{h}.$$

Thus, by (2.10), we have

$$\begin{aligned}&\tilde{\phi}=-\phi_\alpha\tilde{h},\\ &E_\phi(\phi+\varepsilon\tilde{\phi})=E_h(h+\varepsilon\tilde{h}).\end{aligned} \tag{2.15}$$

Hence

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E_\phi(\phi+\varepsilon\tilde{\phi}) &= D_\phi E_\phi\cdot\tilde{\phi}\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E_h(h+\varepsilon\tilde{h}) = D_h E_h\cdot\tilde{h},\end{aligned}\tag{2.16}$$

where $D_hE_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Fréchet differential, i.e., $D_hE_h\cdot\tilde{h}$ is the dual pair which means the first-order variation of E_h at h along the direction of $\tilde{h}$.

By Riesz representation theorem, there exists $\nabla_h E_h \in L^2([0, L],\ \mathrm{d}x)$, such that

$$D_hE_h\cdot\tilde{h} = \int_0^L \nabla_h E_h\tilde{h}\,\mathrm{d}x,$$

where $\nabla_h E_h$ is gradient of $E_h(h)$ in $L^2([0, L],\ \mathrm{d}x)$, which is just what we denoted as $\frac{\delta E_h}{\delta h}$.

Similarly, there exists $\nabla_\phi E_\phi \in L^2([0, 1], |\phi_\alpha|\,\mathrm{d}\alpha)$, such that

$$D_\phi E_\phi\cdot\tilde{\phi} = \int_0^1 \nabla_\phi E_\phi\tilde{\phi}|\phi_\alpha|\,\mathrm{d}\alpha = \int_0^1 -\nabla_\phi E_\phi\tilde{\phi}\phi_\alpha\,\mathrm{d}\alpha.$$

where $\nabla_\phi E_\phi$ is gradient of $E_\phi(\phi)$ in $L^2([0, 1], |\phi_\alpha|\,\mathrm{d}\alpha)$.

Combining (2.15) and (2.16), we get

$$\nabla_\phi E_\phi = -\frac{1}{\phi_\alpha}\nabla_h E_h\circ\phi.$$

Again we define $\frac{\delta E_\phi}{\delta\phi}$ as gradient of $E_\phi(\phi)$ in $L^2([0, 1],\ \mathrm{d}\alpha)$. Noticing (2.15), we have

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E_\phi(\phi+\varepsilon\tilde{\phi}) &= \int_0^1 \frac{\delta E_\phi}{\delta\phi}\tilde{\phi}\,\mathrm{d}\alpha\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} E_h(h+\varepsilon\tilde{h}) = \int_0^L \nabla_h E_h\tilde{h}\,\mathrm{d}x\\ &= \int_0^1 \frac{\delta E_h}{\delta h}\tilde{\phi}\,\mathrm{d}\alpha.\end{aligned}$$

Hence

$$\frac{\delta E_h}{\delta h}\circ\phi = \frac{\delta E_\phi}{\delta\phi} \in L^2([0, 1],\ \mathrm{d}\alpha),$$

and

$$\mu = \frac{\delta E_h}{\delta h}\circ\phi = \nabla_h E_h\circ\phi = -\phi_\alpha\nabla_\phi E_\phi = \frac{\delta E_\phi}{\delta\phi}.$$

Therefore we conclude that (2.11) is equivalent to (2.13). Moreover, we obtain energy identity for (2.13) as

$$\frac{\mathrm{d}E_\phi}{\mathrm{d}t} = \int_0^1 \frac{\delta E_\phi}{\delta \phi}\phi_t \mathrm{d}\alpha = \int_0^1 \frac{1}{\phi_\alpha}\left(\left(\frac{\delta E_\phi}{\delta \phi}\right)_\alpha\right)^2 \mathrm{d}\alpha. \tag{2.17}$$

2.3 Equation for Step Density ρ

Now consider the step density ρ. From the definition, rewriting the energy in terms of ρ, we obtain

$$E_\rho(\rho) := \int_0^L \left(\frac{1}{L}\int_0^L \ln\left|\sin\left(\frac{\pi}{L}(x-y)\right)\right| \rho(x)\rho(y)\,\mathrm{d}y + \rho(x)\ln\rho(x) + \frac{\rho(x)^3}{2}\right)\mathrm{d}x, \tag{2.18}$$

$$\frac{\delta E_\rho}{\delta \rho} = \int_0^L \frac{2}{L}\ln\left|\sin\left(\frac{\pi}{L}(x-y)\right)\right| \rho(y)\,\mathrm{d}y + \ln\rho(x) + 1 + \frac{3}{2}\rho(x)^2,$$

and

$$\left(\frac{\delta E_\rho}{\delta \rho}\right)_x = \mathrm{PV}\int_0^L \frac{2\pi}{L^2}\cot\frac{\pi(x-y)}{L}\rho(y)\,\mathrm{d}y + \frac{\rho_x}{\rho} + 3\rho_x\rho = \mu. \tag{2.19}$$

Similar to the proof of Lemma 2.1, we can define

$$\mathrm{PV}\int_0^L \cot\frac{\pi(x-y)}{L}\rho(y)\,\mathrm{d}y = \lim_{\delta\to 0^+}\left(\int_0^{x-\delta} + \int_{x+\delta}^L\right)\cot\frac{\pi(x-y)}{L}\rho(y)\,\mathrm{d}y.$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\lim_{\delta\to 0^+}&\left(\int_0^{x-\delta} + \int_{x+\delta}^L\right)\ln\left|\sin\frac{\pi(x-y)}{L}\right|\rho(y)\,\mathrm{d}y\\ &= \lim_{\delta\to 0^+}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{x-\delta} + \int_{x+\delta}^L\right)\ln\left|\sin\frac{\pi(x-y)}{L}\right|\rho(y)\,\mathrm{d}y.\end{aligned}$$

Hence we also obtain a variational structure for ρ and (2.4) becomes

$$\rho_t = -\mu_{xxx} = -\left(\frac{\delta E_\rho}{\delta \rho}\right)_{xxxx}. \tag{2.20}$$

This also shows that $\int_0^L \rho\,\mathrm{d}x$ is a constant of motion.

2.4 Equation for *u*

Finally, from definition of u, the energy can be rewritten in terms of u as

$$\begin{aligned} E_u(u) = &\int_0^L \left(\frac{1}{L}\int_0^L \ln\left|\sin\left(\frac{\pi}{L}(x-y)\right)\right| (u_{xx}+b)(u_{yy}+b)\mathrm{d}y \right. \\ &\left. + (u_{xx}+b)\ln(u_{xx}+b) + \frac{(u_{xx}+b)^3}{2}\right)\mathrm{d}x, \end{aligned} \tag{2.21}$$

$$\frac{\delta E_u}{\delta u} = \frac{2\pi}{L}H(u_{xx})_x + \left(\ln(u_{xx}+b) + \frac{3}{2}(u_{xx}+b)^2 + 1\right)_{xx} = \mu_x.$$

Hence we also obtain a variational structure for u and (2.4) becomes

$$u_t = -\frac{\delta E_u}{\delta u}. \tag{2.22}$$

2.5 Equivalence of the Formulations

We end this section with the rigorous justification of the equivalence of the above formulations.

Recall the notations for $W^{k,p}_{\mathrm{per}^\star}(I)$, $W^{k,p}_{\mathrm{per}_0}(I)$ in (1.11) and (1.12). If $k < 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, $W^{k,p}$ is the dual of $W^{-k,q}$. Denote

$$\Phi(\xi) := \begin{cases} \xi\ln\xi + \frac{\xi^3}{2}, & \xi > 0, \\ 0, & \xi = 0, \\ +\infty, & \xi < 0, \end{cases}$$

and

$$\Phi_b(\xi) := \Phi(\xi + b).$$

By the definition (2.18), we have

$$E_\rho(\rho) = \int_0^L \left(\frac{1}{L}\int_0^L \ln\left|\sin\left(\frac{\pi}{L}(x-y)\right)\right| \rho(x)\rho(y)\,\mathrm{d}y + \Phi(\rho)\right)\mathrm{d}x. \tag{2.23}$$

By (2.21), we have

$$E_u(u) = \int_0^L \left(\frac{1}{L}\int_0^L \ln\left|\sin\left(\frac{\pi}{L}(x-y)\right)\right| (u_{xx}+b)(u_{yy}+b)\,\mathrm{d}y + \Phi_b(u_{xx})\right)\mathrm{d}x.$$

Since

$$\frac{\delta E_u(u)}{\delta u} = \frac{2\pi}{L}H(u_{xx})_x + ({\Phi_b}'(u_{xx}))_{xx},$$

Eq. (2.22) can be recast as

$$u_t + \frac{2\pi}{L} H(u_{xx})_x + (\Phi_b{}'(u_{xx}))_{xx} = 0. \tag{2.24}$$

In order to study the problem (1.7) in periodic and mean value zero setup, we establish first, similar to Dal Maso et al. (2014), that

Proposition 2.4 *For any integer $m \geq 1$, any $T > 0$ and some constant $\beta < 0$, the following conditions are equivalent:*

(a) There exists $h \in L^\infty([0,T]; W^{m,3}_{per^\star}(I))$ with $h_t \in L^\infty([0,T]; W^{m-4,3/2}_{per}(I))$ a solution of (1.7) satisfying

$$h_x(x,t) \leq \beta < 0 \quad a.e.\ x \in \mathbb{R},\ t \in [0,T].$$

(b) Set $b := \frac{1}{L} > 0$. There exists $u \in L^\infty([0,T]; W^{m+1,3}_{per_0}(I))$ with $u_t \in L^\infty([0,T]; W^{m-3,3/2}_{per_0}(I))$ a solution of (2.24) satisfying

$$u_{xx}(x,t) + b \geq -\beta > 0 \quad a.e.\ x \in \mathbb{R},\ t \in [0,T].$$

(c) There exists $\rho \in L^\infty([0,T]; W^{m-1,3}_{per}(I))$ with $\rho_t \in L^\infty([0,T]; (W^{m-5,3/2}_{per}(I)))$ a solution of (2.20) satisfying

$$\rho(x,t) \geq -\beta > 0 \quad a.e.\ x \in \mathbb{R},\ t \in [0,T],$$

and

$$\int_0^L \rho(x,t)\,\mathrm{d}x = 1.$$

Proof Step 1. For (a)⇒(c), we simply take

$$\rho(t,x) := -h_x(t,x) = u_{xx}(t,x) + b \tag{2.25}$$

and then (2.19) shows that ρ satisfies (c).

For (c)⇒(a), we take

$$h(x,t) = -\int_0^x \rho(s,t)\mathrm{d}s + k_2(t),$$

with

$$k_2(t) = \frac{1}{L}\int_0^L \int_0^x \rho(y,t)\,\mathrm{d}y\,\mathrm{d}x.$$

Then $h_x = -\rho$ and $h \in L^\infty([0,T]; W^{m,3}_{\mathrm{per}^\star}(I))$, with mean value zero.

Noticing (2.19) again, we have

$$h_{xt} = -\rho_t = \left(\frac{\delta E_\rho}{\delta \rho}\right)_{xxxx} = \left(\frac{\delta E_h}{\delta h}\right)_{xxx},$$

in distribution sense. Integrating from 0 to x, for a.e. $t \in [0, T]$, there exists a constant $c(t)$ such that

$$h_t = \left(\frac{\delta E_h}{\delta h}\right)_{xx} + c(t).$$

That is, for any test function $\varphi \in W^{3,3}_{\text{per}}(I)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle h, \varphi\rangle = \left\langle \frac{\delta E_h}{\delta h}, \varphi_{xx}\right\rangle + \langle c(t), \varphi\rangle .$$

Taking $\varphi = 1$, we get $c(t) = 0$, for $a.e.\ t \in [0, T]$. Hence h is the solution of (1.7).

Step 2. For (a)⇒(b), we take

$$h^T(x, t) = h(x, t) + bx,$$

with $b = \frac{1}{L}$. From (1.3) and (1.7), we know h^T is L-periodic function with respect to x.

Denote

$$k_0 = \frac{1}{L}\int_0^L h^T(s, 0)\mathrm{d}s,$$
$$k_1(t) = \frac{1}{L}\int_0^L \int_0^x h^T(y, t)\,\mathrm{d}y\,\mathrm{d}x - k_0\frac{L}{2}.$$

Set

$$u(x, t) = \int_0^x \left(-h^T(y, t) + k_0\right)\,\mathrm{d}y + k_1(t). \tag{2.26}$$

We know u is L-periodic function with mean value zero. To prove such u satisfies (2.24), we can proceed just the same as Step 1.

Note we also have

$$u_x = -h - bx + k_0, \tag{2.27}$$
$$u_{xx} = -h_x - b. \tag{2.28}$$

For (b)⇒(a), we simply take

$$h = -u_x - bx. \tag{2.29}$$

Then (2.21) and (2.22) show that h satisfies (b). □

Proposition 2.5 *For any integer $m \geq 2$, the following conditions are equivalent:*
(i) There exists $h \in L^\infty([0,T]; W^{1,\infty}_{per^\star}(I) \cap W^{m,2}(I))$ with $h_t \in L^\infty([0,T]; W^{-3,\infty}_{per}(I))$ a solution of (1.7) satisfying

$$h_x(x,t) \leq \beta_1 < 0 \quad a.e.\ x \in \mathbb{R},\ t \in [0,T], \tag{2.30}$$

for some $\beta_1 < 0$.
(ii) There exists $\phi \in L^\infty([0,T]; W^{1,\infty}_{per^\star}([0,1]) \cap W^{m,2}([0,1]))$ with $\phi_t \in L^\infty([0,T]; W^{-3,\infty}_{per}([0,1]))$ a solution of (2.13) satisfying

$$\phi_\alpha(\alpha,t) \leq \beta_2 < 0 \quad a.e.\ \alpha \in \mathbb{R},\ t \in [0,T], \tag{2.31}$$

for some $\beta_2 < 0$.

Proof Notice condition (2.30), (2.31). By inverse function theorem, h and ϕ are inverse functions of each other. Noticing (1.10) and (2.10), $h \in L^\infty([0,T]; W^{1,\infty}_{\text{per}^\star}(I))$ with condition (2.30) implies that $\phi \in L^\infty([0,T]; W^{1,\infty}_{\text{per}^\star}([0,1]))$ with condition (2.31).

From the differentiation of inverse function, we also know

$$\phi^{(m)} \leq C(\beta_1)(h^{(m)} + \sum_{0 \leq \alpha_i \leq m-1} h^{(\alpha_1)} h^{(\alpha_2)} \cdots h^{(\alpha_m)}).$$

Since $W^{m,2} \hookrightarrow W^{(m-1),\infty}$, we have

$$\int_0^L |\phi^{(m)}|^2 \, \mathrm{d}\alpha \leq C(\beta_1)(\|h\|^2_{W^{m,2}} + \|h\|^m_{W^{m,2}}).$$

Hence $h \in L^\infty([0,T]; W^{m,2}(I))$ with condition (2.30) implies that $\phi \in L^\infty([0,T]; W^{m,2}([0,1]))$ with condition (2.31), vice versa. $\square$

3 Local Strong Solution and Proof of Theorem 1.1

We continue studying the properties of the continuum PDE. From now on, denote

$$\varphi^{(n)}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \varphi(x),$$

and c as a generic constant whose value may change from line to line. We first establish the existence and uniqueness of the local strong solution to (2.24).

