

Patched peakon weak solutions of the modified Camassa–Holm equation

Yu Gao ^{a,c,*}, Lei Li ^b, Jian-Guo Liu ^c

^a Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, PR China

^b Institute of Natural Sciences, School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, 200240, PR China

^c Department of Mathematics and Department of Physics, Duke University, Durham, NC 27708, USA

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ABSTRACT

In this paper, we study traveling wave solutions and peakon weak solutions of the modified Camassa–Holm (mCH) equation with dispersive term $2ku_x$ for $k \in \mathbb{R}$. We study traveling wave solutions through a Hamiltonian system obtained from the mCH equation by using a nonlinear transformation. The typical traveling wave solutions given by this Hamiltonian system are unbounded or multi-valued. We provide a method, called patching technic, to truncate these traveling wave solutions and patch different segments to obtain patched bounded single-valued peakon weak solutions which satisfy jump conditions at peakons. Then, we study some special peakon weak solutions constructed by the fundamental solution of the Helmholtz operator $1 - \partial_{xx}$, which can also be obtained by the patching technic. At last, we study some length and total signed area preserving closed planar curve flows that can be described by the mCH equation when $k = 1$, for which we give a Hamiltonian structure and use the patched periodic peakon weak solutions to investigate loops with peakons.

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1. Introduction

This paper is concerned with traveling wave solutions and peakon weak solutions to the following nonlinear partial differential equation:

$$m_t + 2ku_x + [(u^2 - u_x^2)m]_x = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R}, \quad t > 0. \quad (1.1)$$

This equation is referred to as the modified Camassa–Holm (mCH) equation with cubic nonlinearity, which was introduced by several different authors [1–4]. When $k = 0$, sometimes the mCH equation is also called the cubic Camassa–Holm equation or the Fokas–Olver–Rosenau–Qiao equation. The parameter $k \in \mathbb{R}$ characterizes the magnitude of the linear dispersion. The mCH equation is a water wave equation and a suitable approximation of the incompressible irrotational Euler system. The functions u and m represent, respectively, the velocity of the fluid and its potential density [4]. The weak solutions can be non-zero or even unbounded as $|x| \rightarrow \infty$. If a weak solution is periodic with period ℓ where $0 < \ell < +\infty$, then it can be regarded as a weak solution of the mCH equation on the torus $\mathbb{T}_\ell := [-\ell/2, \ell/2] \cong \mathbb{R}/(\ell\mathbb{Z})$. We also denote $\mathbb{T}_\ell = \mathbb{R}$ when $\ell = +\infty$ for convenience.

* Corresponding author at: Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, PR China.

E-mail addresses: yugao@hit.edu.cn (Y. Gao), leili2010@sjtu.edu.cn (L. Li), jliu@phy.duke.edu (J.-G. Liu).

When $\ell < +\infty$, the fundamental solution $G(x)$ for the Helmholtz operator $1 - \partial_{xx}$ is given by

$$G(x) = \begin{cases} \frac{\cosh(x+\ell/2)}{2 \sinh(\ell/2)}, & \text{for } x \in [-\ell/2, 0), \\ \frac{\cosh(x-\ell/2)}{2 \sinh(\ell/2)}, & \text{for } x \in [0, \ell/2), \end{cases} \quad (1.2)$$

and when $\ell = +\infty$, we have

$$G(x) = \frac{1}{2}e^{-|x|}, \quad \text{for } x \in \mathbb{R}. \quad (1.3)$$

The velocity function u can be written as a convolution of m with G

$$u(x, t) = G * m = \int_{\mathbb{T}_\ell} G(x - y)m(y, t)dy, \quad x \in \mathbb{T}_\ell.$$

For smooth solutions to the mCH equation (1.1), there are two conserved quantities (called Hamiltonian functionals of the mCH equation)

$$\begin{aligned} H_0 &= \int_{\mathbb{T}_\ell} mudx, \\ H_1 &= \frac{1}{4} \int_{\mathbb{T}_\ell} \left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4 + 4ku^2 \right) dx. \end{aligned} \quad (1.4)$$

(1.1) can be written in the bi-Hamiltonian form [3,5],

$$m_t = -((u^2 - u_x^2)m)_x = J \frac{\delta H_0}{\delta m} = K \frac{\delta H_1}{\delta m},$$

where

$$J = -\partial_x m \partial_x^{-1} m \partial_x - k \partial_x, \quad K = \partial_x^3 - \partial_x$$

are compatible Hamiltonian operators. (If the Hamiltonian operators J and K are compatible, then any constant coefficient linear combination $\alpha J + \beta K$ is also a Hamiltonian operator.) The Hamiltonian pair J, K is non-degenerate in the sense that one of the associated Poisson structure is symplectic. According to the fundamental theorem of Magri [6], any bi-Hamiltonian system associated with a non-degenerate Hamiltonian pair induces a hierarchy of commuting Hamiltonians and flows, and, provided there are enough functionally independent Hamiltonians, is therefore completely integrable. The mCH equation (1.1) is a complete integrable system and it possesses a Lax pair and may be solved by means of the inverse scattering transform (IST) method [4].

In this paper, one of our main purposes is to introduce a patching technic as described below to construct peakon weak solutions from traveling wave solutions to the mCH equation (1.1), so that we can change unbounded or multi-valued solutions into single-valued bounded weak solutions. To this end, we first give some jump conditions (see Theorem 2.1) for piecewise smooth weak solutions to (1.1). Then, we study traveling wave solutions in the form $u(x, t) = \phi(x - ct) = \phi(\xi)$ ($\xi := x - ct$) with traveling speed $c \in \mathbb{R}$. Under the traveling wave assumption, Eq. (1.1) can be reduced to a third order autonomous ODE (see (3.1)). After a first integral, this third order ODE can be reduced to a second order ODE (see (3.2)). Then, after some transformation this second order ODE can be rewritten as a Hamiltonian system with a Hamiltonian H (see (3.6)). Then, we study the structure of level sets of H in the phase plane (see Proposition 3.2). We show that traveling wave solutions to the mCH equation can be constructed from level sets of H (see Proposition 3.3). Consider a level set $H = h$ for some constant h . There are two particular cases for solutions corresponding to $H = h$: the case $k \leq 0$ and the case $k > 0$. When $k \leq 0$ and h is big enough, there is a trajectory in the phase plane given by $H = h$ extending to infinity (see Theorem 3.1) and this corresponds to a single-valued traveling wave solution to (1.1) that satisfies $\lim_{|\xi| \rightarrow +\infty} |\phi(\xi)| = +\infty$. For the case $k > 0$, the trajectories in the phase plane given by $H = h$ are some closed curves and when the traveling speed $c < 0$ or h big enough for $c > 0$, these closed curves yield multi-valued (see Definition 3.2) periodic traveling wave solutions (see Theorem 3.1).

When $k > 0$, the patching technic is used to obtain a patched single-valued periodic traveling peakon weak solution from a multi-valued periodic traveling wave solution. Briefly speaking, we truncate the level set $H = h$ by curves $\phi^2 - \frac{1}{3}v^2 = c$ in phase plane. The two intersection points on the same branch of $\phi^2 - \frac{1}{3}v^2 = c$ glue together and form a patched single-valued periodic traveling wave weak solution with peakon(s). This truncating method guarantees that the jump condition is satisfied at the peakons and hence a weak solution (see Picture 8). In this paper, this method is called ‘the patching technic’ which can also be used to obtain patched bounded traveling peakon weak solutions for the case $k \leq 0$.

By the patching technic, we can obtain some special peakon weak solutions of the form

$$u(x, t) = pG(x - ct), \quad m(x, t) = p\delta(x - ct), \quad x \in \mathbb{R}, \quad t \geq 0, \quad (1.5)$$

where $p \neq 0$ is a constant describing the amplitude of a peakon and the traveling speed is given by $c = \frac{1}{6}p^2$ (see Remark 4.2). (1.5) is a peakon weak solution to the dispersionless mCH equation (1.1) (with $k = 0$) in \mathbb{R} and it was also obtained in [5]. On the periodic domain \mathbb{T}_ℓ ($\ell < +\infty$), the mCH equation also has peakon weak solutions with similar form as (1.5) but the traveling speed is given by (see Proposition 4.1)

$$c = \frac{1}{4}p^2 \left[\coth^2 \left(\frac{\ell}{2} \right) - \frac{1}{3} \right].$$

As $\ell \rightarrow \infty$, this speed is consistent with the speed $\frac{1}{6}p^2$ in \mathbb{R} . Notice that a solution given by (1.5) satisfies $m(x, t) = 0$ in the smooth region of u and the superposition of such solutions gives a new solution in the smooth region. Hence, the superposition of N such solutions with different traveling speeds can form an N -peakon weak solution if the jump conditions are satisfied along the trajectories of peakons (see Proposition 4.2).

When $k < 0$, there are peakon weak solutions similar to (1.5) of the form (see Proposition 4.3)

$$u(x, t) = pG(x - ct) - \sqrt{-k}, \quad m(x, t) = p\delta(x - ct) - \sqrt{-k}. \quad (1.6)$$

When $\ell < \infty$, the traveling speed is given by

$$c = \left[\frac{p}{2} \coth \left(\frac{\ell}{2} \right) - \sqrt{-k} \right]^2 - \frac{p^2}{12},$$

and when $\ell = +\infty$, it is given by

$$c = \left(\frac{p}{2} - \sqrt{-k} \right)^2 - \frac{p^2}{12}.$$

This kind of peakon weak solutions can also be obtained by the patching technic (see Remark 4.4). Notice that in the whole space \mathbb{R} , this kind of solutions will not vanish as $x \rightarrow \infty$.

For the case $k > 0$, the mCH equation (1.1) does not have peakon weak solutions of the form similar as (1.5) or (1.6) (see Proposition 4.4). However, the mCH equation (1.1) with $k > 0$ has peakon weak solutions in other forms, which can be obtained by the patching technic (see Fig. 8).

Specially, when $k = 1$, the mCH equation (1.1) can be used to describe planar curve flows which preserve arc-length [5]. In this paper, we show that in periodic case, the mCH equation ($k = 1$) describes some dynamics of closed plane curves whose length and total signed area (defined by (5.4)) are both preserved. This property is also shared with the modified Korteweg–de Vries equation. Moreover, we give a Hamiltonian structure for the curve dynamics (see Proposition 5.1) and use the patched peakon weak solutions to investigate loops with peakons (see Fig. 11).

For more results about local well-posedness and blow-up behaviors of the strong solutions to (1.1), one can refer to [7,8,5,9,10]. In [11], Zhang used a method of dissipative approximation to prove the existence and uniqueness of global entropy weak solutions u in $W^{2,1}(\mathbb{R})$ for the dispersionless mCH equation (1.1) ($k = 0$). In [12], a sticky particle model is provided and a solution to this model yields a sticky N -peakon weak solution which is a superposition of N peakons in the form (1.5). By some space–time BV estimates, the mean field limit of this model gives a global weak solution u to the dispersionless mCH equation in \mathbb{R} and u, u_x are space–time BV functions. When $k > 0$, Matsuno [13] studied multi-soliton solutions to (1.1) in \mathbb{R} by using the Hirota bilinear transformation method [14]. Both smooth and singular multi-soliton solutions are studied. The underlying structure of the associated tau-functions constituting the N -soliton solution is the same as that of the N -soliton solution of a model equation for shallow-water waves introduced by Ablowitz et al. [15] (see [13, Section 2]).

Let us remark that the traveling wave solutions to one-dimensional nonlinear shallow water models and the Camassa–Holm (CH) equation [16] (with dispersive term $2ku_x, k \in \mathbb{R}$)

$$u_t - u_{xxt} + 3uu_x + 2ku_x = 2u_x u_{xx} + uu_{xxx}, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.7)$$

have been thoroughly discussed in literature (see [17–20] for examples). In [17], a class of peaked traveling wave solutions of the form $u(x, t) = \sum_{n=1}^{\infty} a_n e^{-n|x-qt|}$ was constructed for a nonlinear shallow water models, and the stability of these solutions was also proved. The authors of [18–20] studied the traveling wave solutions of the CH equation (1.7), another classical model from shallow water problems. The idea of studying traveling wave solutions of CH equation using level sets of the Hamiltonian for a

reduced system appeared in [18], and part of our idea is indeed motivated by [18]. The Hamiltonian we used (Eq. (3.5)) is of fourth degree and introduces many challenging issues compared with the quadratic one used in [18]. Lenells [19] did a thorough work in classifying all the weak traveling wave solutions to the CH equation. A patching and gluing idea also appeared in [19] for the CH equation, where the composite wave is obtained by jointing cuspons or peakons with the same speed at their crests. Note that one of the main aims of our patching technic is to obtain bounded, single-valued solutions to the mCH equation from unbounded or multi-valued ones, while the composite waves in [19] are not out of this consideration. Furthermore, the segments we use are from any kind of solutions (not necessarily cuspons or peakons), and the joint points are not necessarily the crests.

The rest of this article is organized as follows. In Section 2, we give the definition of weak solutions to the mCH equation (1.1) and obtain some jump conditions for piecewise smooth weak solutions. In Section 3, we deduce an ODE system to study traveling wave solutions to (1.1). Traveling wave solutions are given by the level set of the first integral of the ODE system. Then, we construct peakon weak solutions from traveling wave solutions by the patching technic. In Section 4, we show a special class of peakon weak solutions to (1.1) when $k \leq 0$. These peakon weak solutions can also be obtained by the patching technic given in Section 3. In Section 5, we give some criteria for preserving length and total signed area for closed curve flows. Moreover, we show that some length and total signed area preserving curve flows in \mathbb{R}^2 can be described by the mCH equation with $k = 1$. These curve flows have a Hamiltonian structure. Lastly, we use the patched traveling peakon solutions in Section 3 to investigate loops with peakons.

2. Jump conditions for piecewise smooth weak solutions

The physical solutions of (1.1) are usually piecewise smooth and contain some non-smooth points. Due to the non-smooth points, these solutions must be understood in the weak sense. In this section, we first introduce the definition of weak solutions and obtain jump conditions for piecewise smooth weak solutions to the mCH equation (1.1). Our patching technic in Section 3.3 is based on these jump conditions.

Definition 2.1. For $\ell < +\infty$ and $u_0 \in W^{1,\infty}(\mathbb{T}_\ell)$, a function

$$u \in L^\infty(0, T; W^{1,\infty}(\mathbb{T}_\ell)) \cap C([0, T]; H^1(\mathbb{T}_\ell)),$$

is a weak solution of the mCH equation (1.1) subject to initial data $u(x, 0) = u_0(x)$ if

$$\langle u, \varphi \rangle := \mathcal{L}(u, \varphi) + \int_{\mathbb{T}_\ell} u_0 \varphi(x, 0) dx - \int_{\mathbb{T}_\ell} u_0 \varphi_{xx}(x, 0) dx = 0 \quad (2.1)$$

where

$$\begin{aligned} \mathcal{L}(u, \varphi) := & \int_0^T \int_{\mathbb{T}_\ell} u(\varphi_t - \varphi_{xxt}) dx dt \\ & + \int_0^T \int_{\mathbb{T}_\ell} (2ku + u^3 + uu_x^2) \varphi_x dx dt \\ & - \frac{1}{3} \int_0^T \int_{\mathbb{T}_\ell} u^3 \varphi_{xxx} dx dt - \frac{1}{3} \int_0^T \int_{\mathbb{T}_\ell} u_x^3 \varphi_{xx} dx dt. \end{aligned}$$

holds for all $\varphi \in C_c^\infty([0, T] \times \mathbb{T}_\ell)$. If $T = +\infty$, we call u as a global weak solution of the mCH equation.

When $\ell = +\infty$ ($\mathbb{T}_\ell = \mathbb{R}$) and $u_0 \in W_{loc}^{1,\infty}(\mathbb{R})$, we say a function

$$u \in L^\infty(0, T; W_{loc}^{1,\infty}(\mathbb{R})) \cap C([0, T]; H_{loc}^1(\mathbb{R})),$$

is a weak solution of the mCH equation (1.1) subject to initial data $u(x, 0) = u_0(x)$ if (2.1) holds for all $\varphi \in C_c^\infty([0, T] \times \mathbb{R})$.

It is straightforward to check that the definition is consistent. In other words, a weak solution on \mathbb{T}_ℓ can be identified as a weak solution on $\mathbb{T}_{n\ell}$ for $n = 1, 2, \dots$ or \mathbb{R} if we extend the function periodically. Reversely, a periodic weak solution on \mathbb{R} with period ℓ , (or a periodic weak solution on \mathbb{T}_{ℓ_1} with smaller period $\ell < \ell_1$) can be regarded as a weak solution on \mathbb{T}_ℓ . This consistency allows us to identify the periodic weak solutions to (1.1) with weak solutions on \mathbb{T}_ℓ .

Now, we state and prove the following results about jump conditions in this section.

Theorem 2.1. Assume $0 < \ell < +\infty$. Suppose the region $\mathbb{T}_\ell \times [0, T]$ is divided by N (no intersection) curves $\{(x_i(t), t) : 0 \leq t \leq T\}_{i=1}^N$ into N non-overlap open area $\{U_i\}_{i=1}^N$, where N is a positive integer. The boundary between U_i and U_{i+1} is $x_i(t)$ and $x_N(t)$ is the boundary between U_N and $U_{N+1} = U_1$. Assume that $u_i \in C^3(U_i) \cap C^1(\bar{U}_i)$ solves the mCH equation with $u_i(x, 0) = u_0(x)$ in U_i and $m_i = u_i - \partial_{xx} u_i$ for $1 \leq i \leq N$. Denote

$$v_\ell^i(t) := \partial_x u_i(x_i(t), t), \quad v_r^i(t) := \partial_x u_{i+1}(x_i(t), t)$$

and assume the following two limits exist

$$\begin{cases} \lim_{x \rightarrow x_i(t)^-} m_i [u_i^2 - (\partial_x u_i)^2 - x_i'(t)] =: A_\ell^i(t), \\ \lim_{x \rightarrow x_i(t)^+} m_{i+1} [u_{i+1}^2 - (\partial_x u_{i+1})^2 - x_i'(t)] =: A_r^i(t). \end{cases}$$

Then, the function u satisfying $u|_{U_i} = u_i$ is a weak solution to the mCH equation subject to $u(x, 0) = u_0(x)$ if the following conditions are satisfied:

1. $u \in C(\mathbb{T}_\ell \times [0, T])$.
2. For each i , it holds that

$$\begin{cases} \frac{d}{dt} x_i(t) = u(x_i(t), t)^2 - \frac{1}{3} [v_\ell^i(t)]^2 - \frac{1}{3} v_\ell^i(t) v_r^i(t) \\ \quad - \frac{1}{3} [v_r^i(t)]^2, \\ \frac{d}{dt} [v_\ell^i(t) - v_r^i(t)] = A_\ell^i(t) - A_r^i(t) \end{cases} \quad (2.2)$$

or

$$v_\ell^i = v_r^i, \quad A_\ell^i = A_r^i. \quad (2.3)$$

When $\ell = +\infty$ ($\mathbb{T}_\ell = \mathbb{R}$), we have the same jump conditions.

Proof. By the fact $u_i \in C^1(\bar{U}_i)$ and Condition 1, it is clear that

$$u \in L^\infty(0, T; W^{1,\infty}(\mathbb{T}_\ell)) \cap C([0, T]; H_{loc}^1(\mathbb{T}_\ell)).$$

For the Condition 2, we first assume (2.2). Take $\varphi(x, t) \in C_c^\infty([0, T] \times \mathbb{T}_\ell)$. For initial data, we have

$$\begin{aligned} \int_{\mathbb{T}_\ell} u_0 \varphi(x, 0) dx - \int_{\mathbb{T}_\ell} u_0 \varphi_{xx}(x, 0) dx &= \sum_{i=1}^N \int_{x_{i-1}(0)}^{x_i(0)} m_i(x, 0) \varphi(x, 0) dx \\ &+ \sum_{i=1}^N \varphi(x_i(0), 0) (v_\ell^i(0) - v_r^i(0)), \end{aligned} \quad (2.4)$$

where $x_0 = x_N$. From Definition 2.1, we have

$$\begin{aligned} \mathcal{L}(u, \varphi) = & \int_0^T \int_{\mathbb{T}_\ell} u(\varphi_t - \varphi_{xxt}) dx dt \\ & + \int_0^T \int_{\mathbb{T}_\ell} (2ku + u^3 + uu_x^2) \varphi_x dx dt \\ & - \frac{1}{3} \int_0^T \int_{\mathbb{T}_\ell} u^3 \varphi_{xxx} dx dt - \frac{1}{3} \int_0^T \int_{\mathbb{T}_\ell} u_x^3 \varphi_{xx} dx dt. \end{aligned} \quad (2.5)$$

With some calculations, we obtain

$$\begin{aligned} \int_0^T \int_{\mathbb{T}_\ell} u(\varphi_t - \varphi_{xxt}) dx dt &= \int_0^T \int_{\mathbb{T}_\ell} u\varphi_t dx dt \\ &+ \sum_{i=1}^N \int_0^T \int_{x_{i-1}(t)}^{x_i(t)} u_x \varphi_{xt} dx dt \\ &= \int_0^T \sum_{i=1}^N (v_\ell^i(t) - v_r^i(t)) \varphi_t(x_i(t), t) dt \\ &+ \sum_{i=1}^N \iint_{U_i} m_i(x, t) \varphi_t dx dt \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} -\frac{1}{3} \int_0^T \int_{\mathbb{T}_\ell} u^3 \varphi_{xxx} dx dt - \frac{1}{3} \int_0^T \int_{\mathbb{T}_\ell} u_x^3 \varphi_{xx} dx dt \\ = \sum_{i=1}^N \int_0^T (v_\ell^i - v_r^i) \varphi_x(x_i(t), t) \frac{d}{dt} x_i(t) dt \\ - \sum_{i=1}^N \iint_{U_i} (2uu_x^2 + u^2u_{xx} - u_x^2u_{xx}) \varphi_x dx dt, \end{aligned} \tag{2.7}$$

where we used the jump condition (2.2) for the last step. On each U_i , u_{xx} agrees with $(u_i)_{xx}$, which is a continuous function. Combining (2.5), (2.6) and (2.7) gives

$$\begin{aligned} \mathcal{L}(u, \varphi) &= \sum_{i=1}^N \iint_{U_i} m_i(x, t) \varphi_t dx dt \\ &+ \sum_{i=1}^N \iint_{U_i} (2ku + m_i(u^2 - u_x^2)) \varphi_x dx dt \\ &+ \sum_{i=1}^N \int_0^T [(v_\ell^i(t) - v_r^i(t)) \varphi_t(x_i(t), t) \\ &+ (v_\ell^i(t) - v_r^i(t)) \varphi_x(x_i(t), t) \frac{d}{dt} x_i(t)] dt \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

For $I_1 + I_2$, we have

$$\begin{aligned} I_1 + I_2 &= \sum_{i=1}^N \iint_{U_i} (m_i \varphi_t + (2ku + m_i(u^2 - u_x^2)) \varphi_x) dx dt \\ &= \sum_{i=1}^N \iint_{U_i} ((m_i \varphi)_t + ((2ku + m_i(u^2 - u_x^2)) \varphi)_x) dx dt \\ &= \sum_{i=1}^N \oint_{\partial U_i} -m_i \varphi dx + (2ku_i + m_i(u_i^2 - u_{ix}^2)) \varphi dt. \end{aligned}$$

The boundary integral on $t = 0$ together with the initial value terms yields

$$-\sum_{i=1}^N \int_{x_{i-1}(0)}^{x_i(0)} m_i(x, 0) \varphi(x, 0) dx.$$

The integral on $t = T$ is zero since φ vanishes there. By the continuity of u and the jump condition, we then find

$$\begin{aligned} I_1 + I_2 &= -\sum_{i=1}^N \int_{x_{i-1}(0)}^{x_i(0)} m_i(x, 0) \varphi(x, 0) dx \\ &+ \sum_{i=1}^N \int_0^T [m_i x'_{i-1}(t) - m_i(u_i^2 - u_{ix}^2)] \varphi \Big|_{x=x_{i-1}} dt \end{aligned}$$

$$\begin{aligned} &- \sum_{i=1}^N \int_0^T [m_i x'_i(t) - m_i(u_i^2 - u_{ix}^2)] \varphi \Big|_{x=x_i(t)} dt \\ &= -\sum_{i=1}^N \int_{x_{i-1}(0)}^{x_i(0)} m_i(x, 0) \varphi(x, 0) dx + \sum_{i=1}^N \int_0^T \varphi(x_i(t), t) (A_\ell^i - A_r^i) dt. \end{aligned}$$

By the fact

$$I_3 = \sum_{i=1}^N \int_0^T (v_\ell^i(t) - v_r^i(t)) \frac{d}{dt} \varphi(x_i(t), t) dt,$$

(2.4) and Condition 2, we have:

$$\begin{aligned} \mathcal{L}(u, \varphi) &= -\sum_{i=1}^N \int_{x_{i-1}(0)}^{x_i(0)} m_i(x, 0) \varphi(x, 0) dx \\ &+ \sum_{i=1}^N \int_0^T \frac{d}{dt} [\varphi(x_i(t), t) (v_\ell^i(t) - v_r^i(t))] dt \\ &= -\sum_{i=1}^N \int_{x_{i-1}(0)}^{x_i(0)} m_i(x, 0) \varphi(x, 0) dx \\ &- \sum_{i=1}^N \varphi(x_i(0), 0) (v_\ell^i(0) - v_r^i(0)) \\ &= -\int_{\mathbb{T}_\ell} u_0 \varphi(x, 0) dx + \int_{\mathbb{T}_\ell} u_0 \varphi_{xx}(x, 0) dx. \end{aligned}$$

Therefore, by Definition 2.1, u is a weak solution.

For the second condition, if we instead assume (2.3), we find

$$\begin{aligned} \mathcal{L}(u, \varphi) &= \sum_{i=1}^N \iint_{U_i} m_i(x, t) \varphi_t dx dt \\ &+ \sum_{i=1}^N \iint_{U_i} (2ku + m_i(u^2 - u_x^2)) \varphi_x dx dt \\ &= -\sum_{i=1}^N \int_{x_{i-1}(0)}^{x_i(0)} m_i(x, 0) \varphi(x, 0) dx \\ &= -\int_{\mathbb{T}_\ell} u_0 \varphi(x, 0) dx + \int_{\mathbb{T}_\ell} u_0 \varphi_{xx}(x, 0) dx. \end{aligned}$$

Hence it is also a weak solution.

For the case $\mathbb{T}_\ell = \mathbb{R}$ ($\ell = +\infty$), the proof is similar and we omit it. \square

3. Traveling wave solutions of the mCH equation

Consider traveling wave solutions of the form $u(x, t) = \phi(\xi)$, where $\xi := x - ct$ and $c \in \mathbb{R}$ is a traveling speed. Hence, $m = u - u_{xx} = \phi - \phi''$. Then, the mCH equation is reduced to the following ODE:

$$c(\phi''' - \phi') + 2k\phi' + ((\phi - \phi'')(\phi^2 - (\phi')^2))' = 0. \tag{3.1}$$

Integrating once yields the following equation

$$(c - \phi^2 + (\phi')^2)\phi'' - c\phi + 2k\phi + \phi(\phi^2 - (\phi')^2) = g, \tag{3.2}$$

where g is an integrating constant.

If ϕ is a solution for g , then $-\phi$ is a solution for $-g$. Therefore, the structure of solutions with positive g will be the same as that of the solutions for negative g . Hence, we assume from here on that

$$g \geq 0. \tag{3.3}$$

Introducing $v = \phi'$, from (3.2) we can deduce the following first order system

$$\begin{cases} \frac{d\phi}{d\xi} = v, \\ \frac{dv}{d\xi} = \frac{g - 2k\phi}{c - \phi^2 + v^2} + \phi. \end{cases} \quad (3.4)$$

Hence, we have

$$\frac{d\phi}{dv} = \frac{v(c - \phi^2 + v^2)}{g - 2k\phi + \phi(c - \phi^2 + v^2)},$$

which implies

$$(cv + v^3)dv - (g - 2k\phi + c\phi - \phi^3)d\phi - \frac{1}{2}d(\phi^2 v^2) = 0.$$

Hence, the first integral of System (3.4) is given by

$$H = \frac{(\phi^2 - v^2)^2}{4} - \frac{1}{2}c(\phi^2 - v^2) + k\phi^2 - g\phi. \quad (3.5)$$

Changing of variable $d\xi = (c - \phi^2 + v^2)d\tau$, we have the following Hamiltonian system

$$\begin{cases} \frac{d\phi}{d\tau} = \frac{\partial H}{\partial v} = v(c - \phi^2 + v^2), \\ \frac{dv}{d\tau} = -\frac{\partial H}{\partial \phi} = g - 2k\phi - \phi(\phi^2 - v^2 - c). \end{cases} \quad (3.6)$$

Because Systems (3.4) and (3.6) have the same first integral H , the two systems have the same topological phase portraits except the hyperbola $\phi^2 - v^2 - c = 0$. The level sets of H are solution trajectories of System (3.6) and hence give traveling wave solutions of the mCH equation (except the hyperbola $\phi^2 - v^2 = c$). In the following, we first discuss the critical points and level sets of the first integral H . Then, we deduce traveling wave solutions from level sets of H . At last, we provide a patching technic to construct single-valued traveling peakon weak solutions from traveling wave solutions given by level sets of H .

3.1. Critical points and level sets

The critical points of System (3.6) in the phase plane are important for determining the structures of the level sets of H . The following observation enables us to classify the critical points of System (3.6) and the proof is provided in Appendix A.

Lemma 3.1. *Suppose the Hessian of H is non-degenerate at a critical point (ϕ_*, v_*) of System (3.6). Then, (ϕ_*, v_*) is a local extremum of H if and only if it is a center point of System (3.6) in the phase plane while it is a saddle point of H if and only if it is saddle point in the phase plane of System (3.6).*

As mentioned in the above, Systems (3.4) and (3.6) have the same topological phase portraits except the hyperbola curves $\phi^2 - v^2 = c$. The critical points on curves $\phi^2 - v^2 = c$ for system (3.6) can be summarized as follows (we only consider $k \neq 0$ here and the $k = 0$ case is simple) and the proof is again provided in Appendix A:

Proposition 3.1. *When $g^2 - 4k^2c < 0$, there is no critical point on $\phi^2 - v^2 = c$. When $g^2 - 4k^2c > 0$, there are two critical points for System (3.6) on $\phi^2 - v^2 = c$. If $k > 0$, they are center points. If $k < 0$, they are saddle points.*

The critical points on $\phi^2 - v^2 = c$ do not give stationary solutions to System (3.4), but they determine the local structures of the level sets near them. Other critical points of (3.6) are also critical points of (3.4), and they correspond to stationary solutions of System (3.4) or constant traveling wave solutions of the mCH

equation. The critical points of System (3.6) that are not on $\phi^2 - v^2 = c$ are of the form $(\phi_*, 0)$ ($\phi_*^2 \neq c$), where

$$-\phi_*^3 - (2k - c)\phi_* + g = 0. \quad (3.7)$$

At this time, we have

$$\frac{\partial^2 H}{\partial v \partial \phi}(\phi_*, 0) = 0, \quad \frac{\partial^2 H}{\partial \phi^2}(\phi_*, 0) = 3\phi_*^2 + 2k - c,$$

$$\frac{\partial^2 H}{\partial v^2}(\phi_*, 0) = c - \phi_*^2.$$

Hence

$$\begin{aligned} \left(\frac{\partial^2 H}{\partial v \partial \phi}\right)^2 - \frac{\partial^2 H}{\partial \phi^2} \frac{\partial^2 H}{\partial v^2} &= (3\phi_*^2 + 2k - c)(\phi_*^2 - c) \\ &= -(c + 4k)\phi_*^2 + 3g\phi_* - c(2k - c). \end{aligned}$$

By choosing different c , k and g , $(\phi_*, 0)$ can be either a center point or a saddle point.

We do not plan to give a thorough discussion for these critical points. The discussion is straightforward. For example, in the case $g = 0$ and $c - 2k > 0$, there are three critical points with $v = 0$: $(0, 0)$ and $(\pm\sqrt{c - 2k}, 0)$.