Theorem 3.1 *Assume $u^0 \in W^{m,2}_{per_0}(I)$, $u^0_{xx} + b \geq \eta$, where η is a positive constant, $m \in \mathbb{Z}$, $m \geq 7$. Then there exists time T_m depending on η, $\|u^0\|_{W^{m,2}_{per_0}}$ such that*

$$u \in L^\infty([0,T_m]; W^{m,2}_{per_0}(I)) \cap L^2([0,T_m]; W^{m+2,2}_{per_0}(I)) \cap C([0,T_m]; W^{m-4,2}_{per_0}(I)),$$
$$u_t \in L^\infty([0,T_m]; W^{m-4,2}_{per_0}(I)) \cap L^2([0,T_m]; L^2_{per_0}(I))$$

is the unique strong solution of (2.24) *with initial data* u^0*, and* u *satisfies*

$$u_{xx} + b \geq \frac{\eta}{2}, \quad a.e.\ t \in [0, T_m],\ x \in [0, L].$$

Proof We first make the a priori assumption

$$\min_{x\in I}(u_{xx} + b) \geq \frac{\eta}{2} > 0, \quad a.e.\ t \in [0, T_m], \tag{3.1}$$

in which T_m will be determined later. We will prove the existence of local strong solution under (3.1) in Steps 1 and 2, and then justify (3.1) in Step 3.

Let J_δ be the standard $C_c^\infty(I)$ mollifier. Denote $\bar{u}^\delta = J_\delta * u^\delta$.

Define $E_u^\delta(u) := E_u(J_\delta * u)$. Then

$$\frac{\delta E_u^\delta(u^\delta)}{\delta u^\delta} = J_\delta * \left.\frac{\delta E_u(u)}{\delta u}\right|_{\bar{u}^\delta}.$$

We study problem

$$\begin{cases} u_t^\delta = -\frac{\delta E_u^\delta(u^\delta)}{\delta u^\delta}, \\ u^\delta(0) = J_\delta * u^0, \end{cases} \tag{3.2}$$

which is

$$\begin{cases} u_t^\delta = \big(J_\delta * \big(-\frac{2\pi}{L} H(\bar{u}_{xx}^\delta)\big)\big)_x - (J_\delta * \Phi_b{}'(\bar{u}_{xx}^\delta))_{xx}, \\ u^\delta(0) = J_\delta * u^0. \end{cases} \tag{3.3}$$

Step 1. We devote to obtain some a priori estimates, which will be used to prove the convergence of u^δ in (3.2).

Taking u as a test function in (2.24) gives

$$\int_0^L u_t u \,\mathrm{d}x = \int_0^L \frac{2\pi}{L} H(u_{xx}) u_x - \left(\ln(u_{xx} + b) + \frac{3}{2}(u_{xx} + b)^2\right) u_{xx} \,\mathrm{d}x.$$

Notice that

$$\int_0^L H(u_{xx}) u_x \,\mathrm{d}x \leq \int_0^L \frac{3}{4} u_{xx}^2 + 2u^2 \,\mathrm{d}x \leq \int_0^L \frac{1}{8} u_{xx}^3 + 2u^2 \,\mathrm{d}x + C(L),$$

and that

$$\int_0^L \ln(u_{xx} + b) u_{xx} \,\mathrm{d}x \leq C(\eta, L) + \frac{1}{8} \int_0^L u_{xx}^3 \,\mathrm{d}x,$$

due to (3.1). We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L u^2 \,\mathrm{d}x + \int_0^L u_{xx}^3 \,\mathrm{d}x \leq c \int_0^L u^2 \,\mathrm{d}x + C(\eta, L).$$

Then for some $T_1 > 0$, Grönwall's inequality implies that

$$\begin{aligned}
&\|u\|_{L^\infty([0,T_1];L^2(I))} \leq C(\eta, L, \|u^0\|_{W^{m,2}_{\mathrm{per}_0}}, T_1),\\
&\|u_{xx}\|_{L^2([0,T_1];L^3(I))} \leq C(\eta, L, \|u^0\|_{W^{m,2}_{\mathrm{per}_0}}, T_1).
\end{aligned} \tag{3.4}$$

Here and in the following, $C(\eta, L, \|u^0\|_{W^{m,2}_{\mathrm{per}_0}}, T_1)$ is a constant depending only on $\eta,\ L,\ \|u^0\|_{W^{m,2}_{\mathrm{per}_0}}$ and T_1.

Recall (2.25). We use $\rho = u_{xx} + b$ from now.

Since

$$\frac{\mathrm{d}E_u(u)}{\mathrm{d}t} + \int_0^L \left(\frac{\delta E_u(u)}{\delta u}\right)^2 \mathrm{d}x = 0,$$

we have

$$E_u(u) \leq E_u(u_0) < +\infty. \tag{3.5}$$

Also notice

$$\begin{aligned}
&\left|\int_0^L \int_0^L \ln\left|\sin\frac{\pi}{L}(x-y)\right| \rho(x)\rho(y)\,\mathrm{d}x\,\mathrm{d}y\right|\\
&\leq \left(\int_0^L \int_0^L \ln^2\left|\sin\frac{\pi}{L}(x-y)\right| \mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \int_0^L \rho^2(x)\,\mathrm{d}x\\
&\leq \frac{1}{8}\int_0^L \rho^3\,\mathrm{d}x + C(L),
\end{aligned} \tag{3.6}$$

and

$$\left|\int_0^L \rho\ln\rho\,\mathrm{d}x\right| \leq \frac{1}{8}\int_0^L \rho^3\,\mathrm{d}x + C(\eta, L). \tag{3.7}$$

These, together with (3.5), give that

$$\frac{1}{4}\sup_{0\leq t\leq T_1}\int_0^L \rho^3\,\mathrm{d}x < E_\rho(0) + C(\eta, L). \tag{3.8}$$

Now we devote to get a higher-order a priori estimate for $m \geq 4$.

Divide m times in Eq. (2.24) and then take $u^{(m)}$ as a test function, which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot{W}^{m,2}} = \int_0^L -\frac{2\pi}{L}H(\rho)^{(m+1)}u^{(m)} - f(\rho)^{(m+2)}u^{(m)}\,\mathrm{d}x, \tag{3.9}$$

where

$$f(\rho) = \Phi'(\rho) = \ln\rho + 1 + \frac{3}{2}\rho^2.$$

For the first term in (3.9), we have

$$\left|\int_0^L -H(\rho)^{(m+1)}u^{(m)}\,\mathrm{d}x\right| = \left|\int_0^L -H(\rho)^{(m)}\rho^{(m-1)}\,\mathrm{d}x\right| \\ \le \frac{1}{8}\int_0^L \rho^{(m)2}\,\mathrm{d}x + 2\int_0^L \rho^{(m-1)2}\,\mathrm{d}x \\ \le \frac{1}{4}\int_0^L \rho^{(m)2}\,\mathrm{d}x + c\int_0^L \rho^{(m-2)2}\,\mathrm{d}x. \tag{3.10}$$

For the second term in (3.9), we have

$$\int_0^L -f(\rho)^{(m+2)}u^{(m)}\,\mathrm{d}x = \int_0^L -f(\rho)^{(m)}\rho^{(m)}\,\mathrm{d}x \\ = \int_0^L -(f'(\rho)\rho_x)^{(m-1)}\rho^{(m)}\,\mathrm{d}x \\ = \int_0^L -f'(\rho)\rho^{(m)2}\,\mathrm{d}x + \int_0^L \sum_{k=0}^{m-2} C_k f'(\rho)^{(m-1-k)}\rho_x^{(k)}\rho^{(m)}\,\mathrm{d}x. \tag{3.11}$$

Note that

$$f'(\rho) = 3\rho + \frac{1}{\rho} \ge 2\sqrt{3}, \quad \text{for} \rho > 0,$$

so the first term on the right-hand side of (3.11) is strictly negative. We will use it to control the other terms later.

Now we carefully estimate the last term in (3.11). Denote

$$M_1 := \int_0^L \sum_{k=0}^{m-2} C_k f'(\rho)^{(m-1-k)}\rho_x^{(k)}\rho^{(m)}\,\mathrm{d}x \\ \le \|\rho^{(m)}\|_{L^2}\left[\sum_{k=0}^{m-2} C_k \|f'(\rho)^{(m-1-k)}\rho_x^{(k)}\|_{L^2}\right].$$

First the chain rule gives

$$f'(\rho)^{(m-1-k)} = \sum_{\beta_1+\beta_2+\cdots+\beta_\mu = m-1-k} C_\beta \rho^{(\beta_1)}\rho^{(\beta_2)}\cdots\rho^{(\beta_\mu)} f^{(\mu+1)}(\rho).$$

Due to (3.1), we know

$$f^{(\mu+1)}(\rho) \le \frac{C_\mu}{\rho^{\mu+1}} \le \frac{C_\mu}{\eta^{\mu+1}}, \quad \text{for } \mu \ge 1.$$

Also noticing that

$$\|\rho^{(m-3)}\|_{L^\infty} \le c\|\rho\|_{W^{m-2,2}},$$

we have

$$\begin{aligned}
&\|f'(\rho)^{(m-1-k)}\|_{L^4} \le C(\eta,m)\|\rho\|^{m-1}_{W^{m-2,2}}, \quad \text{for } 2 \le k \le m-2,\\
&\|f'(\rho)^{(m-2)}\|_{L^4} \le C(\eta,m)(\|\rho\|^{m-1}_{W^{m-2,2}} + \|\rho^{(m-2)}\|_{L^4}), \quad \text{for } k=1,
\end{aligned}$$

and

$$\|f'(\rho)^{(m-1)}\|_{L^4} \le C(\eta,m)(\|\rho\|^{m-1}_{W^{m-2,2}} + \|\rho^{(m-2)}\|_{L^4} + \|\rho^{(m-1)}\|_{L^4}), \quad \text{for } k=0.$$

Second by interpolating, we know

$$\|\rho^{(m-2)}\|_{L^4} \le c\|\rho^{(m-2)}\|^{\frac{7}{8}}_{L^2}\|\rho^{(m)}\|^{\frac{1}{8}}_{L^2}, \tag{3.12}$$

$$\|\rho^{(m-1)}\|_{L^4} \le c\|\rho^{(m-2)}\|^{\frac{3}{8}}_{L^2}\|\rho^{(m)}\|^{\frac{5}{8}}_{L^2}, \tag{3.13}$$

and for $\mu < m-2$,

$$\|\rho^{(\mu)}\|_{L^4} \le c\|\rho^{(m-2)}\|_{L^4} + c\|\rho\|_{L^4} \le c\|\rho^{(m-2)}\|^{\frac{7}{8}}_{L^2}\|\rho^{(m)}\|^{\frac{1}{8}}_{L^2} + c\|\rho\|_{W^{m-2,2}}. \tag{3.14}$$

Thus (3.12), (3.13) and (3.14) show that

$$\begin{aligned}
&\sum_{k=0}^{m-2} C_k\|f'(\rho)^{(m-1-k)}\rho_x^{(k)}\|_{L^2}\\
&\quad\le \sum_{k=0}^{m-2} C_k\|f'(\rho)^{(m-1-k)}\|_{L^4}\|\rho_x^{(k)}\|_{L^4}\\
&\quad\le c\|f'(\rho)^{(m-2)}\|_{L^4}\|\rho_{xx}\|_{L^4}\\
&\qquad+\sum_{k=1}^{m-2} C(k,\eta,m)\|\rho\|^{m-1}_{W^{m-2,2}}(\|\rho^{(m-2)}\|^{\frac{7}{8}}_{L^2}\|\rho^{(m)}\|^{\frac{1}{8}}_{L^2}\\
&\qquad+c\|\rho\|_{W^{m-2,2}})\\
&\qquad+C(\eta,m)\|\rho\|^{m-1}_{W^{m-2,2}}\|\rho^{(m-2)}\|^{\frac{3}{8}}_{L^2}\|\rho^{(m)}\|^{\frac{5}{8}}_{L^2}
\end{aligned} \tag{3.15}$$

For the first term, we have

$$\begin{aligned}
&\|f'(\rho)^{(m-2)}\|_{L^4}\|\rho_{xx}\|_{L^4}\\
&\quad\le C(\eta,m)(\|\rho\|_{W^{m-2,2}}^{m-1}+\|\rho^{(m-2)}\|_{L^4})(\|\rho^{(m-2)}\|_{L^4}+\|\rho\|_{W^{m-2,2}})\\
&\quad\le C(\eta,m)\Big[\|\rho\|_{W^{m-2,2}}^{m}+(\|\rho\|_{W^{m-2,2}}^{m-1}+1)\|\rho^{(m-2)}\|_{L^2}^{\frac{7}{8}}\|\rho^{(m)}\|_{L^2}^{\frac{1}{8}}\\
&\qquad+\|\rho^{(m-2)}\|_{L^2}^{\frac{7}{4}}\|\rho^{(m)}\|_{L^2}^{\frac{1}{4}}\Big],
\end{aligned}\tag{3.16}$$

where we used (3.12) and (3.14).

Notice that (3.8) gives $\|\rho\|_{L^\infty(0,T_1;L^2(I))}\le C(\eta,L)$. By interpolating, (3.15) and (3.16) lead to

$$M_1\le C(\eta,m)\Big[\|\rho^{(m)}\|_{L^2}^{\frac{5}{8}}\|\rho\|_{\dot W^{m-2,2}}^{m}+\|\rho^{(m)}\|_{L^2}^{\frac{1}{8}}\|\rho\|_{\dot W^{m-2,2}}^{m}\tag{3.17}$$

$$+\|\rho^{(m)}\|_{L^2}^{\frac{1}{4}}\|\rho\|_{\dot W^{m-2,2}}^{m+1}+\|\rho\|_{\dot W^{m-2,2}}^{m}+C(\eta,L)\Big]\|\rho^{(m)}\|_{L^2}\tag{3.18}$$

$$\le\frac{1}{8}\|\rho^{(m)}\|_{L^2}^2+C(\eta,m)\|\rho\|_{\dot W^{m-2,2}}^{10m}+C(\eta,L).\tag{3.19}$$

Combining (3.10), (3.11), (3.17) and Grönwall's inequality, we finally obtain

$$\begin{aligned}
\|u\|_{L^\infty([0,T_1];W^{m,2}_{\mathrm{per}_0}(I))}&\le C(\eta,L,\|u^0\|_{W^{m,2}_{\mathrm{per}_0}},T_1),\\
\|u\|_{L^2([0,T_1];W^{m+2,2}_{\mathrm{per}_0}(I))}&\le C(\eta,L,\|u^0\|_{W^{m,2}_{\mathrm{per}_0}},T_1).
\end{aligned}$$

Step 2. Define $F_\delta: W^{m+2,2}_{\mathrm{per}_0}\to W^{m+2,2}_{\mathrm{per}_0}$ with

$$F_\delta(u^\delta):=\left(J_\delta*\left(-\frac{2\pi}{L}H\left(\bar u^\delta_{xx}\right)\right)\right)_x-(J_\delta*\Phi_b{}'(\bar u^\delta_{xx}))_{xx}.$$

We can easily check that F_δ is locally Lipschitz continuous in $W^{m+2,2}(I)$ for $m\ge 1$. Hence by Majda and Bertozzi (2002, Theorem 3.1), we know (3.3) has a unique local solution $u^\delta\in C^1([0,T_0];W^{m+2,2}_{\mathrm{per}_0}(I))$ and those estimates in Step 1 hold true uniformly in δ. That is, for T_0, we have

$$\|u^\delta\|_{L^\infty([0,T_0];W^{m,2}_{\mathrm{per}_0}(I))}\le C(\eta,L,\|u^0\|_{W^{m,2}_{\mathrm{per}_0}},T_0),\tag{3.20}$$

$$\|u^\delta\|_{L^2([0,T_0];W^{m+2,2}_{\mathrm{per}_0}(I))}\le C(\eta,L,\|u^0\|_{W^{m,2}_{\mathrm{per}_0}},T_0).\tag{3.21}$$

Since

$$E_u^\delta(u^\delta(T))+\int_0^T\int_0^L u_t^{\delta 2}\,\mathrm{d}x\,\mathrm{d}t=E_u^\delta(u^\delta(0)),$$

we also have

$$\|u_t^\delta\|_{L^2([0,T_0]\times I)} \le C(\eta, L, \|u^0\|_{W^{m,2}_{\mathrm{per}_0}}). \tag{3.22}$$

Notice $W^{m+2,2} \hookrightarrow W^{m+1,2}$ compactly and $W^{m+1,2} \hookrightarrow L^2$. Therefore as $\delta \to 0$, we can use Lions–Aubin's compactness lemma to obtain there exists a subsequence, still denoted as u^δ, such that

$$u^\delta \to u, \quad \text{in } L^2([0,T_0]; W^{m+1,2}_{\mathrm{per}_0}(I)).$$

And (3.20), (3.21) and (3.22) show that

$$\begin{aligned}
&u \in L^\infty([0,T_0]; W^{m,2}_{\mathrm{per}_0}(I)) \cap L^2([0,T_0]; W^{m+2,2}_{\mathrm{per}_0}(I)),\\
&u_t \in L^\infty([0,T_0]; W^{m-4,2}_{\mathrm{per}_0}(I)).
\end{aligned}$$

Thus we can take limit in (3.3) and u satisfies (2.24) almost everywhere, i.e., u is the local strong solution of (2.24).

Since

$$\begin{aligned}
&\|u_t\|_{L^2([0,T_0]\times I)} \le \liminf_{\delta\to 0} \|u_t^\delta\|_{L^2([0,T_0]\times I)} \le C(\eta, L, \|u^0\|_{W^{m,2}_{\mathrm{per}_0}}),\\
&u_t \in L^2([0,T_0]\times I),
\end{aligned}$$

by Evans (1998, Theorem 4, p. 288), we actually have

$$u \in C([0,T_0]; W^{1,2}_{\mathrm{per}_0}(I)).$$

Step 3. We justify the a priori assumption (3.1). Note that

$$u_{xx}(x,t) = u_{xx}(0) + \int_0^t u_{xxt}(x,\tau)\mathrm{d}\tau, \tag{3.23}$$

and $u^0_{xx} + b \ge \eta$, so Step 2 and Sobolev embedding theorem lead to

$$u_{xxt} \in L^\infty([0,T_0], W^{m-6,2}(I)) \hookrightarrow L^\infty([0,T_0], L^\infty(I)),$$

for $m \ge 7$. Then

$$\left|\int_0^t u_{xxt}(x,\tau)\mathrm{d}\tau\right| \le t\|u_{xxt}\|_{L^\infty([0,T_0],L^\infty(I))} \le \frac{\eta}{2}, \quad t \in [0, T_m],$$

where $T_m < T_0$ depends only on η, L and $\|u^0\|_{W^{m,2}(I)}$. This, together with (3.23), gives (3.1). □

By using the above Theorem 3.1, we now prove Theorem 1.1.

Proof of Theorem 1.1 Step 1 (Existence). Assume $h^0 \in W^{m,2}_{\text{per}^\star}(I)$, $h^0_x \leq \beta$, for some constant $\beta < 0$, $m \in \mathbb{Z}$, $m \geq 6$. From (2.26), there exists $u^0 \in W^{m+1,2}_{\text{per}^\star}(I)$ satisfying $u^0_{xx} + b \geq -\beta$. Then by Theorem 3.1, there exists $T_m > 0$, such that there exists a unique u satisfying (2.24) with the following regularity:

$$u \in L^\infty([0, T_m]; W^{m+1,2}_{\text{per}_0}(I)) \cap L^2([0, T_m]; W^{m+3,2}_{\text{per}_0}(I)) \cap C([0, T_m]; W^{m-3,2}_{\text{per}_0}(I)),$$
$$u_t \in L^\infty([0, T_m]; W^{m-3,2}_{\text{per}_0}(I)),$$

and u satisfies

$$u_{xx} + b \geq -\frac{\beta}{2}, \quad a.e.\ t \in [0, T_m],\ x \in [0, L].$$

Let $h := -u_x - bx$. Hence we can get the existence of solution to (1.7) satisfying (1.13) and the regularity stated in Theorem 1.1.

Step 2 (Uniqueness). Now we assume h_1, h_2 are two solutions of (1.7) satisfying (1.13) and the same regularity stated in Theorem 1.1. Subtract h_2-equation from h_1-equation and multiply $h_1 - h_2$ on both sides. Then integration by parts shows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (h_1 - h_2)^2\,\mathrm{d}x &= \int_0^L (h_{1t} - h_{2t})(h_1 - h_2)\,\mathrm{d}x \\
&= \int_0^L -\frac{2\pi}{L} H(h_{1x} - h_{2x})(h_{1xx} - h_{2xx}) + \left[\left(3h_{1x} + \frac{1}{h_{1x}}\right) h_{1xx}\right. \\
&\quad \left. - \left(3h_{2x} + \frac{1}{h_{2x}}\right) h_{2xx}\right](h_{1xx} - h_{2xx})\,\mathrm{d}x \\
&= \int_0^L -\frac{2\pi}{L} H(h_{1x} - h_{2x})(h_{1xx} - h_{2xx}) \\
&\quad + \left(3h_{2x} + \frac{1}{h_{2x}}\right)(h_{1xx} - h_{2xx})^2 \\
&\quad + \left(3h_{1x} + \frac{1}{h_{1x}} - 3h_{2x} - \frac{1}{h_{2x}}\right) h_{1xx}(h_{1xx} - h_{2xx})\,\mathrm{d}x \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.24}$$

Since

$$3h_{2x} + \frac{1}{h_{2x}} \leq -2\sqrt{3}, \quad \text{due to } h_{2x} < 0, \tag{3.25}$$

the second term on the right-hand side of (3.24) is strictly negative, which will be used to control the other two terms. For I_1, notice the property of Hilbert transform $\|H(u)\|_{L^p} \leq c\|u\|_{L^p}$ for $1 < p < \infty$; see Butzer and Nessel (2011, Proposition 9.1.3). We can use Young's inequality and interpolating to obtain

$$I_1 \leq \int_0^L \frac{1}{4}(h_{1xx} - h_{2xx})^2 + c(h_1 - h_2)^2\,\mathrm{d}x. \tag{3.26}$$

To estimate I_3, first notice that h_{1xx} is bounded by $\|h_1(0)\|_{W^{m,2}}$ and that

$$|h_{1x}| \geq -\frac{\beta}{2} > 0, \quad |h_{2x}| \geq -\frac{\beta}{2} > 0,$$

due to (1.13). Hence

$$\int_0^L \left[\left(3h_{1x} - 3h_{2x} + \frac{1}{h_{1x}} - \frac{1}{h_{2x}}\right) h_{1xx}\right]^2 \mathrm{d}x \leq C(\beta, \|h_1(0)\|_{W^{m,2}})(h_{1x} - h_{2x})^2 \,\mathrm{d}x,$$

where $C(\beta, \|h_1(0)\|_{W^{m,2}})$ depends only on $\beta,\ \|h_1(0)\|_{W^{m,2}}$. Then Young's inequality and interpolating show that

$$I_3 \leq \int_0^L C(\beta, \|h_1(0)\|_{W^{m,2}})(h_1 - h_2)^2 + \frac{1}{4}(h_{1xx} - h_{2xx})^2 \,\mathrm{d}x, \tag{3.27}$$

where $C(\beta, \|h_1(0)\|_{W^{m,2}})$ depends only on $\beta,\ \|h_1(0)\|_{W^{m,2}}$. Now combining (3.25), (3.26), (3.27) with (3.24) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (h_1 - h_2)^2 \,\mathrm{d}x \leq C(\beta, \|h_1(0)\|_{W^{m,2}}) \int_0^L (h_1 - h_2)^2 \,\mathrm{d}x.$$

Then by Grönwall's inequality, we have

$$\int_0^L (h_1 - h_2)^2 \,\mathrm{d}x \leq C(\beta, \|h_1(0)\|_{W^{m,2}}, T_m) \int_0^L (h_1(0) - h_2(0))^2 \,\mathrm{d}x, \tag{3.28}$$

where $C(\beta, \|h_1(0)\|_{W^{m,2}}, T_m)$ depends only on $\beta,\ \|h_1(0)\|_{W^{m,2}}$ and T_m. This gives the uniqueness of the solution to (1.7). □

3.1 Stability of Linearized ϕ-PDE

Now we set up the stability of linearized ϕ-PDE under assumption

$$h_x(0) \in W^{m,2}_{\mathrm{per}_0}(I), \quad h_x(0) \leq 2\beta < 0,$$

with $m \geq 6$.

Recall Theorem 1.1 and Proposition 2.5. There exists $T_m > 0$, such that

$$\phi(\alpha, t) \in L^\infty([0, T_m]; W^{6,\infty}_{\mathrm{per}^\star}(0, 1)) \tag{3.29}$$

is the strong solution of (2.13) and there exists constants $m_1,\ m_2 > 0$ such that

$$\phi_\alpha \leq -m_1 < 0, \quad |\phi^{(i)}| \leq m_2,\ i = 1, \ldots, 6. \tag{3.30}$$

Recall Eq. (2.13):

$$\phi_t = -\phi_\alpha \mu_{xx} = -\partial_\alpha \left(\frac{1}{\phi_\alpha} \left(\frac{\delta E}{\delta \phi} \right)_\alpha \right),$$

where

$$\frac{\delta E}{\delta \phi} = \frac{2\pi}{L^2} \mathrm{PV} \int_0^1 \cot \frac{\pi(\phi(\alpha) - \phi(\beta))}{L} \, \mathrm{d}\beta - \frac{\phi_{\alpha\alpha}}{\phi_\alpha^2} - 3\frac{\phi_{\alpha\alpha}}{\phi_\alpha^4}.$$

We want to show that the linearized ϕ-PDE is stable, which will be used in the construction of high-order consistency solution (Sect. 6.2).

For ϕ, $\tilde{\phi}$ satisfying Eq. (2.13), set $\phi + \varepsilon\psi = \tilde{\phi}$. Denote

$$A := -\frac{\phi_{\alpha\alpha}}{\phi_\alpha^2} - 3\frac{\phi_{\alpha\alpha}}{\phi_\alpha^4} + \frac{2\pi}{L^2} \mathrm{PV} \int_0^1 \cot \frac{\pi(\phi(\alpha) - \phi(\beta))}{L} \, \mathrm{d}\beta, \tag{3.31}$$

and

$$\begin{aligned} B :=& \left(-\frac{1}{\phi_\alpha^2} - 3\frac{1}{\phi_\alpha^4} \right) \psi_{\alpha\alpha} + \left(\frac{2\phi_{\alpha\alpha}}{\phi_\alpha^3} + \frac{12\phi_{\alpha\alpha}}{\phi_\alpha^5} \right) \psi_\alpha \\ & - \frac{2\pi^2}{L^3} \mathrm{PV} \int_0^1 \sec^2 \frac{\pi}{L} (\phi(\alpha) - \phi(\beta))(\psi(\alpha) - \psi(\beta)) \, \mathrm{d}\beta. \end{aligned} \tag{3.32}$$

So the linearized equation of ϕ-PDE (2.13) is

$$\psi_t = -\partial_\alpha \left(-\frac{\psi_\alpha}{\phi_\alpha^2} \partial_\alpha A + \frac{\partial_\alpha B}{\phi_\alpha} \right). \tag{3.33}$$

Proposition 3.2 *Assume $\psi(0) \in L^2_{per}([0, 1])$ and $m_1, m_2 > 0$ defined in (3.30). Let $T_m > 0$ be the maximal existence time for strong solution ϕ in (3.29). The linearized equation (3.33) is stable in the sense*

$$\|\psi(\cdot, t)\|_{L^2_{per}([0,1])} \leq C(m_1, m_2, T_m) \|\psi(\cdot, 0)\|_{L^2_{per}([0,1])}, \quad \text{for } t \in [0, T_m], \tag{3.34}$$

where $C(m_1, m_2, T_m)$ is a constant depending only on m_1, m_2 and T_m.

Proof Step 1. We perform without the Hilbert transform term $\frac{2\pi}{L^2} \mathrm{PV} \int_0^1 \cot \frac{\pi(\phi(\alpha)-\phi(\beta))}{L} \, \mathrm{d}\beta$. Then A, B in (3.33) become

$$A := -\frac{\phi_{\alpha\alpha}}{\phi_\alpha^2} - 3\frac{\phi_{\alpha\alpha}}{\phi_\alpha^4},$$

and

$$B := \left(-\frac{1}{\phi_\alpha^2} - 3\frac{1}{\phi_\alpha^4} \right) \psi_{\alpha\alpha} + \left(\frac{2\phi_{\alpha\alpha}}{\phi_\alpha^3} + \frac{12\phi_{\alpha\alpha}}{\phi_\alpha^5} \right) \psi_\alpha.$$

Because ψ is 1-periodic function with respect to α, we have

$$
\begin{aligned}
\psi_t =& -\partial_\alpha\left(-\frac{\psi_\alpha}{\phi_\alpha^2}A_\alpha+\partial_\alpha\left(\frac{B}{\phi_\alpha}\right)-\left(\frac{1}{\phi_\alpha}\right)_\alpha B\right)\\
=& -\partial_{\alpha\alpha}\left(\frac{B}{\phi_\alpha}\right)+\partial_\alpha\left(\frac{\psi_\alpha}{\phi_\alpha^2}A_\alpha-\frac{\phi_{\alpha\alpha}}{\phi_\alpha^2}B\right)\\
=& -\partial_{\alpha\alpha}\left[\left(-\frac{1}{\phi_\alpha^3}-\frac{3}{\phi_\alpha^5}\right)\psi_{\alpha\alpha}+\left(\frac{2\phi_{\alpha\alpha}}{\phi_\alpha^4}+\frac{12\phi_{\alpha\alpha}}{\phi_\alpha^6}\right)\psi_\alpha\right]\\
&+\partial_\alpha\left[\left(\frac{\phi_{\alpha\alpha}}{\phi_\alpha^4}+\frac{3\phi_{\alpha\alpha}}{\phi_\alpha^6}\right)\psi_{\alpha\alpha}+\left(-\frac{2\phi_{\alpha\alpha}^2}{\phi_\alpha^5}-\frac{12\phi_{\alpha\alpha}^2}{\phi_\alpha^7}+\frac{A_\alpha}{\phi_\alpha^2}\right)\psi_\alpha\right].
\end{aligned}
$$

Multiplying both sides by ψ and integration by parts show that

$$
\begin{aligned}
\int_0^1\psi\psi_t\,\mathrm{d}\alpha =& \int_0^1\left[\left(\frac{1}{\phi_\alpha^3}+\frac{3}{\phi_\alpha^5}\right)\psi_{\alpha\alpha}^2-\left(\frac{3\phi_{\alpha\alpha}}{\phi_\alpha^4}+\frac{15\phi_{\alpha\alpha}}{\phi_\alpha^6}\right)\psi_\alpha\psi_{\alpha\alpha}\right.\\
&\left.+\left(\frac{2\phi_{\alpha\alpha}^2}{\phi_\alpha^5}+\frac{12\phi_{\alpha\alpha}^2}{\phi_\alpha^7}-\frac{A_\alpha}{\phi_\alpha^2}\right)\psi_\alpha^2\right]\mathrm{d}\alpha. \qquad (3.35)
\end{aligned}
$$

From Young's inequality, for any $\delta, \varepsilon > 0$, we have

$$
\psi_{\alpha\alpha}\psi_\alpha \le \varepsilon\psi_{\alpha\alpha}^2+\frac{1}{4\varepsilon}\psi_\alpha^2, \qquad (3.36)
$$

and

$$
\int_0^1\psi_\alpha^2\,\mathrm{d}\alpha \le \int_0^1\left(\delta\psi_{\alpha\alpha}^2+\frac{1}{4\delta}\psi^2\right)\mathrm{d}\alpha. \qquad (3.37)
$$

Note that ϕ_α is negative and from (3.29), (3.30), we know

$$
\frac{1}{\phi_\alpha^3}+\frac{3}{\phi_\alpha^5} \le -\left(\frac{1}{m_2^3}+\frac{1}{m_2^5}\right).
$$

Now choose ε, δ in (3.36) and (3.37) such that the last two terms in (3.35) can be controlled by $\int_0^1 -\left(\frac{1}{m_2^3}+\frac{1}{m_2^5}\right)\psi_{\alpha\alpha}^2 + C(m_1, m_2)\psi^2\,\mathrm{d}\alpha$. Therefore combining (3.36), (3.37) and (3.30), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\psi^2\,\mathrm{d}\alpha + C(m_2)\int_0^1\psi_{\alpha\alpha}^2\,\mathrm{d}\alpha \le \int_0^1 C(m_1, m_2)\psi^2\,\mathrm{d}\alpha, \qquad (3.38)
$$

where $C(m_2),\ C(m_1, m_2) > 0$ are constants depending on $m_1,\ m_2$.