For $(0, 0)$, we have

$$\left(\frac{\partial^2 H}{\partial v \partial \phi}\right)^2 - \frac{\partial^2 H}{\partial \phi^2} \frac{\partial^2 H}{\partial v^2} = c(c - 2k).$$

It is a saddle point if $c > 0$ while a center point if $c < 0$. Similarly, for $(\pm\sqrt{c - 2k}, 0)$, we have

$$\left(\frac{\partial^2 H}{\partial v \partial \phi}\right)^2 - \frac{\partial^2 H}{\partial \phi^2} \frac{\partial^2 H}{\partial v^2} = -4k(c - 2k).$$

They are center points if $k > 0$ and saddle points if $k < 0$.

The critical points on the one hand may give stationary solutions and on the other hand determine local structures of level sets. To obtain a sense of the structures of the level sets, we sketch some typical level sets of H for $k = -1$, $k = 0$ and $k = 1$. The curve $\phi^2 - \frac{v^2}{3} = c$ (solid line) and $\phi^2 - v^2 = c$ (dashed line) are also plotted for more information and these curves are important for the patching technic in Section 3.3.

Below, we fix parameters (c, g, k) and consider those h values such that the level set $H = h$ contains more than one point. Denote

$$\Sigma \subset \{(\phi, v) : H(\phi, v) = h\} \quad (3.8)$$

as a connected component that has more than one point. In this paper, without mentioning explicitly, the parameters (c, g, k) are fixed but arbitrary. To relate the connected component Σ to traveling wave solutions of the mCH equation, we first study the structure of Σ .

Denote the upper half and lower half of the phase plane respectively by

$$P_+ := \{(\phi, v) : v \geq 0\} \text{ and } P_- := \{(\phi, v) : v \leq 0\}. \quad (3.9)$$

We give some lemmas that are useful to study the structure of Σ .

Lemma 3.2. *Assume Σ is a connected component of level set $H = h$ for some constant $h \in \mathbb{R}$. Then, Σ intersects with $v = 0$ for at most four points. If Σ intersects with $v = 0$, then Σ is symmetric about $v = 0$.*

The first claim follows from the fact that H is a polynomial of degree four. The proof of this lemma is simple and we omit it.

Lemma 3.3. *Denote $P_1 := \{(\phi, v) : \phi^2 - v^2 - c \leq 0\}$ and $P_2 := \{(\phi, v) : \phi^2 - v^2 - c \geq 0\}$. Assume $(k, g) \neq (0, 0)$. Consider*

the two branches of $H = h$ given by

$$\begin{aligned} \Gamma_+ &:= \{(\phi, v) : v^2 = \phi^2 - c + \sqrt{c^2 - 4(k\phi^2 - g\phi - h)}\}, \\ \Gamma_- &:= \{(\phi, v) : v^2 = \phi^2 - c - \sqrt{c^2 - 4(k\phi^2 - g\phi - h)}\}. \end{aligned} \quad (3.10)$$

Then, the following statements hold:

- (i) $\Gamma_+ \subset P_1$ and $\Gamma_- \subset P_2$.
- (ii) If Γ_+ and Γ_- are connected, then the joint is on the hyperbola $\phi^2 - v^2 = c$ and they can be connected at most four points. If Γ_+ (Γ_-) intersects $\phi^2 - v^2 = c$ at a point with $v \neq 0$, then it connects to Γ_- (Γ_+).
- (iii) Assume $\Gamma_+ \cap P_+$ is not empty. Then, this arc is the graph of a function $v = v(\phi)$. And any endpoint of $\Gamma_+ \cap P_+$ must be: (a) on $v = 0$, (b) on $\phi^2 - v^2 = c$ with $v \neq 0$, or (c) at infinity. This statement is also true for $\Gamma_+ \cap P_-$ and $\Gamma_- \cap P_{\pm}$.

Proof. Step 1. Statement (i) is obvious.

Step 2. For Statement (ii), assume $(\phi_0, v_0) \in \Gamma_+ \cap \Gamma_-$ is a point where Γ_+ and Γ_- are connected. By the definitions of Γ_+ and Γ_- , we have

$$c^2 - 4(k\phi_0^2 - g\phi_0 - h) = 0 \text{ and } \phi_0^2 - v_0^2 = c,$$

which means the joint is on the hyperbola $\phi^2 - v^2 = c$. Since $(k, g) \neq (0, 0)$, there are at most two ϕ_0 such that $c^2 - 4(k\phi_0^2 - g\phi_0 - h) = 0$. Each ϕ_0 gives two values of v_0 (at most). Hence, the two components can be connected at most four points.

Next, assume Γ_+ intersects $\phi^2 - v^2 = c$ at a point (ϕ_1, v_1) with $v_1 \neq 0$. Hence, $\phi_1^2 - c = v_1^2 > 0$ and $\sqrt{c^2 - 4(k\phi_1^2 - g\phi_1 - h)} = 0$, which implies $(\phi_1, v_1) \in \Gamma_-$. Similar arguments are performed if Γ_- intersects with the hyperbola.

Step 3. For Statement (iii), consider the curve Γ_+ in P_+ . It is the graph of function

$$v(\phi) = \sqrt{\phi^2 - c + \sqrt{c^2 - 4(k\phi^2 - g\phi - h)}}.$$

Assume D is the largest domain that $v(\phi)$ is defined on. If ∂D contains some points $\tilde{\phi}$, then we have $v(\tilde{\phi}) = 0$ or

$\sqrt{c^2 - 4(k\tilde{\phi}^2 - g\tilde{\phi} - h)} = 0$. Otherwise, there is a neighborhood such that we can extend the domain, contradicting with the fact that D is the largest domain. The same argument applies to $\Gamma_+ \cap P_-$ and $\Gamma_- \cap P_{\pm}$. \square

Now, we are able to conclude the structure of the connected component Σ .

Proposition 3.2. Suppose Σ is a connected component of a level set of $H = h$ that does not contain $(\phi_0, 0)$ with $\phi_0^2 = c$. Then, one of the following two statements holds:

- (i) $\Sigma = \Gamma$ and Γ is a simple curve that is either a closed loop, or with both sides tending to infinity in the phase plane.
- (ii) $\Sigma = \cup_{i=1}^N \Gamma_i$, with Γ_i being a simple curve as described in (i) If $\Gamma_i \cap \Gamma_j \neq \emptyset$ for $i \neq j$, then any $p \in \Gamma_i \cap \Gamma_j$ is a critical point of System (3.6).

If the two sides of Γ_i tend to infinity, then both $|v|$ and $|\phi|$ tend to infinity. Moreover, for $\Sigma \cap P_+$ ($\Sigma \cap P_-$) containing more than one point, it can be represented by a graph of some function $v = f(\phi)$ if and only if Σ does not intersect $\phi^2 - v^2 = c$.

The claims in Proposition 3.2 are clear by Lemma 3.3, but the complete proof is tedious and we put it into Appendix B.

Remark 3.1. By using the Poincaré–Bendixson Theorem, we can conclude that if Σ is bounded and does not pass through a critical point, it must be a simple loop. When $k > 0$, from (3.10) we can tell that the level set of $H = h$ is bounded for any $h \in \mathbb{R}$. Hence, the connected component of level sets $H = h$ when $k > 0$ is a simple

loop (see Fig. 3), if it contains more than one point and does not pass through a critical point.

3.2. Traveling wave solutions

From here on, we will use the same notation Γ to represent a simple curve in the connected component Σ as described in Proposition 3.2 that does not pass through a point $(\phi_0, 0)$ with $\phi_0^2 = c$. Due to $v = \frac{d\phi}{d\xi}$, we have

$$d\xi = \frac{1}{v} d\phi. \quad (3.11)$$

To be convenient, we consider those simple curves Γ for which we can define a period

$$T := \int_{\Gamma} \frac{1}{v} d\phi, \quad (3.12)$$

where the orientation of Γ is chosen such that $T \in [0, \infty]$. Note that it is possible that $T = \infty$.

Remark 3.2. Requiring that T is defined excludes some U-shaped curves, as shown in Figs. 1(b) and 2(b). T cannot be defined for such a curve with any orientation. For example, consider the U-shaped curve Γ in P_+ from Fig. 2(b), with clockwise orientation. $\Gamma = \gamma_1 \cup \gamma_2$, $\gamma_1 \subset \Gamma_+$ and $\gamma_2 \subset \Gamma_-$. It is clear that $\int_{\gamma_1} \frac{1}{v} d\phi = +\infty$ and $\int_{\gamma_2} \frac{1}{v} d\phi = -\infty$. Hence, $T = \int_{\gamma_1 \cup \gamma_2} \frac{1}{v} d\phi$ is not defined on such a curve.

There are some key points on Γ that need careful investigation. For a point (ϕ, v) , we denote

$$\Gamma_{\delta}(\phi, v) := \Gamma \cap B((\phi, v), \delta), \quad \delta > 0, \quad (3.13)$$

with the same orientation as Γ . The following two lemmas tell us the behaviors of the solutions given by Γ near some key points:

Lemma 3.4. Suppose Γ passes through $(\phi_*, 0)$ and $(\phi_*, 0)$ is not a critical point, then Γ is perpendicular with $v = 0$ axis at $(\phi_*, 0)$ and for $\delta > 0$ small enough, we have

$$\int_{\Gamma_{\delta}(\phi_*, 0)} \frac{1}{v} d\phi < \infty.$$

Proof. Suppose $(\phi_*, 0)$ is such a point. Note that $\nabla H(\phi_*, 0) = (\partial_{\phi} H, \partial_v H)|_{(\phi_*, 0)} \neq 0$ since there is no critical point on Γ . At $(\phi, v) = (\phi_*, 0)$, we have

$$\partial_v H(\phi_*, 0) = -v(\phi^2 - v^2) + c v|_{(\phi_*, 0)} = 0.$$

Hence, $\partial_{\phi} H \neq 0$ and by the implicit function theorem we can solve ϕ in terms of v near $(\phi_*, 0)$ as $\phi = \phi(v)$. Hence, $\phi(0) = \phi_*$, $\phi'(0) = 0$. Taylor expansion implies that $1/v \sim |\phi - \phi_*|^{-1/p}$ where $p \geq 2$. Hence, $\int \frac{1}{v} d\phi$ is integrable near this point. \square

There may be some trajectories that connect saddles. We have the following lemma.

Lemma 3.5. Suppose $(\phi_*, v_*) \in \Gamma$ is a non-degenerate critical point of (3.6). If $v_* \neq 0$, then the critical point is on $\phi^2 - v^2 = c$ and there exists $\delta > 0$ such that

$$\int_{\Gamma_{\delta}(\phi_*, v_*)} \frac{1}{v} d\phi < \infty.$$

If $v_* = 0$ and $\phi_*^2 \neq c$, then for any $\delta > 0$

$$\left| \int_{\Gamma_{\delta}(\phi_*, v_*) \cap \{v \geq 0\}} \frac{1}{v} d\phi \right| = \left| \int_{\Gamma_{\delta}(\phi_*, v_*) \cap \{v \leq 0\}} \frac{1}{v} d\phi \right| = \infty.$$

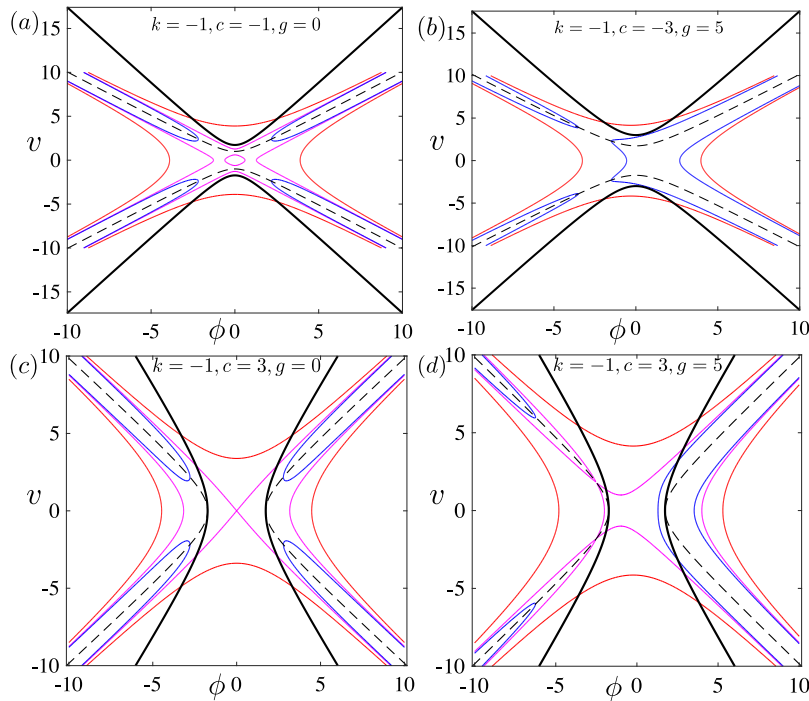


Fig. 1. Level sets of H with different parameters c , g and h for $k = -1$. The black solid lines are $\phi^2 - \frac{v^2}{3} = c$ and the dashed lines are $\phi^2 - v^2 = c$. (a) and (c) are for $g = 0$ in which case the level sets are symmetric about both ϕ and v axis. (b) and (d) are for $g > 0$ where the level sets are skewed.

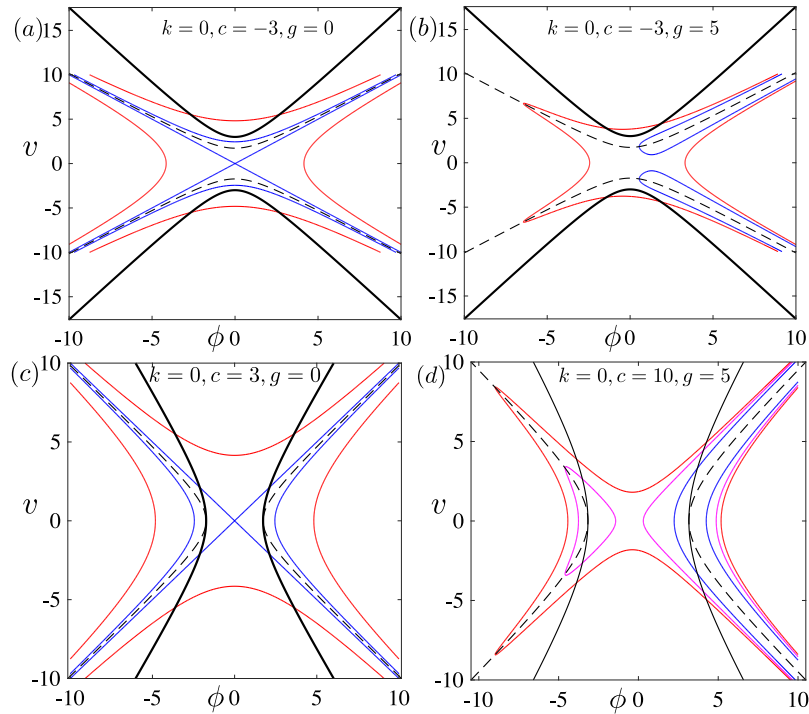


Fig. 2. Level sets of H with different parameters c , g and h for $k = 0$. The hyperbolas (solid lines and dashed lines) are the same as in Fig. 1. (a) and (c) are for $g = 0$, while (b) and (d) are for $g > 0$. In Figure (b), there are components that are 'U' shaped with both branches extending to infinity on one side of $v = 0$. In Figure (d), there is a component that forms a loop.

Proof. Note that (ϕ_*, v_*) must be a saddle point. If $v_* \neq 0$, from (3.6) we can see that this critical point must be on the curve $\phi^2 - v^2 - c = 0$. It is clear that $\int \frac{1}{v} d\phi$ is integrable near the critical point.

If $v_* = 0$ and $\phi_*^2 - v_*^2 - c \neq 0$, then (ϕ_*, v_*) is also a critical point of System (3.4). The existence and uniqueness theorem of System (3.4) holds in the neighborhood of this critical point. As a

result, $\Delta\xi = \int \frac{1}{v} d\phi$ must be infinite near this critical point because the critical point corresponds to a stationary solution of System (3.4). \square

Now, let us define the multi-valued solution to the mCH equation for our convenience of discussion here.