By Grönwall's inequality, we finally achieve the stability for ψ in the sense of (3.34).

Step 2. If we consider Hilbert transform, then $A,\ B$ are defined in (3.31) and (3.32). First notice that change of variable from h to ϕ does not affect the Cauchy principal value integral and that $h_x < 0$. Then for any $\alpha \in [0, 1]$, by variable substitution, we have

$$
\begin{aligned}
&\mathrm{PV}\int_0^1 \frac{\pi}{L}\cot\left(\frac{\pi}{L}\left(\phi(\alpha)-\phi(\beta)\right)\right)\mathrm{d}\beta = -\,\mathrm{PV}\int_0^1 \sum_{k\in\mathbb{Z}} \frac{1}{\phi(\beta)-\phi(\alpha)-kL}\,\mathrm{d}\beta\\
&\quad = -\,\mathrm{PV}\int_{-\infty}^{+\infty} \frac{1}{\phi(\beta)-\phi(\alpha)}\,\mathrm{d}\beta = \mathrm{PV}\int_{-\infty}^{+\infty}\frac{h_y}{y-x}\,\mathrm{d}y\\
&\quad = \mathrm{PV}\sum_{k\in\mathbb{Z}}\int_{-\frac{L}{2}+kL}^{\frac{L}{2}+kL}\frac{h_y}{y-x}\,\mathrm{d}y = \frac{\pi}{L}\,\mathrm{PV}\int_{-\frac{L}{2}}^{\frac{L}{2}} h_y\cot\left(\frac{y-x}{L}\pi\right)\mathrm{d}y\\
&\quad = -\pi H(h_x)\circ\phi,
\end{aligned}
\tag{3.39}
$$

where we used the relation for Hilbert kernel

$$
\sum_{k\in\mathbb{Z}}\frac{1}{x+kL} = \frac{\pi}{L}\cot\left(\frac{\pi}{L}x\right).
$$

Hence

$$
\left(\mathrm{PV}\int_0^1 \cot\frac{\pi}{L}(\phi(\alpha)-\phi(\beta))\,\mathrm{d}\beta\right)_\alpha = -L(H(h_{xx})\circ\phi)\phi_\alpha
$$

is L^p bounded due to the property of Hilbert transform $H(u)_x = H(u_x)$ for $u_x \in L^p$ with $1 < p < \infty$.

Second, using the periodicity of ψ, integration by parts shows that

$$
\begin{aligned}
&\frac{\pi}{L}\,\mathrm{PV}\int_0^1 \sec^2\frac{\pi}{L}(\phi(\alpha)-\phi(\beta))(\psi(\alpha)-\psi(\beta))\,\mathrm{d}\beta\\
&\quad = \mathrm{PV}\int_0^1 \cot\frac{\pi(\phi(\alpha)-\phi(\beta))}{L}\left[-\frac{\psi_\alpha(\beta)}{\phi_\alpha^2(\beta)} - \frac{(\psi(\alpha)-\psi(\beta))\phi_{\alpha\alpha}(\beta)}{\phi_\alpha^2(\beta)}\right]\mathrm{d}\beta.
\end{aligned}
$$

For any $\varepsilon > 0$, by Young's inequality, we have

$$
\begin{aligned}
&\int_0^1 \mathrm{PV}\int_0^1 \psi_{\alpha\alpha}(\alpha)(\psi(\alpha)-\psi(\beta))\sec^2\frac{\pi}{L}(\phi(\alpha)-\phi(\beta))\,\mathrm{d}\beta\,\mathrm{d}\alpha\\
&\quad \le 2\varepsilon\int_0^1 \psi_{\alpha\alpha}^2\,\mathrm{d}\alpha + \frac{c}{\varepsilon}\int_0^1\left[\mathrm{PV}\int_0^1 \cot\frac{\pi(\phi(\alpha)-\phi(\beta))}{L}\right.\\
&\qquad \left.\times\left(-\frac{\psi_\alpha(\beta)}{\phi_\alpha^2(\beta)} - \frac{(\psi(\alpha)-\psi(\beta))\phi_{\alpha\alpha}(\beta)}{\phi_\alpha^2(\beta)}\right)\mathrm{d}\beta\right]^2\mathrm{d}\alpha.
\end{aligned}
$$

Similar to (3.39), we have

$$\begin{aligned}
&\mathrm{PV}\int_0^1 \cot\frac{\pi(\phi(\alpha)-\phi(\beta))}{L}\left(-\frac{\psi_\alpha(\beta)}{\phi_\alpha^2(\beta)}-\frac{(\psi(\alpha)-\psi(\beta))\phi_{\alpha\alpha}(\beta)}{\phi_\alpha^2(\beta)}\right)\mathrm{d}\beta\\
&\quad=\left[H\left(-\frac{\psi_\alpha}{\phi_\alpha^3}\circ h\right)+H\left(\frac{\phi_{\alpha\alpha}\psi}{\phi_\alpha^3}\circ h\right)+\psi(\alpha)H\left(\frac{-\phi_{\alpha\alpha}}{\phi_\alpha^3}\circ h\right)\right]\circ\phi.
\end{aligned}$$

Then notice the property of Hilbert transform $\|H(u)\|_{L^p}\le c\|u\|_{L^p}$ for $1<p<\infty$; see Butzer and Nessel (2011, Proposition 9.1.3). For any $\varepsilon,\delta>0$, by Hölder's inequality and interpolating, we have

$$\begin{aligned}
&\int_0^1 \mathrm{PV}\int_0^1 \psi_{\alpha\alpha}(\alpha)(\psi(\alpha)-\psi(\beta))\sec^2\frac{\pi}{L}(\phi(\alpha)-\phi(\beta))\,\mathrm{d}\beta\,\mathrm{d}\alpha\\
&\le 2\varepsilon\int_0^1\psi_{\alpha\alpha}^2\,\mathrm{d}\alpha+\frac{c}{\varepsilon}\int_0^1\left[H\left(-\frac{\psi_\alpha}{\phi_\alpha^3}\circ h\right)+H\left(\frac{\phi_{\alpha\alpha}\psi}{\phi_\alpha^3}\circ h\right)+\psi(\alpha)H\left(\frac{-\phi_{\alpha\alpha}}{\phi_\alpha^3}\circ h\right)\right]^2\circ\phi\,\mathrm{d}\alpha\\
&\le 2\varepsilon\int_0^1\psi_{\alpha\alpha}^2\,\mathrm{d}\alpha+\frac{c}{\varepsilon}\int_0^1\left[\frac{\psi_\alpha^2}{\phi_\alpha^5}+\frac{\phi_{\alpha\alpha}^2\psi^2}{\phi_\alpha^5}\right]\mathrm{d}\alpha+\left(\int_0^1\psi^4(\alpha)\,\mathrm{d}\alpha\right)^{\frac12}\left(\int_0^1\frac{\phi_{\alpha\alpha}^4}{\phi_\alpha^{11}}\,\mathrm{d}\alpha\right)^{\frac12}\\
&\le\left(2\varepsilon+\frac{\delta}{\varepsilon}\right)\int_0^1\psi_{\alpha\alpha}^2\,\mathrm{d}\alpha+\frac{C(m_1,m_2)}{\varepsilon\delta}\int_0^1\psi(\alpha)^2\,\mathrm{d}\alpha
\end{aligned}$$

where $C(m_1,m_2)$ depends only on m_1, m_2. Here we used variable substitution twice and (3.30).

Then we can perform just like Step 1 to get (3.38) and complete the proof of Proposition 3.2. □

4 Modified BCF-Type Model

We want to rigorously study the continuum limit of a BCF-type model and figure out the convergence rate. From now on, we assume the initial data $x_i(0)$ satisfying

$$x_i(0)<x_{i+1}(0),\quad \text{for } i=1,\ldots,N. \tag{4.1}$$

As mentioned in Introduction, we need to modify the ODE as follows

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}=\frac{1}{a}\left(\frac{f_{i+1}-f_i}{x_{i+1}-x_i}-\frac{f_i-f_{i-1}}{x_i-x_{i-1}}\right),\quad i=1,\ldots,N, \tag{4.2}$$

where the chemical potential

$$f_i:=-\frac{2}{L}\sum_{j\neq i}\frac{a}{x_j-x_i}+\left(\frac{1}{x_{i+1}-x_i}-\frac{1}{x_i-x_{i-1}}\right)+\left(\frac{a^2}{(x_{i+1}-x_i)^3}-\frac{a^2}{(x_i-x_{i-1})^3}\right), \tag{4.3}$$

for $i=1,\ldots,N$. Notice (4.2) with (4.3) is exactly the ODE (1.8) with (1.9), so we refer to (4.2) in the following.

From now on, keep in mind the relation between the Hilbert kernel and Cauchy kernel is

$$\sum_{k\in\mathbb{Z}} \frac{1}{x+kL} = \frac{\pi}{L}\cot\left(\frac{\pi}{L}x\right). \tag{4.4}$$

The corresponding energy is

$$E^N := a^2 \sum_{1\le i<j\le N} \frac{2}{L}\ln\left|\sin\left(\frac{\pi}{L}(x_j-x_i)\right)\right| + a\sum_{i=0}^{N}\left(-\ln\left|\frac{x_i-x_{i+1}}{a}\right| + \frac{a^2}{2}\frac{1}{(x_i-x_{i+1})^2}\right). \tag{4.5}$$

Since as $a \to 0$, we have $x_i = O(a)$, so the contribution of the various terms in E^N is on the same order.

We have

$$f_i = \frac{1}{a}\frac{\partial E^N}{\partial x_i},$$

and energy identity

$$\frac{\mathrm{d}E^N}{\mathrm{d}t} + \sum_{i=1}^{N}\frac{(f_{i+1}-f_i)^2}{x_{i+1}-x_i} = 0, \tag{4.6}$$

which is analogous to (2.17).

We will first study some properties of (4.2) and obtain the consistence result in Sect. 5. Then we construct an auxiliary solution with high-order consistency in Sect. 6.2, which is important when we prove the convergence rate of the modified ODE system. After those preparations, the proof of Theorem 1.2 will be given in Sect. 6.3.

4.1 Global Solution of ODE

In this section, we will prove that for any fixed $N \ge 2$, the ODE system (4.2) has a global-in-time solution.

Proposition 4.1 *Assume initial data satisfy* (4.1). *Then for any $N \ge 2$, the ODE system* (4.2) *has a global-in-time solution.*

Proof Let $T_{\max}$ be the maximal existence time. Then if $T_{\max} < +\infty$, from standard extension theory for ODE, we know either two steps collide, i.e., there exists i such that $x_i(T_{\max}) = x_{i+1}(T_{\max})$, or step reaches infinity, i.e., $x_i(T_{\max}) = +\infty$.

Denote

$$\ell_{min}(t) := \min_{i\in\mathbb{N}}\{x_{i+1}(t) - x_i(t)\},$$

and we state a proposition that we have a positive lower bound for $\ell_{min}(t)$. We will prove this proposition later.

Proposition 4.2 *For any $N \geq 2$, assume initial data satisfy (4.1) and system (4.2) has initial energy $E^N(0)$. Then for any time t the solution of (4.2) exists, we have*

$$\ell_{min}(t) \geq C(N) > 0,$$

where $C(N)$ is a constant depending only on N.

By Proposition 4.2, we have

$$\ell_{\min}(T_{\max}) \geq \lim_{t \to T_{\max}} \ell_{\min}(t) \geq C(N) > 0,$$

which contradicts with $x_i(T_{\max}) = x_{i+1}(T_{\max})$.

On the other hand, combining Proposition 4.2 with Eq. (4.2) gives

$$\max_{1 \leq i \leq N} |\dot{x}_i| \leq C(N),$$

where $C(N)$ is a constant depending only on N. Hence there will be no finite time blowup and we conclude $T_{\max} = +\infty$. □

Proof of Proposition 4.2. First from (4.6), we know, for any time t the solution exists,

$$E^N(t) \leq E^N(0).$$

Let $0 < \ell^\star \leq 1$ small enough. Then

$$\frac{2\pi}{L^2} \cot \frac{\pi}{L}\ell - \frac{1}{2}\frac{a^2}{\ell^3} < 0, \quad \text{for } 0 < \ell \leq \ell^\star.$$

Thus, at least for $0 < \ell \leq \min\{\ell^\star, \frac{L}{2}\}$, we know

$$g(\ell) := \frac{2}{L} \ln \left(\sin \frac{\pi}{L}\ell \right) + \frac{a^2}{4\ell^2}$$

is positive, i.e.,

$$\frac{2}{L} \ln \sin \frac{\pi}{L}\ell + \frac{a^2}{4\ell^2} > 0.$$

Hence

$$\frac{2}{L} \ln \sin \frac{\pi}{L}\ell + \frac{a^2}{2\ell^2} > \frac{a^2}{4\ell^2},$$

and

$$\frac{2}{L} \ln \left(\sin \frac{\pi}{L}\ell \right) - \ln \left(\frac{\ell}{a} \right) + \frac{a^2}{2\ell^2} > \frac{a^2}{4\ell^2} + \ln a \geq c_0(N),$$

where $c_0(N)$ is a constant depending only on N. Then we obtain

$$E^N \geq a^2\left[\frac{2}{L}\ln\left(\sin\left(\frac{\pi}{L}\ell_{min}\right)\right) - \ln(\ell_{min}) + \ln a + \frac{a^2}{2\ell_{min}^2} + \left(\frac{N(N-1)}{2} - 1\right)c_0(N)\right]$$
$$\geq \frac{a^4}{4\ell_{min}^2} + c_1(N),$$

where $c_1(N)$ is a constant depending only on N.

Therefore we have

$$\frac{1}{\ell_{min}^2} \leq C(N, E^N(0)),$$

where $C(N, E^N(0))$ is a positive constant depending only on N and initial data.

So we finally get

$$\ell_{min} \geq \min\left\{\frac{L}{2}, \ell^\star, \frac{1}{\sqrt{C(N, E^N(0))}}\right\}.$$

□

5 Consistency

In this section, we study the local consistency between exact solution ϕ of Eq. (2.13) and solution x of Eq. (4.2). From now on, we always assume there exists a constant $\beta < 0$ such that the initial data satisfy

$$h_x(0) \in W^{m,2}_{\mathrm{per}_0}(I), \quad h_x(0) \leq 2\beta < 0,$$

with $m \geq 6$.

From Theorem 1.1, we know there exists $T_m > 0$, for $t \in [0, T_m]$, $h(x,t) \in L^\infty([0,T]; W^{6,\infty}_{\mathrm{per}^\star}(\mathbb{R}))$ is the strong solution of (1.7) and

$$h_x \leq \beta < 0. \tag{5.1}$$

Also by Proposition 2.5, we know $\phi(\alpha, t)$ is the strong solution of (2.13) satisfying (3.29) and (3.30).