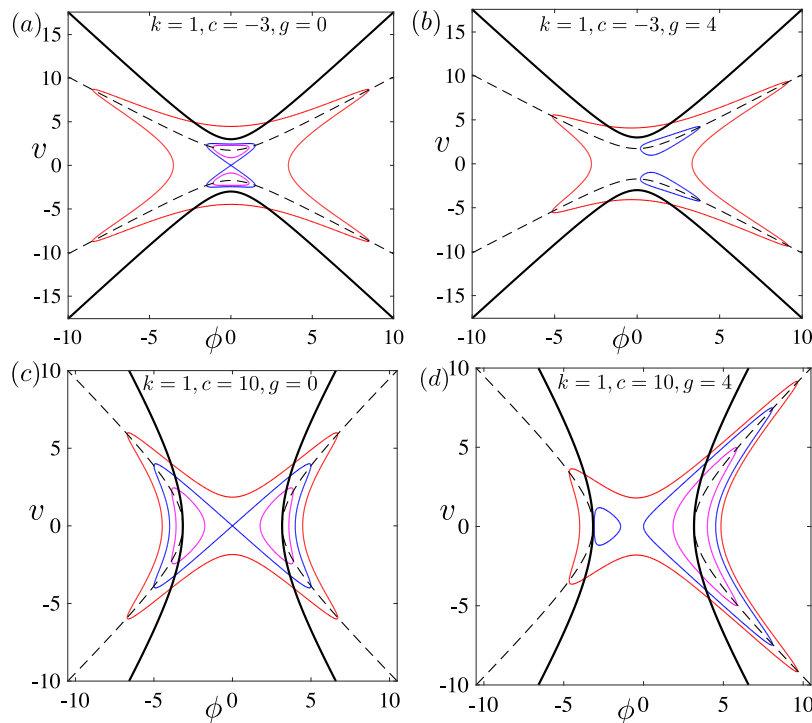


Fig. 3. Level sets of H with different parameters c , g and h for $k = 1$. The hyperbolas (solid lines and dashed lines) are the same as in Fig. 1. (a) and (c) are for $g = 0$, while (b) and (d) are for $g > 0$. We can see that all the simple curve (a curve that does not cross itself) forms a closed loop (see Theorem 3.1 for rigorous proof).

Definition 3.1. For any $a, b \in \mathbb{R}$, denote $J = (a, b)$. We say $\text{sgn}(J) := \text{sgn}(b - a)$ and $x \in J$ if $x \in (\min(a, b), \max(a, b))$. For a function u defined on $(\min(a, b), \max(a, b))$, we also say it is defined on J and define the integral on J as

$$\int_J u d\xi = \text{sgn}(J) \int_{\min(a,b)}^{\max(a,b)} u d\xi.$$

Definition 3.2. We say $u : \mathbb{R} \times [0, T] \rightarrow 2^{\mathbb{R}}, (x, t) \mapsto u(x, t)$ is a multi-valued solution to (1.1) subject to initial data $u(x, 0) = u_0(x)$ if there exist signed intervals $J_n(t) = (a_n(t), b_n(t))$, $n \in \mathbb{Z}$, (see Fig. 4) and a single-valued function $u_n(\cdot, t)$ defined on each $J_n(t)$ with $\gamma_n(t) = \{(x, u_n(x, t)), x \in J_n(t)\}$ such that

- $a_n, b_n \in C^1[0, T]$, $b_n(t) = a_{n+1}(t)$, $\lim_{n \rightarrow -\infty} a_n(t) = -\infty$, $\lim_{n \rightarrow +\infty} b_n(t) = +\infty$
- γ_n are non-intersecting and $\forall t \in [0, T], \cup_n \bar{\gamma}_n = \{(x, u) : x \in \mathbb{R}, u \in u(x, t)\}$.
- $\|u_n\|_{W^{1,\infty}(J_n(t))}$ is bounded as functions of t . For any $[a, b] \times [t_1, t_2] \subset \{(x, t) : x \in J_n(t)\}$, it holds $u_n \in C(t_1, t_2; H^1([a, b]))$. $u_n(b_n(t), t) = u_{n+1}(a_{n+1}(t), t)$.
- It holds that $\forall \varphi \in C_c^\infty([0, T], C_c^\infty(\mathbb{R}))$:

$$\begin{aligned} & \int_0^T \sum_n \int_{J_n} u(\varphi_t - \varphi_{xxt}) dx dt \\ & + \int_0^T \sum_n \int_{J_n} (2ku + u^3 + uu_x^2) \varphi_x dx dt \\ & - \frac{1}{3} \int_0^T \sum_n \int_{J_n} u^3 \varphi_{xxx} dx dt - \frac{1}{3} \int_0^T \sum_n \int_{J_n} u_x^3 \varphi_{xxx} dx dt \\ & = - \sum_n \int_{J_n(0)} u_0 \varphi(x, 0) dx + \sum_n \int_{J_n(0)} u_0 \varphi_{xx}(x, 0) dx. \end{aligned}$$

With similar computations as in the proof of Theorem 2.1, we find

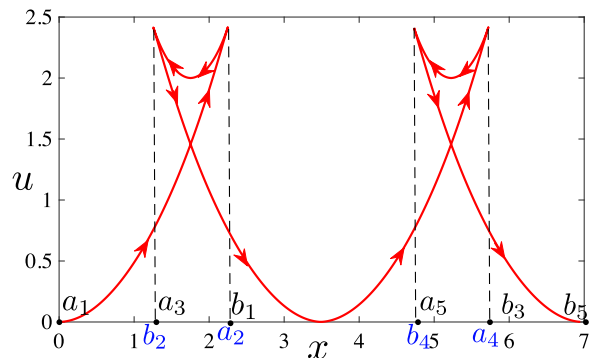


Fig. 4. A typical multi-valued solution $u(x, t)$ (Definition 3.2) at a fixed time t . The intervals $(a_n, b_n)(n = 1, 2, \dots, 5)$ are demonstrated on the x -axis. In the picture, $b_1 > a_1, b_2 < a_2 = b_1$ and $b_3 > a_3 = b_2$ etc.

Lemma 3.6. Use the same notations as in Definition 3.2 and assume that $t \mapsto u_i(\cdot, t)$ solves the mCH equation on the region $\{(x, t) : x \in J_i(t) = (a_i(t), b_i(t))\}$ classically for $i \in \mathbb{Z}$. Let $m_i(x, t) = u_i(x, t) - \partial_{xx} u_i(x, t)$ for $x \in J_i(t)$ and $x_i(t) := a_i(t)$. If the jump conditions (2.2) or (2.3) are satisfied where

$$v_r^i(t) := \lim_{\substack{x \in J_i \\ x \rightarrow a_i(t)}} \partial_x u_i(x, t), \quad v_l^i(t) := \lim_{\substack{x \in J_{i-1} \\ x \rightarrow b_{i-1}(t)}} \partial_x u_{i-1}(x, t)$$

and

$$A_r^i := \lim_{\substack{x \in J_i \\ x \rightarrow a_i(t)}} m_i [u_i^2 - (\partial_x u_i)^2 - a_i'(t)],$$

$$A_l^i := \lim_{\substack{x \in J_{i-1} \\ x \rightarrow b_{i-1}(t)}} m_{i-1} [u_{i-1}^2 - (\partial_x u_{i-1})^2 - a_i'(t)],$$

then u is a multi-valued solution as in Definition 3.2

We now relate the simple curves Γ to traveling wave solutions of the mCH equation:

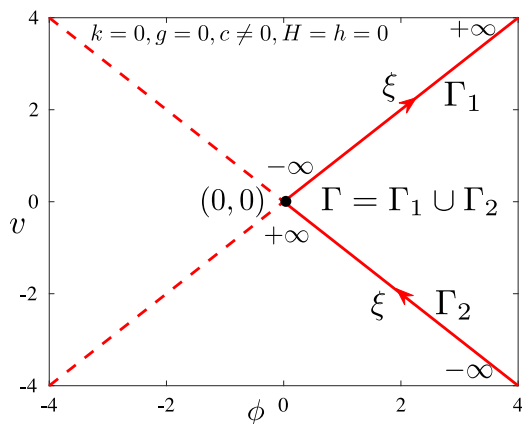


Fig. 5. The curves (both dashed lines and solid lines together) in this figure make up a connected component $(\phi^2 - v^2 = 0)$ of level set $H = 0$ for $g = k = 0$ and $c \neq 0$. $\Gamma = \Gamma_1 \cup \Gamma_2$ is a simple curve as stated in Statement (ii) of Proposition 3.3. $(\phi_*, 0) = (0, 0)$ is a saddle point that divides Γ into Γ_1 and Γ_2 . Γ_1 gives a solution $\phi_1(\xi) = A_1 e^\xi$ to the mCH equation and Γ_2 gives $\phi_2(\xi) = A_2 e^{-\xi}$ for some constants $A_i > 0$, $i = 1, 2$ and $\xi \in \mathbb{R}$.

Proposition 3.3. Suppose Γ is a simple curve described in Proposition 3.2 that does not contain $(\phi_0, 0)$ with $\phi_0^2 = c$. Let T be defined by (3.12) and we assume $T > 0$.

(i) If Γ does not pass any critical point on $v = 0$, then Γ corresponds to a traveling wave solution $u(x, t) = \phi(\xi)$ to the mCH equation (1.1), where $\xi = x - ct$. And if Γ does not cross $\phi^2 - v^2 = c$, then u is a single-valued solution. Moreover, u is a periodic traveling wave solution if and only if Γ is a closed loop.

(ii) If there are critical points of the form $(\phi_*, 0)$ dividing Γ into several segments as $\Gamma = \cup_i \Gamma_i$ (see Fig. 5), then each Γ_i corresponds to a non-periodic traveling wave solution $u(x, t) = \phi(\xi)$ for $-\infty < \xi < \infty$ where $\xi = x - ct$.

Proof. Proof for Statement (i)

Step 1. Preparations.

We assume first that Γ does not pass through any critical point on $v = 0$. Fix $A_0 = (\phi_0, v_0) \in \Gamma$, and use $\Gamma(A_0, (\phi, v))$ to mean the subcurve of Γ from A_0 to (ϕ, v) with the same orientation as when we define T in Eq. (3.12). We define

$$\xi = \int_{\Gamma(A_0, (\phi, v))} \frac{1}{v} d\phi. \tag{3.14}$$

In the case $T \in (0, \infty)$ (or Γ is a loop), this expression is defined for all $\xi \in \mathbb{R}$ if we orbit around the loop repeatedly. This integral then gives a set value mapping $\phi : \mathbb{R} \rightarrow 2^{\mathbb{R}}$, $\xi \mapsto \phi(\xi)$ provided $T > 0$.

Assume $\phi^2 - v^2 = c$ divides Γ into several arcs (may be one segment). From Statement (ii) of Lemma 3.3, we know there are at most five such arcs with four joint points. Moreover, each arc is contained in Γ_+ or Γ_- . Hence, $d\xi$ has a definite sign in each arc. If the sign of $d\xi = \frac{1}{v}d\phi$ is different on two arcs, we get a multi-valued solution, which happens if one arc is below strictly under the other in P_+ or P_- .

The functions $(\phi(\xi), v(\xi))$ solve System (3.4) on each arc and therefore $u(x, t)$ solves the mCH equation in the classical sense, where

$$u(x, t) = \phi(\xi) \text{ and } x - ct = \xi.$$

Step 2. For the case when $\Gamma \cap \{(\phi, v) : \phi^2 - v^2 = c\} = \emptyset$, $d\xi$ has a definite sign and we have a single-valued function.

Step 3. The case for $\Gamma \cap \{(\phi, v) : \phi^2 - v^2 = c\} \neq \emptyset$.

In this case, we may have a multi-valued function. We only have to check the jump conditions across the hyperbola $\phi^2 - v^2 = c$. Assume $A = (\phi_A, v_A) \in \Gamma \cap \{(\phi, v) : \phi^2 - v^2 = c\}$. The jump for $u(x, t)$ happens at

$$x - ct = \xi_A = \int_{\Gamma(A_0, A)} \frac{1}{v} d\phi.$$

Hence, the jump line in x, t plane is given by $x_A(t) = ct + \xi_A$. We have

$$v_\ell = v_r = v_A \text{ and } \frac{d}{dt}x_A(t) = c,$$

which agree with the wave speed. Notice that $m = u - u_{xx} = \phi - \phi'' = \phi - \frac{dv}{d\xi}$. By System (3.4), we have

$$\begin{aligned} \lim_{\xi \neq \xi_A, \xi \rightarrow \xi_A} m(u^2 - \partial_x u^2 - x'_A(t)) &= \lim_{\xi \neq \xi_A, \xi \rightarrow \xi_A} (\phi - \phi'')(\phi^2 - v^2 - c) \\ &= g - 2k\phi_A. \end{aligned}$$

The jump condition (2.3) is verified. Hence, $u(x, t)$ is a weak solution to the mCH equation.

Step 4. Periodicity.

If Γ is a closed loop, we have $0 < T < \infty$. The solution is therefore periodic traveling wave solution.

When Γ is not a closed loop, both sides of Γ tend to infinity. By definition of Γ_\pm (3.10), we have $|v| \sim |\phi|$ as $|\phi| \rightarrow \infty$. This means ξ extends to $-\infty$ on one side and extends to $+\infty$ on the other side by formula (3.14). This yields a solution defined on \mathbb{R} which is not a periodic solution.

Proof for Statement (ii)

Now assume there are critical points of the type $(\phi_*, 0)$ on Γ . If Γ is a loop, both ends of a segment are such critical points. If Γ is an open curve extending to infinity, then the endpoints of each segment are either such critical points or infinity. By Lemma 3.5, the integrals at the two ends of such a segment are infinity. Hence, the function similarly given by (3.14) is defined on \mathbb{R} . That it is a solution can be verified similarly as the first case. \square

Next, we give a theorem to describe the typical traveling wave solutions to the mCH equation (1.1) given by level sets of H . When $k > 0$, the typical solutions are multi-valued periodic traveling wave solutions. And when $k \leq 0$, the typical solutions are unbounded and single-valued, defined on \mathbb{R} .

We have the following theorem.

Theorem 3.1. Consider System (3.6). Assume $S = \{(\phi, v) : H(\phi, v) = h\}$ is nonempty and does not pass through any critical point.

(i) If $k > 0$ for any traveling speed $c \in \mathbb{R}$, each simple curve in S is a loop in the phase plane of System (3.6). There exists $h_0 > 0$ such that for $c < 0$ or $c \geq 0, h > h_0$, there is a loop $\Gamma \subset S$ that gives a multi-valued traveling wave solution of the mCH equation. If $T > 0$, this solution is a periodic solution on \mathbb{R} . (See Fig. 6(a) and (b).)

(ii) If $k \leq 0$, for any traveling speed $c \in \mathbb{R}$, there exists h_0 such that when $h > h_0$, there is a simple curve $\Gamma \subset S$ with both sides tending to infinity which corresponds to a single-valued traveling wave solution to the mCH equation globally on \mathbb{R} and $\lim_{|\xi| \rightarrow \infty} |\phi(\xi)| = +\infty$. (See Fig. 6(c) and (d).)