Denote

$$\begin{aligned}\bar{f_i} := &-\frac{2}{L}\sum_{j\neq i}\frac{a}{\phi_j - \phi_i} + \left(\frac{1}{\phi_{i+1}-\phi_i} - \frac{1}{\phi_i - \phi_{i-1}}\right)\\ &+ \left(\frac{a^2}{(\phi_{i+1}-\phi_i)^3} - \frac{a^2}{(\phi_i - \phi_{i-1})^3}\right).\end{aligned} \tag{5.2}$$

The main result in this section is Theorem 5.1:

Theorem 5.1 *For all $i = 1, \dots, N$, let $\bar{f}_i$ be defined in (5.2), and*

$$v_1(\alpha;\phi) := -\frac{\phi_{\alpha\alpha}}{2\phi_\alpha^2}(\alpha), \quad r_0(\alpha;\phi) := \left(\frac{v_{1\alpha}\phi_{\alpha\alpha}}{\phi_\alpha^2} - \frac{v_{1\alpha\alpha}}{\phi_\alpha}\right)(\alpha). \tag{5.3}$$

Then we have

$$\frac{\mathrm{d}\phi_i}{dt} = \frac{1}{a}\left(\frac{\bar{f}_{i+1} - \bar{f}_i}{\phi_{i+1} - \phi_i} - \frac{\bar{f}_i - \bar{f}_{i-1}}{\phi_i - \phi_{i-1}}\right) + r_0(\alpha_i;\phi)a + R_i a^2, \quad t \in [0, T], \tag{5.4}$$

and

$$|r_0(\alpha_i;\phi)| \le C(\beta, \|h(0)\|_{W^{7,2}(I)}), \quad |R_i| \le C(\beta, \|h(0)\|_{W^{7,2}(I)}), \tag{5.5}$$

where $C(\beta, \|h(0)\|_{W^{7,2}(I)})$ depends on β, $\|h(0)\|_{W^{7,2}(I)}$, and R_i is defined in (5.35). In addition, we have

$$\frac{\mathrm{d}E^N(\phi)}{\mathrm{d}t} + a\sum_{i=1}^N \left(\frac{\bar{f}_{i+1} - \bar{f}_i}{a}\right)^2 \le Ca.$$

To achieve this goal, first we need to set up some notations and lemmas. From (3.29) and (3.30), there exist constants $c_1,\ c_2 > 0$, such that

$$c_1 a \le \phi_{i+1} - \phi_i \le c_2 a. \tag{5.6}$$

Denoting

$$F_i := \frac{1}{a}\left(\frac{\bar{f}_{i+1} - \bar{f}_i}{\phi_{i+1} - \phi_i} - \frac{\bar{f}_i - \bar{f}_{i-1}}{\phi_i - \phi_{i-1}}\right), \tag{5.7}$$

we want to estimate the difference between F_i and $\frac{\mathrm{d}\phi_i}{\mathrm{d}t}$. From PDE (1.7) and (2.10), we have

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}t} = -\frac{\left(-\frac{2\pi}{L}H(h_x) + \left(\frac{1}{h_x} + 3h_x\right)h_{xx}\right)_{xx}}{h_x}\Bigg|_{\phi_i}. \tag{5.8}$$

The main task is then to calculate the term F_i. Let us first estimate $\bar{f}_i$ till order a accuracy by writing

$$\bar{f}_i = I_{1,i} + I_{2,i} + I_{3,i},$$

where

$$\begin{aligned} I_{1,i} &:= -\frac{2}{L}\sum_{j\ne i}\frac{a}{\phi_j - \phi_i} = -\frac{2}{L}\sum_{k\in\mathbb{Z}}\sum_{\substack{j=1\\ j\ne i}}^N \frac{a}{\phi_j - \phi_i + kL}, \\ I_{2,i} &:= \frac{1}{\phi_{i+1} - \phi_i} - \frac{1}{\phi_i - \phi_{i-1}}, \\ I_{3,i} &:= \frac{a^2}{(\phi_{i+1} - \phi_i)^3} - \frac{a^2}{(\phi_i - \phi_{i-1})^3}. \end{aligned} \tag{5.9}$$

To simplify notations, we will henceforth denote

$$\varphi_i = \varphi(x)|_{x=x_i}.$$

Next, we state four lemmas to estimate $I_{1,i},\ I_{2,i},\ I_{3,i}$ one by one, from which we know $O(a)$ error only shows up when estimating the first term $I_{1,i}$ in Lemma 5.6.

Lemma 5.2 *Let $I_{2,i}$ be defined in (5.9) and v_2 be function of α defined as*

$$v_2(\alpha;\phi) := -\frac{\phi^{(4)}}{12\phi_\alpha^2} + \frac{\phi_{\alpha\alpha}}{\phi_\alpha^4}\left(\frac{1}{3}\phi_\alpha\phi^{(3)} - \frac{1}{4}\phi_{\alpha\alpha}^2\right). \tag{5.10}$$

Then we have

$$I_{2,i} = \left.\frac{h_{xx}}{h_x}\right|_{\phi_i} + v_2(\alpha_i;\phi)a^2 + R_{2,i}, \tag{5.11}$$

where $|R_{2,i}| \le a^4 C(\beta, \|h(0)\|_{W^{7,2}(I)})$.

Proof Notice we have

$$\phi_{i+1} = \phi_i - \phi_{\alpha,i}a + \frac{1}{2}\phi_{\alpha\alpha,i}a^2 - \frac{1}{3!}\phi_i^{(3)}a^3 + \frac{1}{4!}\phi_i^{(4)}a^4 - \frac{1}{5!}\phi_i^{(5)}a^5 + \frac{1}{6!}\phi^{(6)}(\xi^+)a^6, \tag{5.12}$$

$$\phi_{i-1} = \phi_i + \phi_{\alpha,i}a + \frac{1}{2}\phi_{\alpha\alpha,i}a^2 + \frac{1}{3!}\phi_i^{(3)}a^3 + \frac{1}{4!}\phi_i^{(4)}a^4 + \frac{1}{5!}\phi_i^{(5)}a^5 + \frac{1}{6!}\phi^{(6)}(\xi^-)a^6, \tag{5.13}$$

where $\xi^+ \in [\alpha_i, \alpha_{i+1}],\ \xi^- \in [\alpha_{i-1}, \alpha_i]$.

Hence using (2.10), we have

$$\begin{aligned}
I_{2,i} &= \frac{1}{\phi_{i+1}-\phi_i} - \frac{1}{\phi_i - \phi_{i-1}} \\
&= \frac{\frac{2\phi_i - \phi_{i+1} - \phi_{i-1}}{a^2}}{\left(\frac{\phi_{i+1}-\phi_i}{a}\right)\left(\frac{\phi_i-\phi_{i-1}}{a}\right)} \\
&= \left(-\phi_{\alpha\alpha,i} - \frac{1}{12}\phi_i^{(4)}a^2 - \frac{1}{6!}(\phi^{(6)}(\xi^+) + \phi^{(6)}(\xi^-))a^4\right) \\
&\quad \cdot \frac{1}{-\phi_{\alpha,i} + \frac{1}{2}\phi_{\alpha\alpha,i}a - \frac{1}{3!}\phi_i^{(3)}a^2 + \frac{1}{4!}\phi_i^{(4)}a^3 - \frac{1}{5!}\phi^{(5)}a^4 + \frac{1}{6!}\phi^{(6)}(\xi^+)a^5} \\
&\quad \cdot \frac{1}{-\phi_{\alpha,i} - \frac{1}{2}\phi_{\alpha\alpha,i}a - \frac{1}{3!}\phi_i^{(3)}a^2 - \frac{1}{4!}\phi_i^{(4)}a^3 - \frac{1}{5!}\phi^{(5)}a^4 - \frac{1}{6!}\phi^{(6)}(\xi^-)a^5} \\
&= \frac{-\phi_{\alpha\alpha,i} - \frac{1}{12}\phi_i^{(4)}a^2 - \frac{1}{6!}(\phi^{(6)}(\xi^+) + \phi^{(6)}(\xi^-))a^4}{(\phi_{\alpha,i}^2 + A_1(\alpha_i;\phi)a^2 + A_{2,i}a^4)}
\end{aligned}$$

$$= \left(\frac{-\phi_{\alpha\alpha}}{\phi_\alpha^2}\right)_i + \left(-\frac{\phi^{(4)}}{12\phi_\alpha^2} + \frac{\phi_{\alpha\alpha}}{\phi_\alpha^4}A_1\right)_i a^2 + A_{3,i}a^4$$
$$= \left(\frac{h_{xx}}{h_x^2}\right)_i + \left(-\frac{\phi^{(4)}}{12\phi_\alpha^2} + \frac{\phi_{\alpha\alpha}}{\phi_\alpha^4}A_1\right)_i a^2 + A_{3,i}a^4,$$

where

$$A_1(\alpha;\phi) = \frac{1}{3}\phi_\alpha\phi^{(3)} - \frac{1}{4}\phi_{\alpha\alpha}^2, \quad |A_{2,i}| \le c, \quad |A_{3,i}| \le c.$$

Denote

$$v_2(\alpha;\phi) = -\frac{\phi^{(4)}}{12\phi_\alpha^2} + \frac{\phi_{\alpha\alpha}}{\phi_\alpha^4}A_1, \tag{5.14}$$

and we complete the proof of Lemma 5.2. □

Now we claim an approximation for periodic Hilbert transform.

Lemma 5.3 *For any $\phi(\alpha_i)$, $i = 1, \dots, N$, we have*

$$\mathrm{PV}\int_0^1 \frac{\pi}{L}\cot\left(\frac{\pi}{L}(\phi(\alpha_i) - \phi(\alpha))\right)\mathrm{d}\alpha = \sum_{j\neq i, j=1}^{N} a\frac{\pi}{L}\cot\left(\frac{\pi}{L}(\phi(\alpha_i) - \phi_j)\right) + \frac{a}{2}\frac{\phi_{\alpha\alpha}}{\phi_\alpha^2} + R_{1,i}, \tag{5.15}$$

where $|R_{1,i}| \le a^4 C(\beta, \|h(0)\|_{W^{7,2}(I)})$.

Proof We use the Euler–Maclaurin expansion in Sidi and Israeli (1988) to estimate $R_{1,i}$. Without loss of generality, we assume $i = 1, \dots, N-1$, that is $\alpha_i \neq 0,\ 1$. For $i = N$, we can change interval $[0, 1]$ to $[-a, 1-a]$ due to periodicity. Using (4.4), we can see

$$\mathrm{PV}\int_0^1 \frac{\pi}{L}\cot\left(\frac{\pi}{L}(\phi(\alpha) - \phi(\alpha_i))\right)\mathrm{d}\alpha$$
$$= \sum_{k\in\mathbb{Z}} \mathrm{PV}\int_0^1 \frac{1}{\phi(\alpha) - \phi(\alpha_i) + kL}\mathrm{d}\alpha$$
$$= \mathrm{PV}\int_0^1 \frac{1}{\phi(\alpha) - \phi(\alpha_i)}\mathrm{d}\alpha + \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\int_0^1 \frac{1}{\phi(\alpha) - \phi(\alpha_i) + kL}\mathrm{d}\alpha$$
$$= T_1 + T_2.$$

Denote

$$\sideset{^\#}{}\sum_{j=0}^{N}\beta_j = \sum_{j=1}^{N-1}\beta_j + \frac{1}{2}\sum_{j=0,N}\beta_j.$$

First we recall Theorem 1 and Theorem 4 in Sidi and Israeli (1988) as follows:

Theorem 5.4 (Theorem 1 of Sidi and Israeli 1988) *Let function* $g(x)$ *be* $2m$ *times differentiable on* $[0, 1]$. *Then*

$$\int_0^1 g(x)\,\mathrm{d}x = a^{\#}\sum_{j=0}^{N} g(x_j) + \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{2\mu!}[g^{(2\mu-1)}|_{x=1}^{x=0}]a^{2\mu} + R_{2m}[g;(0,1)],$$

where

$$R_{2m}[g;(0,1)] = a^{2m}\int_0^1 \frac{\bar{B}_{2m}[\frac{x}{a}] - B_{2m}}{(2m)!} g^{(2m)}(x)\,\mathrm{d}x,$$

B_μ *is the Bernoulli number, and* $\bar{B}_\mu$ *is the periodic Bernoullian function of order* μ.

Theorem 5.5 (Theorem 4 of Sidi and Israeli 1988) *Let function* $G(x)$ *be* $2m$ *times differentiable on* $[0, 1]$, *and let* $g(x) = \frac{G(x)}{x-t}$. *Then*

$$\int_0^1 g(x)\,\mathrm{d}x = a^{\#}\sum_{j=0, x_j \neq t}^{N} g(x_j) + aG'(t) + \sum_{\mu=1}^{m-1} \frac{B_{2\mu}}{2\mu!}[g^{(2\mu-1)}|_{x=1}^{x=0}]a^{2\mu} + \tilde{R}_{2m}[g;(0,1)],$$

where

$$\tilde{R}_{2m}[g;(0,1)] = a^{2m}\,\mathrm{PV}\int_0^1 \frac{\bar{B}_{2m}[\frac{x}{a}] - B_{2m}}{(2m)!} g^{(2m)}(x)\,\mathrm{d}x.$$

For the nonsingular T_2, we apply Theorem 5.4 to obtain

$$T_2 = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\left[a(^{\#}\sum_{j=0}^{N}\frac{1}{\phi(\alpha_j)-\phi(\alpha_i)+kL}) + a^2\frac{B_2}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1}{\phi(\alpha)-\phi(\alpha_i)+kL}\right)\Bigg|_{\alpha=1}^{\alpha=0} + a^4 e_1(k)\right], \tag{5.16}$$

where

$$\begin{aligned}|e_1(k)| &= \left|\int_0^1 \frac{\bar{B}_4[\frac{\alpha}{a}] - B_4}{4!}\frac{\mathrm{d}^4}{\mathrm{d}\alpha^4}\left(\frac{1}{\phi(\alpha)-\phi(\alpha_i)+kL}\right)\mathrm{d}\alpha\right| \\ &\leq c\max_{\alpha\in[0,1]}\frac{\mathrm{d}^4}{\mathrm{d}\alpha^4}\left(\frac{1}{\phi(\alpha)-\phi(\alpha_i)+kL}\right).\end{aligned} \tag{5.17}$$

Due to $\phi_\alpha(1)-\phi_\alpha(0)=0$, the second term in (5.16) becomes

$$\begin{aligned} K_2 &:= \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \frac{B_2}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1}{\phi(\alpha)-\phi(\alpha_i)+kL}\right)\Bigg|_{\alpha=1}^{\alpha=0} \\ &= \sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} \frac{B_2}{2}\phi_\alpha(0)\left(\frac{1}{(kL-\phi(\alpha_i))^2}-\frac{1}{(L+kL-\phi(\alpha_i))^2}\right). \end{aligned} \tag{5.18}$$

To estimate the last term in (5.16), since $\max_{\alpha\in[0,1]}\frac{\mathrm{d}^4}{\mathrm{d}\alpha^4}\left(\frac{1}{\phi(\alpha)-\phi(\alpha_i)+kL}\right)$ in (5.17) is summable with respect to k, we get

$$\left|\sum_{\substack{k\in\mathbb{Z}\\k\neq 0}} e_1(k)\right| \leq C(\beta,\|h(0)\|_{W^{7,2}(I)}). \tag{5.19}$$