Proof. Γ_\pm are given by (3.10). Set

$$f(\phi) := c^2 - 4(k\phi^2 - g\phi - h) = c^2 - 4k\left(\phi - \frac{g}{2k}\right)^2 + \frac{g^2}{k} + 4h.$$

If S is nonempty, there exists $\phi \in \mathbb{R}$ such that

$$f(\phi) \geq 0 \text{ and } \phi^2 - c + \sqrt{f(\phi)} \geq 0.$$

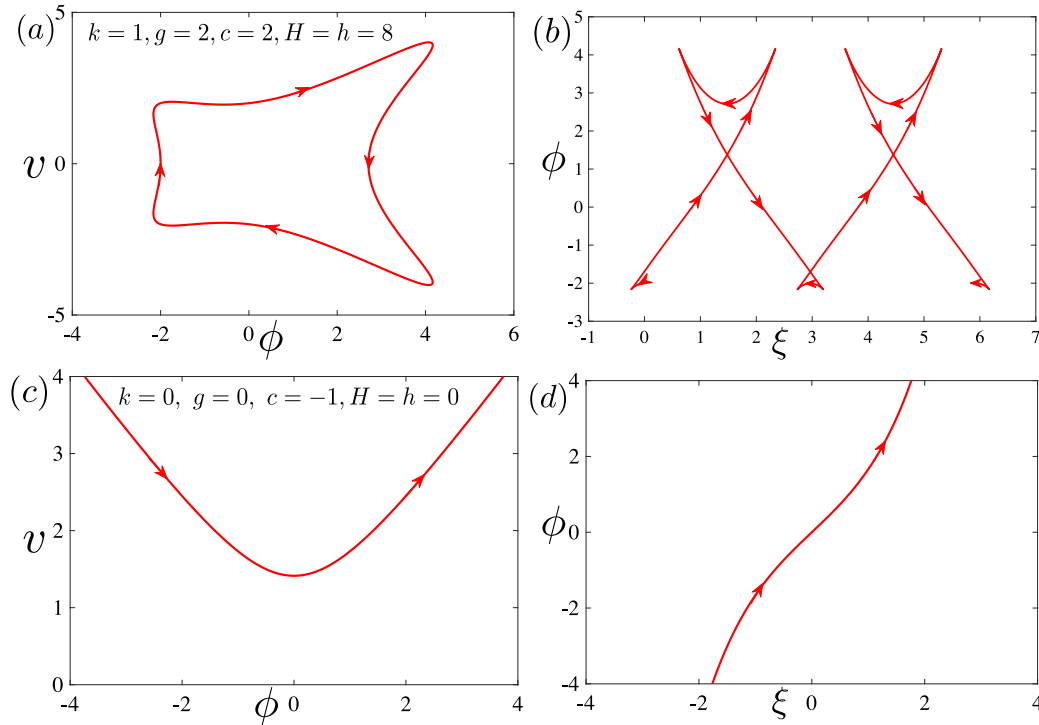


Fig. 6. Illustration of Theorem 3.1. Typical simple curves Γ and the corresponding traveling wave solutions. A closed loop when $k > 0$ in phase plane (a) corresponds to a multi-valued periodic traveling wave solution in (b). A simple curve when $k \leq 0$ in phase plane (c) corresponds to a traveling wave solution (d). (a), (b) are for (i) in Theorem 3.1 and (c), (d) are for (ii) in Theorem 3.1.

(i) For $k > 0$, any level set is bounded. Indeed, ϕ must be bounded to ensure $f(\phi) \geq 0$. Hence, (ϕ, v) is defined in a bounded set of the phase plane.

By Proposition 3.2, for each simple curve inside a connected component of S must be a loop. When $c < 0$ or $h > \frac{4kc - c^2 - 4g\sqrt{c}}{4}$ ($c \geq 0$), Γ_+ intersects with $\phi^2 - v^2 = c$ at a point with $v \neq 0$. Choose Γ to be the simple curve that passes through this intersection. By Proposition 3.3, Γ gives a traveling wave solution. If $T > 0$, it is clearly a periodic traveling wave solution on \mathbb{R} . By Lemma 3.3, $\Gamma_- \cap \Gamma \neq \emptyset$. Since Γ does not pass through any critical point, Γ_- is below Γ_+ and the solution is multi-valued by the proof of Proposition 3.3.

(ii) Consider that $k = 0$. We find that

$$v^2 = \phi^2 - (c \pm \sqrt{c^2 + 4(g\phi + h)}).$$

When $g = 0$, if we choose h such that $c^2 + 4h > 0$. Γ_+ and Γ_- do not intersect. Also, there is a connected component in Γ_+ above $v = 0$ which is a function graph and extends to infinity. Since $d\xi = \frac{1}{v}d\phi > 0$ on this component. T is defined and $T \geq 0$. Hence it is a solution defined on \mathbb{R} .

When $g > 0$, consider $h > \frac{(|c| - c)^2 - (c^2 + 4g\sqrt{|c|})}{4}$. Then,

$$v^2 = \phi^2 - c - \sqrt{c^2 + 4g\phi + 4h} = 0$$

has a root at ϕ_0 satisfying $\phi_0^2 - c > 0$ and $\phi^2 - c - \sqrt{c^2 + 4g\phi + 4h} > 0$ for all $\phi > \phi_0$.

$$\Gamma := \{(\phi, v) : \phi \geq \phi_0, v^2 = \phi^2 - c - \sqrt{c^2 + 4g\phi + 4h}\}$$

yields a single-valued solution defined on \mathbb{R} and $\lim_{|\xi| \rightarrow \infty} |\phi(\xi)| = \infty$ by Proposition 3.3.

Consider $k < 0$, it is clear that when $h > -\frac{g^2}{4k}$, the term in the square root is always positive and Γ_+ ,

$$v^2 = \phi^2 - c + \sqrt{c^2 - 4(k\phi^2 - g\phi - h)},$$

does not cross $\phi^2 - v^2 = c$ and $\phi^2 - c + \sqrt{c^2 - 4(k\phi^2 - g\phi - h)} > 0$ for all ϕ . $\Gamma = \Gamma_+ \cap P_+$ (or $\Gamma = \Gamma_+ \cap P_-$) is a simple curve as described in Proposition 3.2, which gives a single-valued solution defined on \mathbb{R} and $\lim_{\xi \rightarrow \infty} |\phi(\xi)| = \infty$ by Proposition 3.3. \square

Besides the typical solutions mentioned in Theorem 3.1, there are also some other different kinds of solutions. Let us make some complementary discussions.

1. For $k > 0$, the typical solutions obtained in Theorem 3.1 are multi-valued periodic traveling wave solutions (see Fig. 6(b)). There are also single-valued periodic traveling wave solutions to the mCH equation. For example, if $k = 1$, $g = 2$, $c = 4$, the level set $H = -4.1$ does not intersect $\phi^2 - v^2 = c$ and it gives a classical periodic solitary wave solution to the mCH equation.
2. For $k \leq 0$, there may exist a simple curve Γ that is a loop, which gives a periodic solution to the mCH equation. See Fig. 1(a) (single-valued) and Fig. 2(d) (multi-valued).
3. The special case $k = 0$ and $g = 0$ is very important for our discussion in Section 4.

$$H = \frac{1}{4}(\phi^2 - v^2)^2 - \frac{1}{2}c(\phi^2 - v^2). \quad (3.15)$$

The level sets are given by

$$\phi^2 - v^2 = A, \quad \text{for some } A \in \mathbb{R}.$$

If $A < 0$, the two curves of $\phi^2 - v^2 = A$ corresponding to two strictly monotonic solutions to the mCH equation (see Fig. 6(c),(d)):

$$\phi(\xi) = \pm\sqrt{-A} \sinh(\xi + d) \quad \text{for some constant } d \in \mathbb{R}.$$

If $A > 0$, we solve that

$$\phi(\xi) = \pm\sqrt{A} \cosh(\xi + d) \quad \text{for some constant } d \in \mathbb{R}.$$

The case $A = 0$ is very interesting as it passes through the critical point $(0, 0)$ which is a saddle point when the

traveling speed $c \neq 0$. The time spent near $(\phi, v) = (0, 0)$ now is infinity. Later in Section 4, we obtain peakon weak solutions (see Remark 4.2) by the patching technic.

3.3. Patched traveling peakon weak solutions

By Theorem 3.1, the typical solutions for $k > 0$ are multi-valued periodic solutions while the typical solutions for $k \leq 0$ are single-valued solutions which tend to infinity as $|\xi| \rightarrow \infty$. In this subsection, we introduce the patching technic to obtain patched bounded single-valued weak solutions from these two types of solutions. Therefore, our method is also a way to change ‘bad solutions’ into ‘good solutions’.

First, we present a patching criterion to construct a patched periodic weak solution by connecting smooth solutions given by level sets of H .

Proposition 3.4. Fix any $k \in \mathbb{R}$, $c \in \mathbb{R}$ and $g \geq 0$. Suppose curves

$$\gamma^+ \subset \{(\phi, v) : H = h, v \geq 0\}, \quad \gamma^- \subset \{(\phi, v) : H = h', v \leq 0\},$$

for some h and h' ,

are function graphs of two continuous functions $v = f^+(\phi)$, $v = f^-(\phi)$ for $\phi \in [\phi_1, \phi_2]$ ($\phi_1 < \phi_2$) (see Fig. 7(a)). Denote points $A^+ : (\phi_1, v_1^+) = (\phi_1, f^+(\phi_1))$ and $B^+ : (\phi_2, v_2^+) = (\phi_2, f^+(\phi_2))$, $A^- : (\phi_1, v_1^-) = (\phi_1, f^-(\phi_1))$ and $B^- : (\phi_2, v_2^-) = (\phi_2, f^-(\phi_2))$.

If γ^+ and γ^- do not pass through critical points and the end points A^+ , B^+ , A^- and B^- satisfy:

$$(\phi_i)^2 - \frac{1}{3} \left((v_i^+)^2 + v_i^+ v_i^- + (v_i^-)^2 \right) = c, \quad i = 1, 2, \tag{3.16}$$

then $\Gamma = \gamma^+ \cup B^+ B^- \cup \gamma^- \cup A^- A^+$ gives a patched periodic traveling peakon weak solution with wave speed c (see Fig. 7(b)), where PQ represents the line segment between two points P and Q .

Proof. Let Γ be oriented clockwise. We use $\Gamma(A^+, (\phi, v))$ to mean the part of Γ from A^+ to (ϕ, v) with the same orientation as Γ . The relation

$$\xi = \int_{\Gamma(A^+, (\phi, v))} \frac{1}{v} d\phi \tag{3.17}$$

then gives a mapping $\xi \mapsto \phi(\xi)$. First note that $\int_{A^- A^+} \frac{1}{v} d\phi = 0 = \int_{B^+ B^-} \frac{1}{v} d\phi$. On γ^+ , since $1/v > 0$ and $d\phi > 0$, we have $d\xi > 0$. Similarly, $d\xi > 0$ on γ^- . Hence, $\phi(\xi)$ is a single-valued mapping. Since the arcs do not pass through critical points, T defined by (3.12) satisfies

$$T = \int_{\Gamma} \frac{1}{v} d\phi \in (0, \infty).$$

Hence, $\phi(\xi)$ is a periodic function. Denote $T_1 = \int_{\gamma^+} \frac{1}{v} d\phi$ and consider

$$u(x, t) = \begin{cases} u_1(x - ct) = \phi(x - ct), & x - ct = \xi \in (0, T_1); \\ u_2(x - ct) = \phi(x - ct), & x - ct = \xi \in (T_1, T), \end{cases}$$

and $m_i = u_i - \partial_{xx} u_i$ for $i = 1, 2$. The functions u_1 and u_2 solve the mCH equation in the classical sense respectively for $\xi \in (0, T_1)$ and $\xi \in (T_1, T)$. We only have to verify that the jump conditions are satisfied at the joints $A^+ A^-$ and $B^- B^+$, corresponding to $x - ct = \xi = 0$ and $x - ct = \xi = T_1$, or $(x_1(t), t)$, $(x_2(t), t)$ where

$$x_1(t) = ct, \quad x_2(t) = T_1 + ct.$$

From the conditions (3.16), one can easily verify that condition in (2.2) is satisfied. Hence, the patched function is a periodic traveling peakon weak solution. \square

One possible way to construct the arcs γ^+ and γ^- in Proposition 3.4 is to cut curves from the same level set $H = h$. Below we show that the hyperbola $\phi^2 - \frac{1}{3}v^2 = c$ can be used to cut these two arcs from the level set $H = h$ so that the condition for endpoints (3.16) is satisfied automatically.

Corollary 3.1. Fix $k \in \mathbb{R}$ and $c \in \mathbb{R}$. Suppose $\gamma \subset \{(\phi, v) : H = h\} \cap P_+$ is the graph of a function $v = f(\phi)$ that does not pass through any critical point. Assume the endpoints of γ , denoted as A^+ and B^+ , are either on the hyperbola $\phi^2 - \frac{1}{3}v^2 = c$ or on the $v = 0$ axis. Let $\tilde{\gamma}$ be the reflection of γ about $v = 0$ with corresponding endpoints A^-, B^- . Then the loop $\Gamma = \gamma \cup B^+ B^- \cup \tilde{\gamma} \cup A^- A^+$ gives a patched periodic traveling peakon weak solution of the mCH equation with wave speed c .

Proof. Assume $A^+ = (\phi_1, v_1^+)$ and $B^+ = (\phi_2, v_2^+)$. Due to the symmetric of γ and $\tilde{\gamma}$, we have $A^- = (\phi_1, -v_1^+)$ and $B^- = (\phi_2, -v_2^+)$. If A^+, B^+ are on the hyperbola $\phi^2 - \frac{1}{3}v^2 = c$, we know A^- and B^- are also on the hyperbola. At this time, (3.16) is satisfied and the result follows.

Consider that some endpoint, for example A^+ , is on $v = 0$. Then, $A^+ = A^-$. Hence, $\gamma \cup \tilde{\gamma}$ is actually a connected arc of the level set, which forms a strong solution. The jump only happens at $B^+ B^-$ and the jump conditions can be verified similarly as in Proposition 3.4. \square

Now, for any traveling speed $c \in \mathbb{R}$ and any $k \in \mathbb{R}$, we prove the existence of the patched periodic traveling peakon weak solutions corresponding to Corollary 3.1. And then give a figure to illustrate it (See Fig. 8).

Theorem 3.2. For any fixed traveling speed $c \in \mathbb{R}$ and any fixed $k \in \mathbb{R}$, there is a h_0 (depending on k, c and g) such that for any $h > h_0$, there is an arc in the level set $H = h$ satisfying the conditions in Corollary 3.1. Consequently, there are patched single-valued periodic traveling peakon weak solutions with speed c .

Proof. We just consider the following function graph $v = f_1(\phi)$ in Γ_+

$$v = \sqrt{\phi^2 - c + \sqrt{c^2 - 4(k\phi^2 - g\phi - h)}}.$$

Consider

$$D = \{\phi \in \mathbb{R} : c^2 - 4(k\phi^2 - g\phi - h) \geq 0\}.$$

When $k = 0$, D is an interval of \mathbb{R} . When $k < 0$ and $h > \frac{-g^2 - kc^2}{4k}$, we have $D = \mathbb{R}$. When $k > 0$ and $h > \frac{-g^2 - kc^2}{4k}$, D is an interval of \mathbb{R} . Next, we assume h big enough such that D is an interval of \mathbb{R} .

The domain of $v = f_1$ is given by

$$D_{f_1} = \{\phi \in D : \phi^2 - c + \sqrt{c^2 - 4(k\phi^2 - g\phi - h)} \geq 0\}.$$

For each $\tilde{\phi}$ in $\{\phi : \phi^2 \leq c\}$, we can find a $h_1(\tilde{\phi})$ such that

$$\tilde{\phi}^2 - c + \sqrt{c^2 - 4(k\tilde{\phi}^2 - g\tilde{\phi} - h_1(\tilde{\phi}))} \geq 0.$$

Since $\{\phi : \phi^2 \leq c\}$ is a compact set, there exists $h_1 > 0$ such that for all $h > h_1$, $D_{f_1} = D$ is an interval of \mathbb{R} .

We now show that there exists $h_0 \geq h_1$ so that whenever $h > h_0$, $v = f_1(\phi)$ intersects with $\phi^2 - \frac{1}{3}v^2 = c$ at two points ϕ_1, ϕ_2 and $[\phi_1, \phi_2] \subset D$.

Consider the following function

$$f(\phi) = 3(\phi^2 - c) - (\phi^2 - c + \sqrt{c^2 - 4(k\phi^2 - g\phi - h)}).$$

We claim that there is h_2 such that $\forall h \geq h_2, \exists M_1 > 0, [-M_1, M_1] \subset D, f(\pm M_1) > 0$. In the case $k < 0$, when h_0 is large enough, $D = \mathbb{R}$, and the claim is clear since $f(\phi)$ grows like $2\phi^2$. In

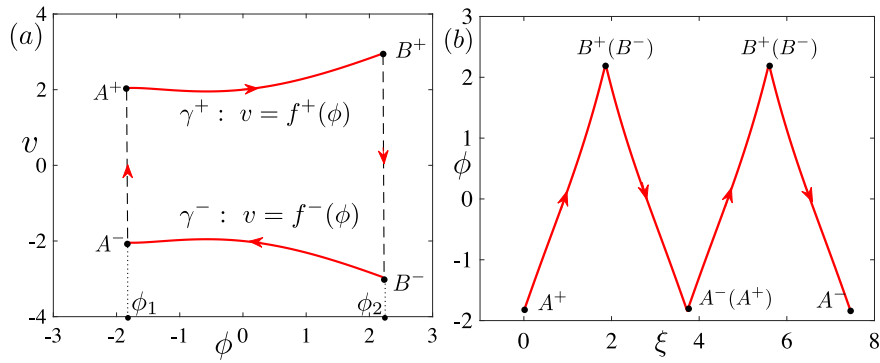


Fig. 7. A sketch to illustrate Proposition 3.4.