Now we deal with the singular term T_1. Denote $G(\alpha):=\frac{\alpha-\alpha_i}{\phi(\alpha)-\phi(\alpha_i)}$. Applying Theorem 5.5 to

$$g(\alpha)=\frac{G(\alpha)}{\alpha-\alpha_i}=\frac{\frac{\alpha-\alpha_i}{\phi(\alpha)-\phi(\alpha_i)}}{\alpha-\alpha_i}=\frac{1}{\phi(\alpha)-\phi(\alpha_i)},$$

then we have

$$\begin{aligned} T_1 &= a\left(\sideset{}{^{\#}}\sum_{j=0,j\neq i}^{N}\frac{1}{\phi(\alpha_j)-\phi(\alpha_i)}\right)-\frac{a}{2}\frac{\phi_{\alpha\alpha}}{\phi_\alpha^2}\bigg|_{\alpha_i} \\ &\quad + a^2\frac{B_2}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1}{\phi(\alpha)-\phi(\alpha_i)}\right)\Bigg|_{\alpha=1}^{\alpha=0}+a^4 e_2, \end{aligned} \tag{5.20}$$

where

$$e_2 := \mathrm{PV}\int_0^1 \frac{\bar{B}_4[\frac{\alpha}{a}]-B_4}{4!}\frac{\mathrm{d}^4}{\mathrm{d}\alpha^4}\left(\frac{1}{\phi(\alpha)-\phi(\alpha_i)}\right)\mathrm{d}\alpha.$$

Due to $\phi_\alpha(1)-\phi_\alpha(0)=0$ again, the third term in (5.20) becomes

$$\begin{aligned} K_1 &:= \frac{B_2}{2}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1}{\phi(\alpha)-\phi(\alpha_i)}\right)\Bigg|_{\alpha=1}^{\alpha=0} \\ &= \frac{B_2}{2}\phi_\alpha(0)\left(\frac{1}{(-\phi(\alpha_i))^2}-\frac{1}{(L-\phi(\alpha_i))^2}\right). \end{aligned} \tag{5.21}$$

Without loss of generality, we can also assume $\alpha_i \leq \frac{1}{2}$. Denote $p(\alpha) := \frac{\bar{B}_4[\frac{\alpha}{a}]-B_4}{4!}$, we have

$$\begin{aligned} e_2 &= \mathrm{PV} \int_0^1 p(\alpha) \frac{\mathrm{d}^4}{\mathrm{d}\alpha^4} \left(\frac{G(\alpha) - G(\alpha_i)}{\alpha - \alpha_i} + \frac{G(\alpha_i)}{\alpha - \alpha_i} \right) \mathrm{d}\alpha \\ &\leq C(\beta, \|h(0)\|_{W^{7,2}(I)}) + \mathrm{PV} \int_0^1 cp(\alpha) \frac{\mathrm{d}^4}{\mathrm{d}\alpha^4} \left(\frac{1}{\alpha - \alpha_i} \right) \mathrm{d}\alpha, \end{aligned} \tag{5.22}$$

where we used the differentiability of $G(\alpha)$. For the last term in (5.22), since α_i is the singular point, we do variable substitution to obtain

$$\begin{aligned} &\mathrm{PV} \int_0^1 cp(\alpha) \frac{\mathrm{d}^4}{\mathrm{d}\alpha^4} \left(\frac{1}{\alpha - \alpha_i} \right) \mathrm{d}\alpha \\ &\quad = \mathrm{PV} \int_{-\alpha_i}^{1-\alpha_i} cp(\alpha + \alpha_i) \frac{\mathrm{d}^4}{\mathrm{d}\alpha^4} \left(\frac{1}{\alpha} \right) \mathrm{d}\alpha \\ &\quad = \mathrm{PV} \int_{-\alpha_i}^{\alpha_i} cp(\alpha + \alpha_i) \frac{1}{\alpha^5} \mathrm{d}\alpha + \int_{\alpha_i}^{1-\alpha_i} cp(\alpha + \alpha_i) \frac{1}{\alpha^5} \mathrm{d}\alpha \\ &\quad = \int_{\alpha_i}^{1-\alpha_i} cp(\alpha + \alpha_i) \frac{1}{\alpha^5} \mathrm{d}\alpha. \end{aligned}$$

Here we used

$$\bar{B}_4\Big[\frac{\alpha + \alpha_i}{a}\Big] = \bar{B}_4\Big[\frac{\alpha}{a}\Big],$$

due to $\frac{\alpha_i}{a}$ is integer. Since $\bar{B}_4(x)$ is even, $cp(\alpha+\alpha_i)\frac{1}{\alpha^5}$ is odd, so the Cauchy principal value integral $\mathrm{PV} \int_{-\alpha_i}^{\alpha_i} cp(\alpha + \alpha_i)\frac{1}{\alpha^5} \mathrm{d}\alpha$ is zero.

Hence we get

$$|e_2| \leq C(\beta, \|h(0)\|_{W^{7,2}(I)}). \tag{5.23}$$

On the other hand, (5.18) and (5.21) show that

$$K_1 + K_2 = \sum_{k \in \mathbb{Z}} \frac{B_2}{2} \phi_\alpha(0) \left(\frac{1}{(kL - \phi(\alpha_i))^2} - \frac{1}{(L + kL - \phi(\alpha_i))^2} \right) = 0.$$

Denote $e := \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} e_1(k) + e_2$. Combining the calculations for T_1 and T_2, we obtain

$$\begin{aligned} \mathrm{PV} \int_0^1 \frac{\pi}{L} \cot\left(\frac{\pi}{L} \left(\phi(\alpha) - \phi(\alpha_i) \right) \right) \mathrm{d}\alpha = &\sum_{j \neq i, j=1}^{N} a \frac{\pi}{L} \cot\left(\frac{\pi}{L} \left(\phi_j - \phi(\alpha_i) \right) \right) \\ &- \frac{a}{2} \frac{\phi_{\alpha\alpha}}{\phi_\alpha^2} \bigg|_{\alpha_i} + e a^4, \end{aligned}$$

with $|e| \leq C(\beta, \|h(0)\|_{W^{7,2}(I)})$. This concludes (5.15) and $|R_{1,i}| \leq a^4 C(\beta, \|h(0)\|_{W^{7,2}(I)})$. $\square$

Notice that change of variable from h to ϕ does not affect the Cauchy principal value integral and that $h_x < 0$. Then similar to (3.39), by (4.4) and variable substitution, we have

$$\begin{aligned}
&\mathrm{PV}\int_0^1 \frac{\pi}{L}\cot\left(\frac{\pi}{L}\left(\phi(\alpha_i)-\phi(\alpha)\right)\right)\,\mathrm{d}\alpha = -\,\mathrm{PV}\int_0^1 \sum_{k\in\mathbb{Z}} \frac{1}{\phi(\alpha)-\phi(\alpha_i)-kL}\,\mathrm{d}\alpha\\
&\quad= -\,\mathrm{PV}\int_{-\infty}^{+\infty} \frac{1}{\phi(\alpha)-\phi(\alpha_i)}\,\mathrm{d}\alpha = \mathrm{PV}\int_{-\infty}^{+\infty} \frac{h_x}{x-\phi_i}\,\mathrm{d}x\\
&\quad= \mathrm{PV}\sum_{k\in\mathbb{Z}} \int_{-\frac{L}{2}+kL}^{\frac{L}{2}+kL} \frac{h_x}{x-\phi_i}\,\mathrm{d}x = \frac{\pi}{L}\,\mathrm{PV}\int_{-\frac{L}{2}}^{\frac{L}{2}} h_x \cot\left(\frac{x-\phi_i}{L}\pi\right)\mathrm{d}x\\
&\quad= -\pi H(h_x)|_{\phi_i}.
\end{aligned}$$

This, combined with Lemma 5.3, leads to

Lemma 5.6 *Let $I_{1,i}$ be defined in (5.9) and v_1 be function of α defined as*

$$v_1(\alpha;\phi) := -\frac{\phi_{\alpha\alpha}}{L\phi_\alpha^2}. \tag{5.24}$$

Then we have

$$I_{1,i} = -\frac{2\pi}{L} H(h_x)\Big|_{\phi_i} + v_1(\alpha_i;\phi)a + R_{1,i}, \tag{5.25}$$

with $|R_{1,i}| \leq a^4 C(\beta, \|h(0)\|_{W^{7,2}(I)})$.

We now turn to estimate $I_{3,i}$.

Lemma 5.7 *Let $I_{3,i}$ be defined in (5.9) and v_3 be function of α defined as*

$$v_3(\alpha;\phi) := \frac{-\frac{5}{2}\phi_{\alpha\alpha}^3 - \frac{1}{4}\phi_\alpha^2\phi^{(4)} + 2\phi_\alpha\phi_{\alpha\alpha}\phi^{(3)}}{\phi_\alpha^6}. \tag{5.26}$$

Then we have

$$I_{3,i} = 3h_{xx}h_x|_{\phi_i} + v_3(\alpha_i;\phi)a^2 + R_{3,i}, \tag{5.27}$$

where $|R_{3,i}| \leq a^4 C(\beta, \|h(0)\|_{W^{7,2}(I)})$.

Proof Using (2.10) and Taylor expansion, it is similar to the proof of Lemma 5.2 that

$$
\begin{aligned}
I_{3,i} &= a^2\left(\frac{1}{(\phi_{i+1}-\phi_i)^3}-\frac{1}{(\phi_i-\phi_{i-1})^3}\right)\\
&= \frac{\frac{2\phi_i-\phi_{i+1}-\phi_{i-1}}{a^2}}{\left(\frac{\phi_{i+1}-\phi_i}{a}\right)^3\left(\frac{\phi_i-\phi_{i-1}}{a}\right)^3}\cdot\left(\left(\frac{\phi_i-\phi_{i-1}}{a}\right)^2+\left(\frac{\phi_{i+1}-\phi_i}{a}\right)^2+\left(\frac{\phi_i-\phi_{i-1}}{a}\right)\left(\frac{\phi_{i+1}-\phi_i}{a}\right)\right)\\
&= \frac{\left(-\phi_{\alpha\alpha,i}-\frac{1}{12}\phi_i^{(4)}a^2-\frac{1}{6!}(\phi^{(6)}(\xi^+)+\phi^{(6)}(\xi^-))a^4\right)\left(3\phi_{\alpha,i}^2+B_{1,i}a^2+B_{2,i}a^4\right)}{\phi_{\alpha,i}^6+C_{1,i}a^2+C_{2,i}a^4}\\
&= \left(-\frac{3\phi_{\alpha\alpha}}{\phi_\alpha^4}\right)_i+\left[\frac{-\frac{5}{2}\phi_{\alpha\alpha}^3-\frac{1}{4}\phi_\alpha^2\phi^{(4)}+2\phi_\alpha\phi_{\alpha\alpha}\phi^{(3)}}{\phi_\alpha^6}\right]_i a^2+C_{3,i}a^4\\
&= (3h_{xx}h_x)_i+\left[\frac{-\frac{5}{2}\phi_{\alpha\alpha}^3-\frac{1}{4}\phi_\alpha^2\phi^{(4)}+2\phi_\alpha\phi_{\alpha\alpha}\phi^{(3)}}{\phi_\alpha^6}\right]_i a^2+C_{3,i}a^4,
\end{aligned}
$$

where

$$
\begin{aligned}
B_{1,i} &= \left(\phi_\alpha\phi^{(3)}+\frac{1}{4}\phi_{\alpha\alpha}^2\right)_i, \quad |B_{2,i}|\le c,\\
C_{1,i} &= \left(-\frac{3}{4}\phi_\alpha^4\phi_{\alpha\alpha}^2+\phi_\alpha^5\phi^{(3)}\right)_i, \quad |C_{2,i}|\le c, \quad |C_{3,i}|\le c.
\end{aligned}
$$

Denote

$$v_3(\alpha;\phi) := \frac{-\frac{5}{2}\phi_{\alpha\alpha}^3-\frac{1}{4}\phi_\alpha^2\phi^{(4)}+2\phi_\alpha\phi_{\alpha\alpha}\phi^{(3)}}{\phi_\alpha^6}.$$

We conclude the proof of Lemma 5.7. □

Denote

$$A(x;h) := \left(-\frac{2\pi}{L}H(h_x)+3h_{xx}h_x+\frac{h_{xx}}{h_x}\right)(x), \tag{5.28}$$

and

$$R_{4,i} := R_{1,i}+R_{2,i}+R_{3,i}.$$

The above three lemmas yield

Lemma 5.8 *For $\bar{f_i}$ defined in (5.2), v_1 defined in (5.24), v_2 defined in (5.10), and v_3 defined in (5.26), we have*

$$\bar{f_i} = A(\phi_i;h)+v_1(\alpha_i;\phi)a+(v_2+v_3)(\alpha_i;\phi)a^2+R_{4,i}, \tag{5.29}$$

where $|R_{4,i}|\le a^4 C(\beta,\|h(0)\|_{W^{7,2}(I)})$.

Now we are ready to prove the main result of this section, Theorem 5.1.

Proof of Theorem 5.1. Step 1. To calculate F_i in (5.7), by (5.29) in Lemma 5.8, we first need to calculate

$$\begin{aligned}
&\frac{A_{i+1}-A_i}{\phi_{i+1}-\phi_i}-\frac{A_i-A_{i-1}}{\phi_i-\phi_{i-1}}\\
&\quad = A_{xx,i}\frac{\phi_{i+1}-\phi_{i-1}}{2}+A_{xxx,i}\frac{(\phi_{i+1}-\phi_{i-1})(\phi_{i+1}+\phi_{i-1}-2\phi_i)}{3!}+r_{1,i}a^4\\
&\quad = -\phi_{\alpha,i}A_{xx,i}a+r_2(\alpha_i;\phi)a^3+r_{3,i}a^4,
\end{aligned} \tag{5.30}$$

where $|r_{1,i}|, |r_{3,i}| \le C(\beta, \|h(0)\|_{W^{7,2}(I)})$ and

$$r_2(\alpha;\phi) := \left(-\frac{1}{3}\phi^{(3)}(A_{xx}\circ\phi)-2\phi_\alpha\phi_{\alpha\alpha}\right)(\alpha).$$

Second, for any smooth function $v(\alpha)$ with respect to α, notice that

$$\begin{aligned}
v_{i+1}-v_i &= v_{\alpha,i}(\alpha_{i+1}-\alpha_i)+\frac{1}{2}v_{\alpha\alpha,i}(\alpha_{i+1}-\alpha_i)^2+\frac{1}{3!}v_i^{(3)}(\xi^+)(\alpha_{i+1}-\alpha_i)^3,\\
v_{i-1}-v_i &= v_{\alpha,i}(\alpha_{i-1}-\alpha_i)+\frac{1}{2}v_{\alpha\alpha,i}(\alpha_{i-1}-\alpha_i)^2+\frac{1}{3!}v_i^{(3)}(\xi^-)(\alpha_{i-1}-\alpha_i)^3.
\end{aligned}$$

Then for other terms in (5.29), we have

$$\begin{aligned}
&\frac{v_{i+1}-v_i}{\phi_{i+1}-\phi_i}-\frac{v_i-v_{i-1}}{\phi_i-\phi_{i-1}}\\
&\quad=\frac{v_{i+1}-v_i}{\alpha_{i+1}-\alpha_i}\frac{h_{i+1}-h_i}{\phi_{i+1}-\phi_i}-\frac{v_i-v_{i-1}}{\alpha_i-\alpha_{i-1}}\frac{h_i-h_{i-1}}{\phi_i-\phi_{i-1}}\\
&\quad=\left[v_{\alpha,i}-\frac{1}{2}v_{\alpha\alpha,i}a+\frac{1}{3!}v^{(3)}(\xi^+)a^2\right]\left[h_{x,i}+h_{xx,i}\frac{\phi_{i+1}-\phi_i}{2}+\frac{1}{3!}h_{xxx}(\eta^+)(\phi_{i+1}-\phi_i)^2\right]\\
&\qquad-\left[v_{\alpha,i}+\frac{1}{2}v_{\alpha\alpha,i}a+\frac{1}{3!}v^{(3)}(\xi^-)a^2\right]\left[h_{x,i}-h_{xx,i}\frac{\phi_i-\phi_{i-1}}{2}+\frac{1}{3!}h_{xxx}(\eta^-)(\phi_i-\phi_{i-1})^2\right]\\
&\quad=r_4(\alpha_i;\phi)a+r_{5,i}a^2,
\end{aligned} \tag{5.31}$$

where $|r_{5,i}| \le C(\beta, \|h(0)\|_{W^{7,2}(I)})$, $\eta^+ \in [\phi_i, \phi_{i+1}]$, $\eta^- \in [\phi_{i-1}, \phi_i]$ and

$$r_4(\alpha;\phi) := \left(\frac{v_\alpha\phi_{\alpha\alpha}}{\phi_\alpha^2}-\frac{v_{\alpha\alpha}}{\phi_\alpha}\right)(\alpha).$$

Denote

$$r_0(\alpha;\phi) := \left(\frac{v_{1\alpha}\phi_{\alpha\alpha}}{\phi_\alpha^2}-\frac{v_{1\alpha\alpha}}{\phi_\alpha}\right)(\alpha), \tag{5.32}$$

and

$$r(\alpha;\phi) := \left(\frac{v_{2\alpha}\phi_{\alpha\alpha}}{\phi_\alpha^2}-\frac{v_{2\alpha\alpha}}{\phi_\alpha}+\frac{v_{3\alpha}\phi_{\alpha\alpha}}{\phi_\alpha^2}-\frac{v_{3\alpha\alpha}}{\phi_\alpha}\right)(\alpha)+r_2(\alpha;\phi). \tag{5.33}$$

Thus for F_i in (5.7), combining (5.30) and (5.31), we get

$$\begin{aligned} F_i &= -\frac{A_{xx}}{h_x}(\phi_i) + r_0(\alpha_i;\phi)a + r(\alpha_i;\phi)a^2 + \frac{R_{4,i+1} - 2R_{4,i} + R_{4,i-1}}{a^2}(h_x(\phi_i) + r_{6,i}a) \\ &= -\frac{A_{xx}}{h_x}(\phi_i) + r_0(\alpha_i;\phi)a + r(\alpha_i;\phi)a^2 + R_{5,i}a^2, \end{aligned} \tag{5.34}$$

where $|r_{6,i}| \leq C(\beta, \|h(0)\|_{W^{7,2}(I)})$, $A(x;h)$ defined in (5.28). To obtain $|R_{5,i}| \leq C(\beta, \|h(0)\|_{W^{7,2}(I)})$, here we also used $|R_{4,i}| \leq a^4 C(\beta, \|h(0)\|_{W^{7,2}(I)})$ due to Lemma 5.8.

Denote

$$R_i := r(\alpha_i;\phi) + R_{5,i}. \tag{5.35}$$

For a small enough, we have $|R_i| \leq |r(\alpha_i;\phi)| + |R_{5,i}| \leq C(\beta, \|h(0)\|_{W^{7,2}(I)})$. Finally, comparing (5.34) with (5.8), we conclude (5.4).

Step 2. Now using (5.4) and Lemma 5.8, we can claim

$$\sum_{i=1}^{N} \bar{f}_i \left(F_i - \frac{\mathrm{d}\phi_i}{\mathrm{d}t}\right) \leq C(\beta, \|h(0)\|_{W^{7,2}(I)}), \tag{5.36}$$

where $C(\beta, \|h(0)\|_{W^{7,2}(I)})$ depends on β, $\|h(0)\|_{W^{7,2}(I)}$.

From (5.36), multiplying $\bar{f}_i$ in (5.4) and summation by parts show that

$$\frac{\mathrm{d}E^N(\phi)}{\mathrm{d}t} + \sum_{i=1}^{N} \frac{\left(\bar{f}_{i+1}(\phi) - \bar{f}_i(\phi)\right)^2}{\phi_{i+1} - \phi_i} \leq C(\beta, \|h(0)\|_{W^{7,2}(I)})a,$$

Then by (5.6), we have

$$\frac{\mathrm{d}E^N(\phi)}{\mathrm{d}t} + a\sum_{i=1}^{N} \left(\frac{\bar{f}_{i+1}(\phi) - \bar{f}_i(\phi)}{a}\right)^2 \leq C(\beta, \|h(0)\|_{W^{7,2}(I)})a,$$

which completes the proof of Theorem 5.1. □

6 Convergence and the Proof of Theorem 1.2

In this section, our goal is to prove Theorem 1.2. The main idea is to first construct an auxiliary solution with high-order consistency (see Sect. 6.2) and then prove the convergence rate for the auxiliary solution, which helps us obtain the convergence rate for the original PDE solution.

6.1 Stability of Linearized x-ODE

First of all, we devote to study the stability of linearized ODE, which is important when we estimate the convergence rate for the auxiliary solution. The procedure here is analogous to the stability result of linearized ϕ-PDE; see Sect. 3.1.

For vector $x,\ y$ satisfying (4.2), set $x = y + \varepsilon z$. We also assume $y_i(t) = \phi(\alpha_i, t)$, and ϕ is the solution of (2.13) satisfying (3.29) and (3.30). Denote

$$M_i = \frac{1}{y_{i+1}-y_i} + \frac{a^2}{(y_{i+1}-y_i)^3} - \frac{1}{y_i - y_{i-1}} - \frac{a^2}{(y_i - y_{i-1})^3} - \frac{2}{L}\sum_{j\neq i}\frac{a}{y_j - y_i}, \tag{6.1}$$

and

$$\begin{aligned} T_i = & -\frac{z_{i+1}-z_i}{(y_{i+1}-y_i)^2} - 3a^2\frac{z_{i+1}-z_i}{(y_{i+1}-y_i)^4} + \frac{z_i - z_{i-1}}{(y_i - y_{i-1})^2} + 3a^2\frac{z_i - z_{i-1}}{(y_i - y_{i-1})^4} \\ & + \frac{2}{L}\sum_{j\neq i}\frac{a(z_j - z_i)}{(y_j - y_i)^2}. \end{aligned} \tag{6.2}$$

Then z satisfies the following linearized equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}z_i = & \frac{1}{a}\left(\frac{T_{i+1}-T_i}{y_{i+1}-y_i} - \frac{T_i - T_{i-1}}{y_i - y_{i-1}}\right) - \frac{1}{a}\Bigg[\frac{z_{i+1}-z_i}{(y_{i+1}-y_i)^2}(M_{i+1}-M_i) \\ & - \frac{z_i - z_{i-1}}{(y_i - y_{i-1})^2}(M_i - M_{i-1})\Bigg]. \end{aligned} \tag{6.3}$$

Proposition 6.1 *Assume $z(0) \in \ell^2$ and $m_1, m_2 > 0$ defined in (3.30). Let $T_m > 0$ be the maximal existence time for strong solution ϕ in (3.29). The linearized equation (6.3) is stable in the sense*

$$\|z(t)\|_{\ell^2} \le C(m_1, m_2, T_m)\|z(0)\|_{\ell^2}, \ \textit{for } t \in [0, T_m], \tag{6.4}$$

where $C(m_1, m_2, T_m)$ is a constant depending only on $m_1,\ m_2$ and T_m.

Proof Step 1. Similar to the proof of Proposition 3.2, first we study the linearized system for (4.2) without the Hilbert transform term $-\frac{2}{L}\sum_{j\neq i}\frac{a}{x_j - x_i}$. Thus $M_i,\ T_i$ in (6.1) and (6.2) become

$$M_i = \frac{1}{y_{i+1}-y_i} + \frac{a^2}{(y_{i+1}-y_i)^3} - \frac{1}{y_i - y_{i-1}} - \frac{a^2}{(y_i - y_{i-1})^3},$$

and

$$T_i = -\frac{z_{i+1}-z_i}{(y_{i+1}-y_i)^2} - 3a^2\frac{z_{i+1}-z_i}{(y_{i+1}-y_i)^4} + \frac{z_i - z_{i-1}}{(y_i - y_{i-1})^2} + 3a^2\frac{z_i - z_{i-1}}{(y_i - y_{i-1})^4}.$$

Since $z_{i+N}=z_i$, multiplying both sides of (6.3) by az_i and taking summation by parts, we have

$$\begin{aligned}\sum_{i=1}^{N} az_i\dot{z}_i &= -\sum_{i=1}^{N}\frac{z_{i+1}-z_i}{y_{i+1}-y_i}(T_{i+1}-T_i)+\sum_{i=1}^{N}(z_{i+1}-z_i)\frac{z_{i+1}-z_i}{(y_{i+1}-y_i)^2}(M_{i+1}-M_i)\\ &= -a\sum_{i=1}^{N}\frac{z_{i+1}-z_i}{a}\frac{\frac{T_{i+1}}{\frac{y_{i+1}-y_i}{a}}-\frac{T_i}{\frac{y_i-y_{i-1}}{a}}}{a}-a\sum_{i=1}^{N}\frac{z_{i+1}-z_i}{a}\frac{\frac{T_i}{\frac{y_i-y_{i-1}}{a}}-\frac{T_i}{\frac{y_{i+1}-y_i}{a}}}{a}\\ &\quad+a\sum_{i=1}^{N}\left(\frac{z_{i+1}-z_i}{a}\right)^2\frac{1}{(\frac{y_{i+1}-y_i}{a})^2}\frac{M_{i+1}-M_i}{a}\\ &= I_1+I_2+I_3.\end{aligned}$$

Next, we will estimate $I_1,\ I_2,\ I_3$ one by one. First we deal with

$$\begin{aligned}I_1 &= -a\sum_{i=1}^{N}\frac{z_{i+1}-z_i}{a}\frac{\frac{T_{i+1}}{\frac{y_{i+1}-y_i}{a}}-\frac{T_i}{\frac{y_i-y_{i-1}}{a}}}{a}\\ &= a\sum_{i=1}^{N}\frac{T_i}{\frac{y_i-y_{i-1}}{a}}\frac{\frac{z_{i+1}-z_i}{a}-\frac{z_i-z_{i-1}}{a}}{a}.\end{aligned}$$

We can see

$$\begin{aligned}T_i &= a^2\frac{z_{i+1}-2z_i+z_{i-1}}{a^2}\left(-\frac{1}{(y_{i+1}-y_i)^2}-\frac{3a^2}{(y_{i+1}-y_i)^4}\right)\\ &\quad+a\left[-\frac{1}{(y_{i+1}-y_i)^2}-\frac{3a^2}{(y_{i+1}-y_i)^4}+\frac{1}{(y_i-y_{i-1})^2}+\frac{3a^2}{(y_i-y_{i-1})^4}\right]\frac{z_i-z_{i-1}}{a}.\end{aligned}$$

Due to Young's inequality, for any $\varepsilon>0$, we have

$$\begin{aligned}a\sum_{i=1}^{N}\left(\frac{z_{i+1}-z_i}{a}\right)^2 &= -a\sum_{i=1}^{N}z_i\frac{z_{i+1}-2z_i+z_{i-1}}{a^2}\\ &\le a\sum_{i=1}^{N}\left(\frac{1}{4\varepsilon}z_i^2+\varepsilon\left(\frac{z_{i+1}-2z_i+z_{i-1}}{a^2}\right)^2\right).\end{aligned}\tag{6.5}$$

Besides, due to $y_i(t)=\phi(\alpha_i,t)$, we have

$$\begin{aligned}&a\left[-\frac{1}{(y_{i+1}-y_i)^2}-\frac{3a^2}{(y_{i+1}-y_i)^4}+\frac{1}{(y_i-y_{i-1})^2}+\frac{3a^2}{(y_i-y_{i-1})^4}\right]\frac{a}{y_i-y_{i-1}}\\ &\le C_0(m_1,m_2),\end{aligned}$$

$$\left(-\frac{1}{(y_{i+1}-y_i)^2}-\frac{3a^2}{(y_{i+1}-y_i)^4}\right)a^2\frac{a}{y_i-y_{i-1}}\leq -C(m_2)$$

for a small enough.

Then for I_1, we have

$$\begin{aligned}I_1 &= a\sum_{i=1}^{N}\frac{T_i}{\frac{y_i-y_{i-1}}{a}}\frac{\frac{z_{i+1}-z_i}{a}-\frac{z_i-z_{i-1}}{a}}{a}\\ &\leq C_1(m_1,m_2)a\sum_i z_i^2-\frac{3}{4}C(m_2)a\sum_i\left(\frac{z_{i+1}-2z_i+z_{i-1}}{a^2}\right)^2.\end{aligned}$$

Let us keep in mind that functions, such as M_i, involving only $\frac{y_{i+1}-y_i}{a}$ can be bounded by a constant depending only on $m_1,\ m_2$. Then similar to the estimate for I_1, together with (6.5), we have

$$I_2\leq C_2(m_1,m_2)a\sum_i z_i^2+\frac{1}{4}C(m_2)a\sum_i\left(\frac{z_{i+1}-2z_i+z_{i-1}}{a^2}\right)^2,$$

and

$$I_3\leq C_3(m_1,m_2)a\sum_i z_i^2+\frac{1}{4}C(m_2)a\sum_i\left(\frac{z_{i+1}-2z_i+z_{i-1}}{a^2}\right)^2.$$

Here $C_i(m_1,m_2),\ i=0,1,2,3$ are positive constants depending only on $m_1,\ m_2$.

Combining estimates for $I_1,\ I_2,\ I_3$, we have

$$\frac{\mathrm{d}\|z(t)\|_{\ell^2}^2}{\mathrm{d}t}+\frac{1}{4}C(m_2)a\sum_i\left(\frac{z_{i+1}-2z_i+z_{i-1}}{a^2}\right)^2\leq C(m_1,m_2)\|z\|_{\ell^2}^2.$$

Then Grönwall's inequality yields (6.4).

Step 2. Now we consider Hilbert transform term $-\frac{2}{L}\sum_{j\neq i}\frac{a}{x_j-x_i}$. Then the terms $M_i,\ T_i$ in (6.3) become (6.1) and (6.2).

First Lemmas 5.3 and 5.6 show that $\sum_{j\neq i}\frac{a}{y_j-y_i}$ can be estimated by $C(m_1,m_2)$ and PV $\int_0^1\cot\frac{\pi}{L}(\phi(\alpha)-\phi(\beta))\,\mathrm{d}\beta$.

Second, from the proof of Lemma 5.3, we know $a\sum_{j\neq i}\frac{z_j-z_i}{(y_j-y_i)^2}$ can be estimated by $C(m_1,m_2)$ and PV $\int_0^1\sec^2\frac{(\phi(\alpha)-\phi(\beta))\pi}{L}(\psi(\alpha)-\psi(\beta))\,\mathrm{d}\beta$, where ψ is the piecewise-cubic interpolant of z.