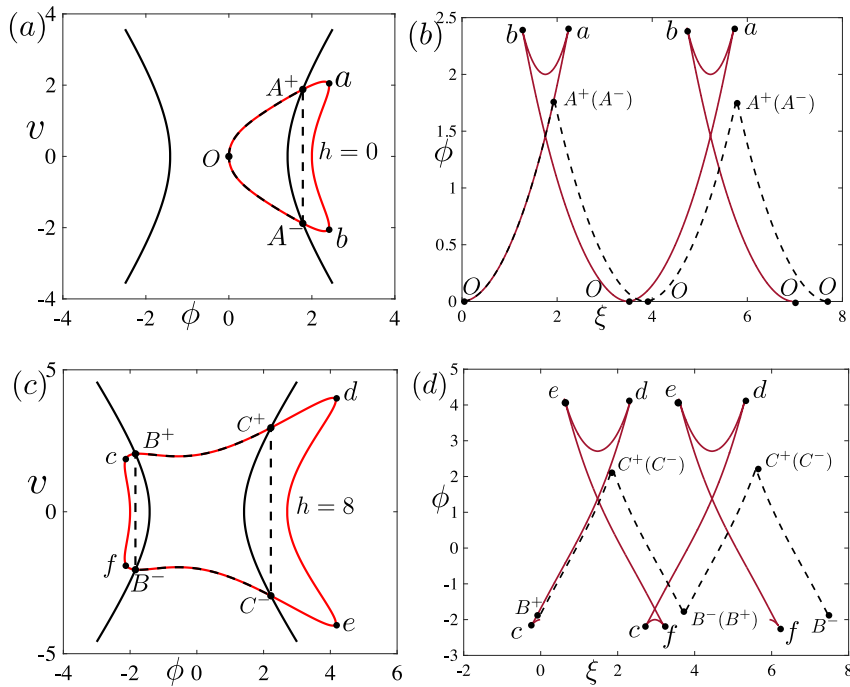


Fig. 8. In phase plane (a), the closed loop \widehat{OabO} (solid line) corresponds to multi-valued periodic traveling wave solution \widehat{OabO} (solid line) in (b). This multi-valued solution was also given in [13, Section 3], where it was called symmetric singular soliton. By the patching technic, we truncate the closed loop in the phase plane (a) at A^+ and A^- and obtain a new closed loop \widehat{OAbO} (dashed line) corresponding to a patched single-valued periodic traveling wave weak solution \widehat{OAbO} with a peakon $A^+(A^-)$ (dashed line) in (b). The points A^+ and A^- are obtained by the interaction between the level set $H = 0$ and the hyperbola $\phi^2 - \frac{1}{3}v^2 = c = 2$ in phase plane. In (b), they glue together and form a peakon $A^+(A^-)$ where the jump condition (2.2) is satisfied (see Corollary 3.1 for more details). The case for (c) and (d) is similar, where B^+ , B^- glue together and form a peakon at bottom while C^+ , C^- glue together and form a peakon at the top.

the case $k = 0$, if $g = 0, D = \mathbb{R}$. If $g > 0, \phi \geq \phi_c = -\frac{c^2+4h}{4g}$. It is clear that $f(\phi_c)$ grows like $2h^2$. Hence, when h is large enough, picking $M_1 = |\phi_c|$ suffices. If $k > 0$, the critical values are $\phi_c^\pm = (g \pm \sqrt{g^2 + 4h + c^2})/2k$. As h increases, $D = [\phi_c^-, \phi_c^+]$ increases and $f(\phi_c^\pm)$ increases to infinity. The claim is also true if we choose $M_1 = \min(|\phi_c^-|, |\phi_c^+|)$.

Now, we compute

$$f(0) = -2c - \sqrt{c^2 + 4h} < 0.$$

When $c > 0$, we have $f(0) < 0$. When $c < 0, h > \frac{3c^2}{4}$ implies $f(0) < 0$.

Choose

$$h_0 = \max\left\{h_1, h_2, \frac{3c^2}{4}\right\}.$$

Whenever $h > h_0, v = f_1(\phi)$ intersects with $\phi^2 - \frac{1}{3}v^2 = c$ at two points ϕ_1, ϕ_2 and $[\phi_1, \phi_2] \subset D$.

The proof then is complete. \square

Remark 3.3. In the proof, we only constructed the cases when the two endpoints are on the hyperbola $\phi^2 - \frac{1}{3}v^2 = c$. Actually, the other situation that one endpoint is on $v = 0$ in Corollary 3.1 can also happen. See Fig. 8(a), (b) for an example. And by the patching technic, we can also obtain peakon weak solutions that are not periodic (see Remark 3.4 and Section 4).

In Fig. 8, we plot two multi-valued traveling wave solutions and the corresponding patched periodic traveling peakon weak solutions for the cases $k = 1, c = 2$ and $g = 2$.

Remark 3.4. Note that we can also patch solutions corresponding to different speeds. For example, consider the level set $H = 0$ with the parameter $g = 0$. Then, $(0, 0) \in \{(\phi, v) : H = 0\}$. As in Fig. 8(b) (with $g = 0$), assume $(\phi(\xi_0), v(\xi_0)) = (0, 0)$. Consider one period in $(\xi_0, \xi_0 + T)$. Construct u as

$$u(x, t) = \begin{cases} 0 & \text{for } x - ct = \xi \notin [\xi_0, \xi_0 + T], \\ \phi(\xi) & \text{for } x - ct = \xi \in [\xi_0, \xi_0 + T]. \end{cases} \quad (3.18)$$

Then, the jump condition (2.3) can be verified easily at the patching areas: ξ_0 and $\xi_0 + T$. Hence, u is a weak solution to (1.1). Based on this kind of peakon solutions, we can construct multi-peakon solutions with different speeds. For example, assume u_1 and u_2 are constructed as (3.18) with different traveling speeds c_1, c_2 and periods T_1, T_2 :

$$u_1(x, t) = \begin{cases} 0, & x - c_1t = \xi \notin [\xi_1, \xi_1 + T_1], \\ \phi_1(\xi), & x - c_1t = \xi \in [\xi_1, \xi_1 + T_1], \end{cases}$$

$$u_2(x, t) = \begin{cases} 0, & x - c_2t = \xi \notin [\xi_2, \xi_2 + T_2], \\ \phi_2(\xi), & x - c_2t = \xi \in [\xi_2, \xi_2 + T_2]. \end{cases}$$

For $\xi_1 + T_1 < \xi_2$, we can construct 2-peakon weak solution as

$$u(x, t) = \begin{cases} u_1(x, t), & x - c_1t \in [\xi_1, \xi_1 + T_1], \\ u_2(x, t), & x - c_2t \in [\xi_2, \xi_2 + T_2], \\ 0, & \text{otherwise.} \end{cases}$$

If $c_1 < c_2$, the left peakon stays far behind the right one. If $c_1 > c_2$, the left peakon catches up the right one in finite time. Whether the left peakon can go beyond the right one without changing shape or not, which is the property of solitons, is left for future study.

4. A class of peakon weak solutions for $k \leq 0$

In this section, we use the fundamental solution $G(x)$ (defined by (1.2) and (1.3)) of Helmholtz operator $1 - \partial_{xx}$ to give some peakon weak solutions to the mCH equation (1.1) (with $k \leq 0$) in \mathbb{T}_ℓ . And we show that some of these peakon weak solutions also can be obtained by the patching technic.

4.1. Peakon weak solutions for $k = 0$

In this subsection, we consider the dispersionless mCH equation (Eq. (1.1) with $k = 0$). When considering the whole space \mathbb{R} (or $\ell = +\infty$), the dispersionless mCH equation has peakon weak solutions of the form (1.5). Here, we present a more general result about this kind of peakon weak solutions for all $0 < \ell \leq +\infty$.

Proposition 4.1. Assume $0 < \ell \leq +\infty$ and $p \neq 0$. Let

$$c = \begin{cases} \frac{1}{4}p^2 \left[\coth^2\left(\frac{\ell}{2}\right) - \frac{1}{3} \right] & \text{if } \ell < +\infty; \\ \frac{1}{6}p^2 & \text{if } \ell = +\infty. \end{cases}$$

Then, $u(x, t) = pG(x - ct)$ is a one peakon weak solution to the dispersionless mCH equation (1.1) ($k = 0$).

Proof. By the definition $G(x)$ ((1.2) and (1.3)), we know

$$m(x, t) = u(x, t) - \partial_{xx}u(x, t) = 0 \text{ for } x \neq ct.$$

Hence, $u(x, t)$ is a classical solution to the dispersionless mCH equation in $\mathbb{R} \times [0, +\infty) \setminus \{(x(t), t) : x(t) = ct, t \geq 0\}$. By Theorem 2.1, we only have to prove $u(x, t)$ satisfies (2.2) or (2.3) along the line $\{(x(t), t) : x(t) = ct, t \geq 0\}$. Here, we prove u satisfies jump condition (2.2). Notice that $u_x(x(t)-, t) = -u_x(x(t)+, t) = \frac{p}{2}$. We have

$$\begin{aligned} & u^2(x(t), t) - \frac{1}{3}u_x^2(x(t)-, t) - \frac{1}{3}u_x^2(x(t)+, t) \\ & - \frac{1}{3}u_x(x(t)-, t)u_x(x(t)+, t) \\ & = \begin{cases} \frac{1}{4}p^2 \left[\coth^2\left(\frac{\ell}{2}\right) - \frac{1}{3} \right] & \text{if } \ell < +\infty; \\ \frac{1}{6}p^2 & \text{if } \ell = +\infty. \end{cases} \end{aligned}$$

This implies the first condition in (2.2)

$$\begin{aligned} \frac{d}{dt}x(t) = c = & u^2(x(t), t) - \frac{1}{3}u_x^2(x(t)-, t) - \frac{1}{3}u_x^2(x(t)+, t) \\ & - \frac{1}{3}u_x(x(t)-, t)u_x(x(t)+, t). \end{aligned}$$

The second condition in (2.2) can be easily checked. Therefore, $u(x, t) = pG(x - ct)$ is a peakon weak solution to the dispersionless mCH equation. \square

Remark 4.1. The traveling speed is determined by the amplitude p and the period ℓ . When the period ℓ tends to infinity, the traveling speed c tends to $\frac{1}{6}p^2$:

$$\frac{1}{6}p^2 = \lim_{\ell \rightarrow +\infty} \frac{1}{4}p^2 \left[\coth^2\left(\frac{\ell}{2}\right) - \frac{1}{3} \right].$$

Remark 4.2. All the peakon weak solutions given by Proposition 4.1 can be obtained by the patching technic introduced in Section 3.3. Here, we only show the case for $\ell = +\infty$. In this case, the solution is reduced to

$$u = pG(x - \frac{1}{6}p^2t) = \frac{1}{2}pe^{-|x - \frac{1}{6}p^2t|}.$$

As shown in Fig. 9, consider the level set of (3.5) for $k = 0, g = 0$ and $c > 0$. At this time $(0, 0)$ is a saddle point of (3.6). The two lines $\phi = \pm v$ belong to the level set of $H = h = 0$ and the time spent near $(\phi, v) = (0, 0)$ is infinity. Consider the two branches of the level set $H = 0$ in the half plane $\phi > 0$. These two branches correspond to two traveling wave solutions to the mCH equation:

$$\phi_1(\xi) = A_1e^\xi \text{ and } \phi_2(\xi) = A_2e^{-\xi}$$

for some constants $A_i > 0, i = 1, 2$.

(When the origin point for $\xi = 0$ is set, we can determine A_i for $i = 1, 2$.) Now, assume the traveling speed $c = \frac{1}{6}p^2$ for $p \neq 0$. The two branches intersect $\phi = \sqrt{\frac{v^2}{3} + c}$ (the right curve of $\phi^2 - \frac{v^2}{3} = c$) at $B^+ = (\sqrt{\frac{3}{2}c}, \sqrt{\frac{3}{2}c})$ and $B^- = (\sqrt{\frac{3}{2}c}, -\sqrt{\frac{3}{2}c})$. Assume $\gamma := \{(\phi, v) : 0 < \phi < \sqrt{\frac{3}{2}c}, v = \phi\}$ and $\tilde{\gamma} := \{(\phi, v) : 0 < \phi < \sqrt{\frac{3}{2}c}, v = -\phi\}$. Consider the patched weak solution given by curve (clockwise)

$$\gamma \cup B^+B^- \cup \tilde{\gamma}.$$

Assume $\xi = 0$ on the segment B^+B^- . Hence, we have

$$\phi_1(0) = \phi_2(0) = \sqrt{\frac{3}{2}c} = A_1 = A_2.$$

This yields $A_i = \frac{|p|}{2}$ and the corresponding patched weak solution is

$$u(x, t) = \phi(x - ct) = \frac{|p|}{2}e^{-|x - ct|} = |p|G(x - ct).$$

This is the one peakon weak solution given by Proposition 4.1. This weak solution is not of finite period since it passes through the critical points $(0, 0)$. The time spent near $(0, 0)$ is infinity. Similarly, the intersection between $\phi = \pm v$ ($\phi < 0$) and the left curve of $\phi^2 - \frac{v^2}{3} = c$ gives the peakon weak solution $u = -|p|G(x - ct)$.

Notice that a solution given by Proposition 4.1 satisfies $m(x, t) = 0$ is in the smooth region of u and the superposition of such solutions u gives a new solution in the smooth region. Hence, the superposition of N such solutions with different traveling speeds forms an N -peakon weak solutions only if one of the jump conditions ((2.2) or (2.3)) is satisfied along the trajectory of each peakon. Indeed, we have the following proposition.

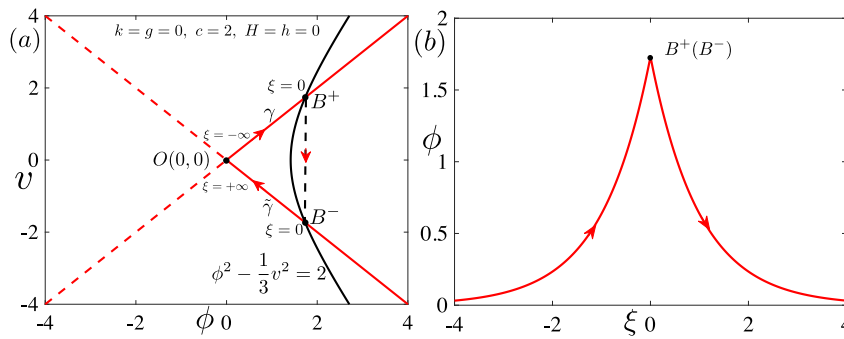


Fig. 9. Illustration of Remark 4.2: the patching technic for peakon weak solutions of the form (1.5). When $g = k = 0$ and traveling speed $c = 2$, the hyperbola $\phi^2 - \frac{1}{3}v^2 = c$ ($\phi > 0$) intersects the level set $H = 0$ at B^+ and B^- in phase plane (a). $(0, 0)$ is a saddle point and $\gamma \cup B^+B^- \cup \hat{\gamma}$ corresponds to the patched peakon weak solution $\phi(\xi) = u(x, t) = 2\sqrt{3}G(x - 2t) = \sqrt{3}e^{-|x-2t|} = \sqrt{3}e^{-|\xi|}$ in (b).