Then using the same arguments in Step 2 of the proof of Proposition 3.2, we can conclude (6.4). □

6.2 Construction of Solution with High-Order Truncation Error

From now on, we proceed under the same hypothesis of Theorem 1.2, i.e., we assume for some $\beta < 0$, the initial datum $h(0)$ is smooth enough and satisfies

$$h_x(0) \leq \beta < 0. \tag{6.6}$$

By Theorem 1.1 and Proposition 2.5, for some constant $m \in \mathbb{N}$ large enough, we know there exists $T_m > 0$, such that

$$\phi(\alpha, t) \in C([0, T_m]; C^m[0, 1]) \tag{6.7}$$

is the strong solution to (2.13). Obviously, there exist $M > 0$, whose values depend only on β and $\|h(0)\|_{W^{m,2}}$, such that

$$\phi_\alpha \leq \frac{\beta}{2} < 0, \quad |\phi^{(i)}| \leq M, \quad \text{for } 1 \leq i \leq m. \tag{6.8}$$

Recalling Eq. (2.13), we define $F(\phi) : C^\infty[0, 1] \to C^\infty[0, 1]$ as an operator

$$F(\phi) := -\partial_\alpha \left(\frac{1}{\phi_\alpha} \left(\frac{\delta E}{\delta \phi} \right)_\alpha \right).$$

Then we have

$$\phi_t = F(\phi). \tag{6.9}$$

For F_i defined in (5.7), denote

$$F_N := \{F_i,\ i = 1, \ldots, N\}, \quad r_N(\phi) := \{r_0(\alpha_i; \phi),\ i = 1, \ldots, N\},$$

where $r_0(\alpha; \phi)$ is the function defined in (5.3). Then for $\phi_N = \{\phi_i,\ i = 1, \ldots, N\}$, Theorem 5.1 shows that

$$\dot{\phi}_N = F_N(\phi_N) + r_N(\phi)a + O(a^2).$$

Now we want to construct $y = \phi + a\psi$, for ψ satisfying the same regularity with ϕ, such that y has a higher truncation error than ϕ. In fact, we state

Proposition 6.2 *Let $T_m > 0$ in (6.7) and ϕ be the solution of (6.9). Then there exists ψ smooth enough such that $\|\psi(\cdot, t)\|_{L^2([0,1])}$ is uniformly bounded for $t \in [0, T_m]$, and*

$$y(\alpha, t) = \phi(\alpha, t) + a\psi(\alpha, t) \tag{6.10}$$

satisfies the ODE system (4.2) till order $O(a^7)$, i.e., the nodal values $y_N = \{y(\alpha_i, t),\ i = 1, \ldots, N\}$ satisfy

$$\dot{y}_N = F_N(y_N) + O(a^7). \tag{6.11}$$

Proof To simplify the calculation, first we show there exists ψ such that

$$\dot{y}_N = F_N(y_N) + O(a^2). \tag{6.12}$$

For $y_N = \phi_N + a\psi_N$, where ψ_N is the nodal values of ψ, Theorem 5.1 shows that

$$F_N(y_N) = F_N(\phi_N + a\psi_N) = F(\phi + a\psi)|_{\alpha=\alpha_i} - r_N(\phi + a\psi)a - O(a^2).$$

Hence y_N satisfies

$$\dot{y}_N - F_N(y_N) = a\dot{\psi}_N + [F(\phi) - F(\phi + a\psi)]|_{\alpha=\alpha_i} + r_N(\phi + a\psi)a + O(a^2).$$

Now by Proposition 3.2, we can choose ψ to be the solution of (6.9)'s linearized system

$$\psi_t = -\partial_\alpha \left(-\frac{\psi_\alpha}{\phi_\alpha^2} \partial_\alpha A + \frac{\partial_\alpha B}{\phi_\alpha} \right) - r_0(\phi), \tag{6.13}$$

where $A,\ B$ are defined in (3.31) and (3.32). After that, (6.12) holds.

To obtain higher-order truncation error construction, we can repeat above processes to get higher-order corrections. We omit the details here. □

6.3 Convergence of ODE and PDE System

In this section, we will combine above results and complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume ϕ is the strong solution of (2.13) satisfying (6.7) and (6.8) with maximal existence time $T_m > 0$. Let $\beta,\ M$ be constants in Eq. (6.8). Recall vector $x(t) = \{x_i(t);\ i = 1, \ldots, N\}$ is the solution of (4.2), and with slight abuse of notation, denote $y(t) := \{y(\alpha_i, t);\ i = 1, \ldots, N\}$ being the constructed vector value function y_N in Proposition 6.2. We will first obtain the convergence rate for $x,\ y$ in Steps 1 and 2, and then obtain the convergence rate for $x,\ \phi$ in Step 3.

Step 1. We first claim that under the a priori assumption

$$\|x(t) - y(t)\|_{\ell^\infty} \le a^{6+\frac{1}{3}}, \quad \text{for } t \in [0, T_m], \tag{6.14}$$

we have

$$\|x(t) - y(t)\|_{\ell^2} \le C(\beta, M, T_m)a^7, \ \text{for } t \in [0, T_m], \tag{6.15}$$

where $C(\beta, M, T_m)$ is a constant depending only on $\beta,\ M,\ T_m$. We will verify the a priori assumption (6.14) in Step 2.

In fact, from Proposition 6.2, we know y has a^7-order consistence error, i.e.,

$$\frac{\mathrm{d}(y - x)}{\mathrm{d}t} = F_N(y) - F_N(x) + O(a^7).$$

Denote the inner product for $x,\ y$ as

$$\langle x, y\rangle := \sum_{i=1}^{N} a x_i y_i.$$

Then for $\beta,\ M$ defined in (6.8), we have

$$\begin{aligned}\langle x-y, \dot{x}-\dot{y}\rangle &= \langle x-y, \nabla F_N(y)(x-y)\rangle + \langle x-y, (x-y)\nabla^2 F_N(y)(x-y)^T\rangle\\ &\quad + C(\beta, M)\langle x-y, a^7\rangle,\end{aligned} \tag{6.16}$$

where $C(\beta, M)$ depends only on $\beta,\ M$.

For the second term in (6.16), we can see

$$\begin{aligned}&\langle x-y, (x-y)\nabla^2 F_N(y)(x-y)^T\rangle\\ &\quad\le \|x-y\|_{\ell^2}\Big\|\sum_{i,j=1}^{N}(x_i-y_i)(x_j-y_j)\partial_{ij}F_N\Big\|_{\ell^2}\\ &\quad\le \|x-y\|_{\ell^2}^2\|x-y\|_{\ell^\infty}\sqrt{\sum_{k=1}^{N}\left(\sum_{i=1}^{N}\left(\sum_{j=1}^{N}(\partial_{ij}F_k)^2\right)^{\frac{1}{2}}\right)^2}\\ &\quad\le \|x-y\|_{\ell^2}^2\|x-y\|_{\ell^\infty} N\max_k\left(\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}(\partial_{ij}F_k)^2}\right),\end{aligned} \tag{6.17}$$

where we used Hölder's inequality in the last step.

Now keep in mind that functions involving only $\frac{y_{i+1}-y_i}{a}$ can be bounded by a constant depending only on $\beta,\ M$, and that

$$\left|\frac{1}{y_j-y_i}\right| \le \max\{\frac{1}{y_{i+1}-y_i}, \frac{1}{y_i-y_{i-1}}\}.$$

We can start to estimate the term $\max_k\left(\sqrt{\sum_i\sum_j(\partial_{ij}F_k)^2}\right)$.

For $k = 1, \ldots, N$, denote

$$\begin{aligned}Q_k :=& \frac{1}{y_{k+1}-y_k}\Bigg[-\sum_{\ell\neq k+1}\frac{a}{y_\ell-y_{k+1}} + \sum_{\ell\neq k}\frac{a}{y_\ell-y_k}\\ &+\left(\frac{1}{y_{k+2}-y_{k+1}} - 2\frac{1}{y_{k+1}-y_k} + \frac{1}{y_k-y_{k-1}}\right)\\ &+\left(\frac{a^2}{(y_{k+2}-y_{k+1})^3} - 2\frac{a^2}{(y_{k+1}-y_k)^3} + \frac{a^2}{(y_k-y_{k-1})^3}\right)\Bigg].\end{aligned}$$

Then $F_k = \frac{1}{a}(Q_k - Q_{k-1})$, and

$$(\partial_{ij} F_k)^2 \le \frac{1}{a}[(\partial_{ij} Q_k)^2 + (\partial_{ij} Q_{k-1})^2].$$

First calculate $\partial_i Q_k$, for $k = 1, \ldots, N$.

$$\partial_i Q_k = \begin{cases} \dfrac{a}{y_{k+1}-y_k}\Big[\dfrac{1}{(y_i-y_{k+1})^2} - \dfrac{1}{(y_i-y_k)^2}\Big], & \text{for } \begin{array}{l} 1 \le i \le k-2, \\ k+3 \le i \le N; \end{array} \\[2ex] \dfrac{a}{y_{k+1}-y_k}\Big[\dfrac{1}{(y_{k-1}-y_{k+1})^2} - \dfrac{1}{(y_{k-1}-y_k)^2}\Big] \\ +\dfrac{1}{y_{k+1}-y_k}\dfrac{1}{(y_k-y_{k-1})^2} + \dfrac{1}{y_{k+1}-y_k}\dfrac{3a^2}{(y_k-y_{k-1})^4}, & \text{for } i = k-1; \\[2ex] \dfrac{a}{(y_{k+1}-y_k)^3} - \dfrac{4}{(y_{k+1}-y_k)^3} - \dfrac{8a^2}{(y_{k+1}-y_k)^5} \\ -\dfrac{y_{k+1}-2y_k+y_{k-1}}{(y_{k+1}-y_k)^2(y_k-y_{k-1})^2} - a^2\dfrac{3(y_{k+1}-y_k)-(y_k-y_{k-1})}{(y_{k+1}-y_k)^2(y_k-y_{k-1})^4}, & \text{for } i = k; \\[2ex] -\dfrac{a}{(y_{k+1}-y_k)^3} + \dfrac{4}{(y_{k+1}-y_k)^3} + \dfrac{8a^2}{(y_{k+1}-y_k)^5} \\ +\dfrac{y_{k+2}-2y_{k+1}+y_k}{(y_{k+1}-y_k)^2(y_{k+2}-y_{k+1})^2} + a^2\dfrac{(y_{k+2}-y_{k+1})-3(y_{k+1}-y_k)}{(y_{k+1}-y_k)^2(y_{k+2}-y_{k+1})^4}, & \text{for } i = k+1; \\[2ex] \dfrac{a}{y_{k+1}-y_k}\Big[\dfrac{1}{(y_{k+2}-y_{k+1})^2} - \dfrac{1}{(y_{k+2}-y_k)^2}\Big] \\ -\dfrac{1}{y_{k+1}-y_k}\dfrac{1}{(y_{k+2}-y_{k+1})^2} - \dfrac{1}{y_{k+1}-y_k}\dfrac{3a^2}{(y_{k+2}-y_{k+1})^4}, & \text{for } i = k+2. \end{cases}$$

Hence

$$\partial_{ij} Q_k = \begin{array}{cc} & \begin{array}{cccccccccc} j=1 & \cdots & k-2 & k-1 & j=k & k+1 & k+2 & \ldots & N-1 & j=N \end{array} \\ \begin{array}{c} \\ i=1 \\ \vdots \\ k-2 \\ k-1 \\ k \\ k+1 \\ k+2 \\ k+3 \\ \vdots \\ i=N \end{array} & \begin{pmatrix} O\left(\frac{1}{a^3}\right) & \mathbf{0} & 0 & 0 & O\left(\frac{1}{a^3}\right) & O\left(\frac{1}{a^3}\right) & 0 & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} & O\left(\frac{1}{a^3}\right) & O\left(\frac{1}{a^3}\right) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & \mathbf{0} & O\left(\frac{1}{a^3}\right) & 0 & O\left(\frac{1}{a^3}\right) & O\left(\frac{1}{a^3}\right) & 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & O\left(\frac{1}{a^4}\right) & O\left(\frac{1}{a^4}\right) & O\left(\frac{1}{a^4}\right) & 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & O\left(\frac{1}{a^4}\right) & O\left(\frac{1}{a^4}\right) & O\left(\frac{1}{a^4}\right) & 0 & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & O\left(\frac{1}{a^4}\right) & O\left(\frac{1}{a^4}\right) & O\left(\frac{1}{a^4}\right) & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & O\left(\frac{1}{a^4}\right) & O\left(\frac{1}{a^4}\right) & O\left(\frac{1}{a^4}\right) & \mathbf{0} & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & O\left(\frac{1}{a^3}\right) & O\left(\frac{1}{a^3}\right) & 0 & O\left(\frac{1}{a^3}\right) & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & O\left(\frac{1}{a^3}\right) & O\left(\frac{1}{a^3}\right) & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ 0 & \mathbf{0} & 0 & 0 & O\left(\frac{1}{a^3}\right) & O\left(\frac{1}{a^3}\right) & 0 & \mathbf{0} & 0 & O\left(\frac{1}{a^3}\right) \end{pmatrix} \end{array},$$

where $\{\partial_{ij} Q_k\}_{i,j=1,\ldots,k-2}$ and $\{\partial_{ij} Q_k\}_{i,j=k+3,\ldots,N}$ are diagonal matrixes with $O(\frac{1}{a^3})$ main diagonal entries and the bold zeros $\mathbf{0}$ represent zero matrices with corresponding dimensions.

For Q_{k-1}, we have a similar Hessian matrix. Notice that only three terms in one row are nonzero and that only at most four terms in one column are of order $\frac{1}{a^4}$. Hence for a small enough, we have

$$\max_k \left(\sqrt{\sum_i \sum_j (\partial_{ij} F_k)^2} \right) \leq C(\beta, M) \frac{1}{a^5}.$$

where $C(\beta, M)$ is a constant depending only on $\beta,\ M$.

Then from (6.17) and the a priori condition (6.14), we have

$$\langle x - y, (x - y)\nabla^2 F_N(y)(x - y)^T \rangle \leq C(\beta, M) a^{\frac{1}{3}} \|x - y\|_{\ell^2}^2.$$

Combining this with (6.16), together with linearized stability in Proposition 6.1, gives

$$\frac{\mathrm{d}\|x - y\|_{\ell^2}^2}{\mathrm{d}t} \leq C(\beta, M)\|x - y\|_{\ell^2}^2 + C(\beta, M) a^7 \|x - y\|_{\ell^2}.$$

Therefore by Grönwall's inequality, we obtain

$$\|x(t) - y(t)\|_{\ell^2} \leq C(\beta, M, T_m)(\|x(0) - y(0)\|_{\ell^2} + a^7), \quad \text{for } t \in [0, T_m], \tag{6.18}$$

where $C(\beta, M, T_m)$ is a constant depending only on $\beta,\ M,\ T_m$. We choose initial data of y such that $y(0) = x(0)$, so (6.18) leads to (6.15).

Step 2. Now we need to verify the a priori assumption (6.14) is true for $t \in [0, T_m]$. In fact,

$$\|x(t) - y(t)\|_{\ell^\infty} \leq \frac{\|x(t) - y(t)\|_{\ell^2}}{\sqrt{a}} \leq C(\beta, M, T_m) a^{7-\frac{1}{2}} \ll a^{6+\frac{1}{3}},$$

for a small enough, $t \in [0, T_m]$. Hence (6.15) actually verifies the a priori condition (6.14).

Step 3. For the exact strong solution ϕ of (2.13), recall the nodal values $\phi_N = \{\phi_i,\ i = 1, \ldots, N\}$. By Proposition 6.2, we know that the constructed function y in (6.10) satisfies

$$\|y(t) - \phi_N(t)\|_{\ell^2} = \|a\psi_N(t)\|_{\ell^2} \leq ca, \quad \text{for } t \in [0, T_m],$$

where we used $\psi(t)$, defined in Proposition 6.2, which is uniformly bounded. This, together with (6.15), shows that

$$\|x(t) - \phi_N(t)\|_{\ell^2} \leq \|x(t) - y(t)\|_{\ell^2} + \|y(t) - \phi_N(t)\|_{\ell^2} \leq C(\beta, M, T_m)a, \quad \text{for } t \in [0, T_m], \tag{6.19}$$

where $C(\beta, M, T_m)$ is a constant depending only on $\beta,\ M,\ T_m$. This completes the proof of Theorem 1.2. □

Acknowledgements We would like to thank the support by the National Science Foundation under Grants DMS-1514826 (JGL), DMS-1454939 (JL), and also through the research network KI-Net RNMS11-07444. We thank Dionisios Margetis, Jeremy Marzuola, Yang Xiang and Aaron N.K. Yip for helpful discussions.

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