Proposition 4.2. Let $p_1, \dots, p_N \in \mathbb{R}$ be N constants and $p_i \neq 0$, $i = 1, \dots, N$. Assume $x_{10} < x_{20} < \dots < x_{N0}$. Consider N trajectories $\{x_i(t)\}_{i=1}^N$ defined by

$$\frac{d}{dt}x_i(t) = u^2(x_i, t) - \frac{1}{3} \left(u_x^2(x_i, t) + u_x(x_i, t)u_x(x_{i-1}, t) + u_x^2(x_{i-1}, t) \right), \quad (4.1)$$

subject to $x_i(0) = x_{i0}$, where

$$u(x, t) = \sum_{i=1}^N p_i G(x - x_i(t))$$

Then, u is an N -peakon weak solution of the mCH equation (1.1) before the first collision time T :

$$T := \sup \{ T_0 : x_1(t) < x_2(t) < \dots < x_N(t) \text{ for } t \in [0, T_0] \}. \quad (4.2)$$

Proof. We only have to check the second condition in (2.2) for $t \in [0, T)$, where T is defined by (4.2). By the assumption that $x_i(t) \neq x_j(t)$ for $i \neq j$ and $t < T$, we have

$$v_\ell^i(t) - v_r^i(t) = u_x(x_i(t)-, t) - u_x(x_i(t)+, t) = p_i.$$

Because $m = 0$ in the smooth region of u , we have

$$A_\ell^i(t) = \lim_{x \rightarrow x_i(t)-} m(x, t) (u^2(x, t) - u_x^2(x, t) - x_i'(t)) = 0$$

and similarly $A_r^i(t) = 0$. Hence, we have

$$\frac{d}{dt}(v_\ell^i(t) - v_r^i(t)) = A_\ell^i(t) - A_r^i(t) = 0, \quad i = 1, \dots, N,$$

which proves the second condition of (2.2). \square

When $\ell < +\infty$ and $x_1(t) < x_2(t) < \dots < x_N(t)$, Eq. (4.1) can be simplified as

$$\frac{d}{dt}x_i = \frac{1}{4}p_i^2 \left(\coth^2(\ell/2) - \frac{1}{3} \right) + \coth(\ell/2)p_i \sum_{j \neq i} p_j G(x_i - x_j) + \left(\sum_{j \neq i} p_j G(x_i - x_j) \right)^2 - \left(\sum_{j \neq i} p_j G_x(x_i - x_j) \right)^2. \quad (4.3)$$

In the case $\ell = \infty$ and $x_1(t) < x_2(t) < \dots < x_N(t)$, Eq. (4.1) is simplified as

$$\frac{d}{dt}x_i = \frac{1}{6}p_i^2 + \frac{1}{2} \sum_{j \neq i} p_i p_j e^{-|x_i - x_j|} + \sum_{1 \leq m < i < n \leq N} p_m p_n e^{-|x_m - x_n|}. \quad (4.4)$$

This ODE system was also obtained in [5].

Remark 4.3. Similar peakon weak solutions can also be obtained for the CH equation (1.7) (with $k = 0$) in \mathbb{T}_ℓ . When $\ell = +\infty$,

$u(x, t) = pG(x - ct)$ is a weak solution to the CH equation if the traveling speed is $c = \frac{p}{2}$ [16]. When consider the periodic domain $\ell < +\infty$, one can prove that the traveling speed for such a peakon weak solution is given by $\frac{p}{2} \coth(\frac{\ell}{2})$ (see Appendix C). Letting $\ell \rightarrow +\infty$ in $\frac{p}{2} \coth(\frac{\ell}{2})$, the speed converges to $\frac{p}{2}$.

The periodic peakon weak solutions to the CH equation was also discussed in [21]. But the traveling speed is given by $\frac{p}{2}$ instead of $\frac{p}{2} \coth(\frac{\ell}{2})$. In [22], Lenells analyzed the trajectory stability of this periodic peakon weak solutions with the expression given by [21], without worrying about the speed of the traveling peakon. The reason that Lenells can successfully obtain the stability result is that he used the method in [23] which only considers the shape stability of peakon weak solutions and notice that different traveling speeds in $pG(x - ct)$ give a same function shape.

4.2. Peakon weak solutions for $k < 0$

If the dispersive term $2ku_x$ is present for $k \in \mathbb{R}$, the CH equation (1.7) has peakon weak solutions of the form [18,20]:

$$u(x, t) = pe^{-|x-(p-k)t|} - k, \quad x \in \mathbb{R}, t \geq 0. \quad (4.5)$$

Clearly, these peakon solutions do not vanish at infinity when $k \neq 0$ ($u \rightarrow -k$ as $|x| \rightarrow \infty$).

For the mCH equation with $k < 0$, we have the following similar result.

Proposition 4.3. Assume $p \neq 0$ and $k < 0$. Let

$$u(x, t) = pG(x - ct) - \sqrt{-k}, \quad m(x, t) = p\delta(x - ct) - \sqrt{-k}, \quad (4.6)$$

where the traveling speed c is given by

$$c = \begin{cases} \left[\frac{p}{2} \coth(\frac{\ell}{2}) - \sqrt{-k} \right]^2 - \frac{p^2}{12} & \text{if } \ell < +\infty; \\ \left(\frac{p}{2} - \sqrt{-k} \right)^2 - \frac{p^2}{12} & \text{if } \ell = +\infty. \end{cases} \quad (4.7)$$

Then, $u(x, t)$ is a weak solution to the mCH equation (1.1) in \mathbb{T}_ℓ .

The proof is just a straightforward verification of the jump conditions proposed in Theorem 2.1 and we omit it.

Remark 4.4. We remark that peakon weak solutions (4.6) can also be obtained by the patching technic described in Section 3.3. Consider the case $\ell = +\infty$. Fix $k < 0$, $k \neq -1$ and $p > 0$. Let c be defined by (4.7). Set the integral constant $g = (c - k)\sqrt{-k}$ in (3.5). There are two branches of level set $H = -k\left(\frac{c}{2} - \frac{k}{4}\right)$ intersects at $(-\sqrt{-k}, 0)$ which is a saddle point of System (3.6). With similar arguments as in Remark 4.2, one can show that these peakon weak solutions can be obtained by the patching technic. When $k = -1$, the critical point $(-1, 0)$ is degenerate, but a patched peakon weak solution of form (4.6) can also be obtained.

As shown by Propositions 4.1 and 4.3, the mCH equation (1.1) with $k \leq 0$ has special peakon weak solutions constructed by the fundamental solution to the Helmholtz operator $1 - \partial_{xx}$. And all these solutions can be constructed by the patching technic. Naturally, one may wonder whether it has such peakon weak solutions when $k > 0$. Next, we give a proposition to show that there are no such peakon weak solutions to (1.1) when $k > 0$.

Proposition 4.4. *If $k > 0$, there is no weak solution to (1.1) of the form $u(x, t) = AG(x - ct) - B$ for any constants $c, A \neq 0$ and B .*

Proof. Suppose there is a weak solution to (1.1) of the form $u(x, t) = AG(x - ct) - B$. Then, $m = u - u_{xx} = -B =: M$ is a constant when $x \neq ct$. Consider the smooth region of u where it is a traveling wave solution. Let $\phi(\xi) := u(x, t)$ and $\xi = x - ct$. By System (3.4), we have

$$m = \phi - \phi'' = \frac{2k\phi - g}{c - \phi^2 + v^2} = M.$$

Since ϕ is not a constant, we find

$$\phi^2 - v^2 - c = \frac{1}{M}(g - 2k\phi).$$

The solution must correspond to a level set of H :

$$\begin{aligned} H &= \frac{1}{4}(\phi^2 - v^2)^2 + \frac{1}{2}c(v^2 - \phi^2) + k\phi^2 - g\phi \\ &= \frac{1}{4}\left(c + \frac{1}{M}(g - 2k\phi)\right)^2 - \frac{1}{2}c\left(c + \frac{1}{M}(g - 2k\phi)\right) \\ &\quad + k\phi^2 - g\phi \\ &= \frac{k(k + M^2)}{M^2}\phi^2 - \frac{g(k + M^2)}{M^2}\phi + \frac{1}{4}\left(c + \frac{g}{M}\right)^2 \\ &\quad - \frac{1}{2}c\left(c + \frac{g}{M}\right) = h. \end{aligned} \tag{4.8}$$

Since ϕ is not a constant, the coefficients of ϕ must be zero on the right hand side but this is impossible if $k > 0$. \square

Remark 4.5. When $k < 0$, (4.8) also implies that there exists traveling wave solutions that makes m is a constant in smooth regions only if $M = \sqrt{-k}$.

5. Planar curve flows and the mCH equation

In this section, we discuss the planar curve flow $\mathcal{C}(t)$ and see how a class of curve flows are related to the mCH equation. Suppose the velocity for a point $p(t; \xi)$ on curve $\mathcal{C}(t)$ is given by $v(t; \xi)$, and thus

$$\frac{dp(t; \xi)}{dt} = v(t; \xi),$$

where $\xi \in [0, \ell]$ is a label for the point, called Lagrangian coordinate for the curve. The Lagrangian coordinate allows us to parametrize $\mathcal{C}(t)$ as $r(\xi, t) := p(t; \xi)$. Since the velocity $v(t; \xi)$ can be decomposed along the tangential and normal directions, we then can write the equation for the planar curve flow as

$$r_t = a\tau + bn, \tag{5.1}$$

where

$$\tau = \tau(\xi, t) = (\tau_1, \tau_2) := r_\xi / |r_\xi|$$

is the unit tangent vector and

$$n = n(\xi, t) := \tau^\perp(\xi, t) = (-\tau_2, \tau_1)$$

is the unit normal vector.

Before we discuss the our planar curve flows, let us make the following remarks: (i) If a and b are Euclidean differential invariants (i.e. they only depend on curvature and its derivatives about the arc-length), the curve flows are intrinsically the same (i.e. if r is the flow for r_0 , then $Pr + b$ is the flow for initial curve $Pr_0 + b$ where P is a rotation and b is a translation). (ii) Geometric properties like total arclength, curvature and the derivatives of curvature only depend on the normal component b . This means if one cares the geometric curve flow only, one can set a to be anything (see [24]). (iii) a affects the stretching of the curve and elastic properties.

We will study curve motions that preserve arc-length and total signed area in detail. For relative discussions, see [24].

5.1. Arc-length preserving motions

The following results give criteria for preserving arc-length and total signed area of closed curve flows given by (5.1).

Lemma 5.1. *Suppose a smooth curve flow $r(\xi, t)$ is a solution to (5.1) with Euclidean differential invariants a and b . $\xi \in [0, \ell]$ is the Lagrangian coordinate and $s = s(\xi, t)$ is the arc-length parameter at time t . Let $\kappa(s, t)$ be the curvature of the curve $r(\xi, t)$. Then, the following statements hold.*

(i) *The curve is non-stretching (i.e. $\partial_t |r_\xi(\xi, t)| \equiv 0$, the length between any two points is unchanged or arc-length is preserved) if and only if*

$$a_s = b\kappa \tag{5.2}$$

holds for any $t > 0$.

(ii) *Suppose the curve is a closed curve, or in other words, $r(\xi, t)$ is a periodic function with period ℓ and $\xi \in [0, \ell]$. Then, we have*

$$\frac{d}{dt}A(t) = - \oint_{\mathcal{C}(t)} b ds, \tag{5.3}$$

where

$$A(t) = \frac{1}{2} \oint_{\mathcal{C}(t)} r \cdot (-nds) \tag{5.4}$$

is the total signed area enclosed by the curve $r(\xi, t)$. In particular, if $\oint_{\mathcal{C}(t)} b ds = 0$, the curve flow preserves the area enclosed.

Proof. (i) Denote

$$\chi(\xi, t) = |r_\xi(\xi, t)|.$$

Because

$$\frac{1}{2} \partial_t \chi^2 = r_\xi \cdot r_{\xi t} = a_\xi \chi - \chi^2 b \kappa,$$

we see that χ is a constant if and only if $a_\xi = b\kappa\chi$ or equivalently (5.2) holds since $ds = \chi d\xi$.

(ii) By direct computation, we find

$$\tau_t = \partial_t \left(\frac{r_\xi}{|r_\xi|} \right) = \frac{1}{\chi} (a\tau_\xi + b_\xi n) = \left(a\kappa + \frac{b_\xi}{\chi} \right) n. \tag{5.5}$$

Further, since $n_t \perp n$, we can write $n_t = h\tau$ for some constant h . By the fact that $\tau \cdot n = 0$, we have $h = -n \cdot \tau_t$ and

$$n_t = - \left(a\kappa + \frac{b_\xi}{\chi} \right) \tau.$$

We now rewrite the formula for the total signed area as

$$A(t) = \frac{1}{2} \oint_{\mathcal{C}(t)} r \cdot (-nds) = -\frac{1}{2} \int_0^\ell r \cdot n \chi d\xi.$$

Hence,

$$\frac{d}{dt}A = -\frac{1}{2} \int_0^\ell b |r_\xi| + r \cdot \tau (-a\kappa |r_\xi| + b_\xi) + r \cdot n (a_\xi - |r_\xi| b \kappa) d\xi$$

Integrating by parts for the terms about a_ξ and b_ξ , we have $A' = -\int_0^\ell b|r_\xi|d\xi = -\oint_{\mathcal{C}(\ell)} bds$. \square

Under the non-stretching assumption (5.2) for curve flow $r(\xi, t)$, we derive an evolution equation for curvature κ . When $s(\xi, 0) = \xi$, by Lemma 5.1 we have $\chi(\xi, t) \equiv 1$ and $s(\xi, t) \equiv \xi$ for any $t \geq 0$. Consequently, (5.5) can be rewritten as

$$\tau_t = (b_s + \kappa a)n.$$

Using the above equation, we can obtain

$$\begin{aligned} (\kappa n)_t = \tau_{st} = \tau_{ts} &= (b_s + \kappa a)_s n + (b_s + \kappa a)n_s \\ &= (b_{ss} + (\kappa a)_s)n - \kappa(b_s + \kappa a)\tau. \end{aligned}$$

Hence, it follows that

$$\kappa_t = n \cdot (\kappa n)_t = b_{ss} + (\kappa a)_s, \tag{5.6}$$

where $n_t \cdot n = 0$ was used.

We give two examples of closed curve flows connecting with two integrable systems which preserve arc-length and total signed area. These two examples can be found in [5].

1. Take $a = \frac{1}{2}\kappa^2$, $b = \frac{1}{\kappa}a_s = \kappa_s$ in (5.1). The arc-length parameter is then preserved for any t . In the case that the curve is closed, $\oint_{\mathcal{C}(\ell)} bds = 0$ and the total signed area enclosed by the curve is preserved. By (5.6), the curvature satisfies

$$\kappa_t = \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s. \tag{5.7}$$

This equation is the modified Korteweg–de Vries (mKdV) equation.

2. In this example, a and b in (5.1) are determined by κ globally by first solving $u - u_{ss} = \kappa$ and then taking $a = -(u^2 - u_s^2) - 2$, $b = a_s/\kappa = -2u_s$. In the case that the curve is closed, we have $\oint_{\mathcal{C}(\ell)} bds = 0$ which means the total signed area is also preserved. By (5.6), u satisfies the following mCH equation with a dispersion term $2u_s$

$$\kappa_t + [(u^2 - u_s^2)\kappa]_s + 2u_s = 0, \quad \kappa = (1 - \partial_{ss})u. \tag{5.8}$$

Hence, the curvature is the function m introduced in (1.1). This example connects a particular curve flows with the mCH equation we are studying in this paper.

Remark 5.1. It is well-known that a planar curve is determined uniquely by its curvature function up to a rigid body motion. In the recent work [25], Bruveris, Michor and Mumford showed that the space of closed plane curves equipped with some Sobolev-type metric is geodesically complete.

5.2. Hamiltonian structure for the arc-length preserving motions

In this section, we consider closed curve flows given by a Hamiltonian system

$$r_t = P \frac{\delta E}{\delta r} \tag{5.9}$$

where E is some functional of the curve r and P is an anti-Hermitian operator so that E is conserved under this motion. One example is

$$P = \lambda(\xi)J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.10}$$

and λ is a scalar. We aim to find Hamiltonian systems of the form (5.9) for the arc-length preserving flows (5.1).

Motivated by the elastic energies in material science, we consider energy functionals depending both on geometric quantities (curvature and its derivatives) and stretching. To define the

stretching, we assume the curve has a relaxed configuration so that it contains no elastic energy under this configuration (one may consider a planar elastic spring for the motivation). Choosing ξ as the arc-length parameter for this relaxed configuration, then the elastic stretching is defined as

$$\chi(\xi, t) = |r_\xi(\xi, t)|. \tag{5.11}$$

We have the following proposition.

Proposition 5.1. Suppose a smooth curve flow $r(\xi, t)$ is a solution to (5.1) with Euclidean differential invariants a and b . $\xi \in [0, \ell]$ is the Lagrangian coordinate chosen as arc-length parameter for the relaxed configuration and $s = s(\xi, t)$ is the arc-length parameter at time t . Let $\kappa(s, t)$ be the curvature of the curve $r(\xi, t)$. Denote $\chi(\xi, t)$ the elastic stretching of $r(\xi, t)$ at time t defined in (5.11).

If $s(\xi, 0) = \xi$ and $a_s = b\kappa$, then the dynamics (5.1) for this curve flow can be rewritten in the following Hamiltonian structure

$$r_t = -\frac{a^2}{\kappa} J \frac{\delta E}{\delta r}, \tag{5.12}$$

where J and E are given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E := \int_0^\ell \frac{1}{a}(\chi - 1)d\xi. \tag{5.13}$$

Consequently, the curve flows given by (5.12) move in the manifold $\{r : E = 0\}$.

In particular, when $a = \frac{1}{2}\kappa^2$ in (5.13), (5.12) gives the curve flows described by the mKdV equation (5.7). When $a = -(u^2 - u_s^2) - 2$ in (5.13) where u satisfies $u - u_{ss} = \kappa$, (5.12) gives the curve flows described by the mCH equation (5.8).

Proof. By Lemma 5.1, if $a_s = b\kappa$, then the curve is non-stretching and the arc-length is preserved. Hence, when $s(\xi, 0) = \xi$, we have $s(\xi, t) \equiv \xi$ for any time $t \geq 0$.

Taking variation of the curve subject to $\delta r(0) = \delta r(\ell)$ gives

$$\delta E = -\int_0^\ell \frac{\delta a}{a^2}(\chi - 1)d\xi - \int_0^\ell \frac{\partial}{\partial \xi} \left(\frac{1}{a} \frac{r_\xi}{|r_\xi|} \right) \cdot \delta r d\xi. \tag{5.14}$$

Along the arc-length preserving curve flow $r(\xi, t)$, we have $\chi \equiv 1$ and the first term in (5.14) is zero. Hence, we have

$$\frac{\delta E}{\delta r} = -\frac{\partial}{\partial s} \left(\frac{1}{a} \tau \right) = \frac{a_s}{a^2} \tau - \frac{\kappa}{a} n. \tag{5.15}$$

As a consequence, we have

$$r_t = a\tau + bn = a\tau + \frac{a_s}{\kappa}n = -\frac{a^2}{\kappa} J \frac{\delta E}{\delta r}.$$

The rest statements can be obtained easily and we omit the proofs. \square

5.3. Loops with peakons

We now study the closed curve flows corresponding to the patched traveling peakon weak solutions to the mCH equation (5.8), which are constructed in Theorem 3.2. Recall that such a patched peakon weak solution $u(s, t) = \phi(\xi)$ ($\xi = s - ct$) is piecewise smooth with jumps in first order derivative. The curvature $\kappa = u - u_{ss} = \phi - \phi''$ (ϕ'' is understood in distribution sense) of the corresponding curve contains Dirac delta mass. If the curve is closed, then we have a loop with peakons due to the Dirac delta mass in curvature.

The curvature $\kappa(s)$ of a closed curve must also be a periodic function and the period T is a divisor of the curve length ℓ . Moreover, it must satisfy the Gauss–Bonnet formula:

$$\int_0^\ell \kappa(s)ds = 2n\pi, \quad n \in \mathbb{Z}. \tag{5.16}$$

For a simple closed loop, we have $n = \pm 1$. If we allow the curve to turn over, then n can be other integers.

However, not every periodic solution κ of Eq. (5.8) corresponds to a closed curve flow. Next, we study what kind of κ gives a closed curve. For convenience, we now introduce the angle function $\theta(s, t)$ which is the integral of $\kappa(s, t)$ about s . Note that κ is the distributional derivative of a piecewise smooth function f . We omit the t dependence for the convenience of discussion. The angle function is then defined as

$$\theta(s) := \int_{(0,s]} \kappa(z)dz = f(s+) - f(0+). \tag{5.17}$$

We now present a criterion on θ that guarantees that the curve is closed. With a modification of results in [26], we have

Lemma 5.2. *Suppose $\theta(s)$ is a piecewise smooth function that is cadlag (right-continuous with left limits) and there exists $T > 0$ such that $\theta(s + T) = \theta(s) + \theta(T)$, $\forall s \geq 0$.*

If $\theta(T) \in 2\pi(\mathbb{Q} \setminus \mathbb{Z})$ so that

$$\theta(T) = \frac{2m\pi}{n}, \quad m, n \in \mathbb{Z}, n > 0, \gcd(m, n) = 1,$$

then

$$\gamma(s) = \int_0^s \exp(i\theta(\tau))d\tau$$

represents a closed curve in the plane with s to be the arc-length parameter. The total length of the curve is nT . If $\theta(T) \in 2\pi\mathbb{Z}$, there are examples that the curves are not closed.

Proof. We show that

$$\gamma(nT + s) - \gamma(s) = 0, \forall s.$$

This is true because

$$\begin{aligned} \int_s^{s+nT} \exp(i\theta(\tau))d\tau &= \sum_{j=0}^{n-1} \int_{s+jT}^{s+(j+1)T} \exp(i\theta(\tau))d\tau \\ &= \sum_{j=0}^{n-1} \int_s^{s+T} \exp(i\theta(\tau + jT))d\tau \\ &= \sum_{j=0}^{n-1} \exp(ij\theta(T)) \int_s^{s+T} \exp(i\theta(\tau))d\tau \\ &= \int_s^{s+T} \exp(i\theta(\tau))d\tau \frac{1 - \exp(im2\pi)}{1 - \exp(im2\pi/n)} = 0. \end{aligned}$$

Then, the total length of γ must be n_1T so that $n_1|n$ for some integer n_1 .

By the definition of $\gamma(s)$, θ is the incline angle of the tangent vector and hence $n_1\theta(T)$ is a multiple of 2π since the curve is closed, or $n|mn_1$. Since $\gcd(n, m) = 1$, we must have $n_1 = n$ and the total length of γ is nT .

In the case $\theta(T) \in 2\pi\mathbb{Z}$, one can refer to [26] for examples where the curves are not closed. \square

Now, we show that some patched periodic traveling peakon weak solutions of (5.8) may correspond to loops with peakons. Fix $k = 1$, $c > 0$ and $g > 0$. By Theorem 3.2, there exists $h_0 = h_0(c, g) > 0$ so that for all $h > h_0$,

$$\Gamma_h := \left\{ (\phi, v) : v^2 = \phi^2 - c + \sqrt{c^2 - 4(\phi^2 - g\phi - h)}, \right. \\ \left. \phi_1 \leq \phi \leq \phi_2, v > 0 \right\}, \tag{5.18}$$

gives a patched periodic traveling peakon solution $u_h(s, t) = \phi_h(\xi)$ ($\xi = s - ct$ and $\xi \in [0, T_h]$, T_h is a periodic), where ϕ_1, ϕ_2 are

determined by

$$\begin{cases} v^2 = \phi^2 - c + \sqrt{c^2 - 4(\phi^2 - g\phi - h)}, v > 0, \\ \phi^2 - \frac{1}{3}v^2 = c. \end{cases} \tag{5.19}$$

Proposition 5.2. *Assume $k = 1$ and fix $c > 0$ and $g > 0$. Let $u_h(s, t) = \phi_h(\xi) : \xi = s - ct \in [0, T_h]$ be the patched periodic weak solution corresponding to Γ_h defined by (5.18) for $h \in (h_0, +\infty)$, where T_h is a period of the patched periodic peakon solution and h_0 is the constant in Theorem 3.2. Let (ϕ_1, v_1^+) and (ϕ_2, v_2^+) be the two endpoints of Γ_h with $\phi_1 < \phi_2$. The following statements hold:*

(i) *Let θ_h be the angle function corresponding to $\kappa_h(\xi) := \phi_h(\xi) - \phi_h(\xi)'$. Then,*

$$\begin{aligned} \theta_h(T_h) &= 2 \left(\int_{\Gamma_h} \frac{2\phi - g}{c - \phi^2 + v^2} \frac{1}{v} d\phi + v_2^+ - v_1^+ \right) \\ &= \int_0^{T_h} \phi_h(\xi) d\xi = 2 \int_{\Gamma_h} \frac{\phi}{v} d\phi. \end{aligned} \tag{5.20}$$

(ii) *$\theta_h(T_h)$ is continuous in h . If $\theta_h(T_h)$ is not a constant for $h \in (h_0, \infty)$, then there exist infinitely many solutions ϕ_h such that*

$$\theta_h(T_h) \in 2\pi(\mathbb{Q} \setminus \mathbb{Z}).$$

Letting $\theta_h(T_h) = 2\pi \frac{n}{N}$ with $n, N \in \mathbb{Z}, N > 0, \gcd(n, N) = 1$, then the solution gives an arc-length preserving loop with peakons. The total length is $L_h = NT_h$ and the curvature κ_h satisfies

$$\int_{(0, L_h]} \kappa_h(\xi) d\xi = 2n\pi.$$

Proof. Since ϕ_h'' is the distributional derivative of ϕ_h' which is periodic and piecewise continuous, by Eq. (5.17), we find that

$$\int_{(0, T_h]} \phi_h'' d\xi = 0.$$

Due to $d\xi = \frac{1}{v} d\phi$, we have

$$\theta_h(T_h) = \int_0^{T_h} \kappa_h(\xi) d\xi = \int_0^{T_h} \phi_h d\xi = 2 \int_{\Gamma_h} \frac{\phi}{v} d\phi.$$

By this expression, the continuity of $\theta_h(T_h)$ on h is clear.

For the other expression, we just consider

$$\begin{aligned} \int_{(0, T_h]} \kappa d\xi &= 2 \left(\int_{\Gamma_h} \kappa \frac{1}{v} d\phi + \int_{\Gamma_h} \phi'' d\xi \right) \\ &= 2 \int_{\Gamma_h} \kappa \frac{1}{v} d\phi + 2(v_2^+ - v_1^+). \end{aligned} \tag{5.21}$$

On Γ_h , by System (3.4) we have

$$\begin{aligned} \phi_h'' &= \phi_h + \frac{g - 2\phi_h}{c - \phi_h^2 + (\phi_h')^2} \\ \Rightarrow \kappa_h(\xi) &= \frac{2\phi - g}{c - \phi^2 + v^2} \Big|_{(\phi, v) = (\phi_h(\xi), \phi_h'(\xi))}. \end{aligned}$$

Since rational numbers are dense in any interval of \mathbb{R} , as long as $\theta_h(T_h)$ is not constant in h , there are infinitely many h such that $\theta_h(T_h) \in 2\pi(\mathbb{Q} \setminus \mathbb{Z})$. According to Lemma 5.2, the claim follows. \square

Next, we give an example to illustrate Proposition 5.2.

Fix $g = 2$ and $c = 2$. Consider the branch Γ_+ (defined by (3.10)) of $H = h$ in the phase plane that $v > 0$. Let $f_{0h}(\phi) := 4 - 4(\phi^2 - 2\phi - h)$ and $f_{1h}(\phi) := \phi^2 - 2 + \sqrt{f_{0h}(\phi)}$. Assume

$$D_{0h} := \{\phi \in \mathbb{R} : f_{0h}(\phi) \geq 0\} \text{ and } D_{1h} := \{\phi \in \mathbb{R} : f_{1h}(\phi) \geq 0\}.$$

Choose $h_0 = 1 + 2\sqrt{2}$ and for any $h > h_0$ we have

$$f_{0h}(\phi) \geq f_{0h}(-\sqrt{2}) \geq 0 \text{ for any } \phi \in [-\sqrt{2}, \sqrt{2}].$$

Moreover, we have

$$f_{1h}(\phi) \geq 0 \text{ for } \phi \in [-\sqrt{2}, \sqrt{2}], \text{ and } [-\sqrt{2}, \sqrt{2}] \subset D_{1h}.$$

Consider the intersections between $\Gamma_+(v > 0)$ and $\phi^2 - \frac{1}{3}v^2 = 2$. For $h \in (1 + \sqrt{2}, +\infty)$, there are two points $\phi_{1h}, \phi_{2h} \in \mathbb{R}$ that satisfy

$$2(\phi^2 - 2) = \sqrt{4 - 4(\phi^2 - 2\phi - h)}.$$

Hence, from (5.20) we have

$$\begin{aligned} \theta_h(T_h) &= 2 \int_{\phi_{1h}}^{\phi_{2h}} \frac{\phi}{\sqrt{f_{1h}(\phi)}} d\phi \\ &= 2 \int_{\phi_{1h}}^{\phi_{2h}} \frac{\phi}{\sqrt{\phi^2 - 2 + \sqrt{4 - 4(\phi^2 - 2\phi - h)}}} d\phi. \end{aligned}$$

Therefore, ϕ_h in Proposition 5.2 is defined on $(1 + 2\sqrt{2}, \infty)$.

The figure $\theta_h(T_h)$ versus h is shown in Fig. 10. Clearly, $\theta_h(T_h)$ is not a constant and therefore there are h such that $\theta_h(T_h) \in 2\pi(\mathbb{Q} \setminus \mathbb{Z})$. Such a solution then gives a closed loop by Lemma 5.2.

However, it is not easy to tell which h satisfies $\theta_h(T_h) \in 2\pi(\mathbb{Q} \setminus \mathbb{Z})$. To illustrate how the peakons happen, let us consider a function ϕ , which is not necessarily a solution to the mCH equation:

$$\phi(\xi) = \begin{cases} \phi_0(\xi), & \xi \in [0, 1], \\ \phi_0(2 - \xi), & \xi \in (1, 2], \end{cases}$$

where $\phi_0(\xi) = \frac{\pi}{3} + \xi^2 + \xi - \frac{5}{6}$ for $\xi \in [0, 1]$. Then, $\kappa = \phi - \phi'' = \frac{\pi}{3} + \xi + \xi^2 - \frac{17}{6}$, $\xi \in (0, 1)$. As a result, the angle function $\theta(s)$ is given by

$$\theta(s) = \begin{cases} \frac{\pi}{3}s + \frac{1}{2}s^2 + \frac{1}{3}s^3 - \frac{17}{6}s, & s \in [0, 1], \\ \frac{\pi}{3}s - \frac{1}{3}(2-s)^3 - \frac{1}{2}(2-s)^2 - \frac{17s}{6} + \frac{23}{3}, & s \in [1, 2]. \end{cases} \quad (5.22)$$

Then, $\theta(2) = \theta(2-) - 2 = \frac{2\pi}{3}$.

In Fig. 11, we plot the $\theta(s)$ function and the corresponding curve. The curve is computed by evaluating the integral $\gamma(s) = \int_0^s \exp(i\theta(\tau))d\tau$ directly.

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Appendix A. Proofs about critical points

We give a brief proof of Lemma 3.1

Proof. Consider the linearized System of (3.6) at the critical point (ϕ_*, v_*) where $\frac{\partial H}{\partial v}(\phi_*, v_*) = \frac{\partial H}{\partial \phi}(\phi_*, v_*) = 0$,

$$\begin{cases} \frac{d\phi}{d\tau} = \frac{\partial^2 H}{\partial v \partial \phi}(\phi - \phi_*) + \frac{\partial^2 H}{\partial v^2}(v - v_*), \\ \frac{dv}{d\tau} = -\frac{\partial^2 H}{\partial v \partial \phi}(v - v_*) - \frac{\partial^2 H}{\partial \phi^2}(\phi - \phi_*). \end{cases}$$

The eigenvalues of the linearized system satisfy

$$\lambda^2 - \left(\frac{\partial^2 H}{\partial v \partial \phi}\right)^2 + \frac{\partial^2 H}{\partial \phi^2} \frac{\partial^2 H}{\partial v^2} = 0.$$

When

$$\left(\frac{\partial^2 H}{\partial v \partial \phi}\right)^2 - \frac{\partial^2 H}{\partial \phi^2} \frac{\partial^2 H}{\partial v^2} < 0,$$

(ϕ_*, v_*) is a center point and $H(\phi_*, v_*)$ is a local extremum of H . When

$$\left(\frac{\partial^2 H}{\partial v \partial \phi}\right)^2 - \frac{\partial^2 H}{\partial \phi^2} \frac{\partial^2 H}{\partial v^2} > 0,$$

(ϕ_*, v_*) is a saddle point of System (3.6) and it is a saddle point of H . \square

We now give a brief proof for Proposition 3.1

Proof. Assume that (ϕ_*, v_*) is a critical point which satisfies $\phi_*^2 - v_*^2 = c$. From the second equation in System (3.6), we obtain $-2k\phi_* + g = 0$. Hence,

$$\phi_* = \frac{g}{2k} \text{ and } v_*^2 = \phi_*^2 - c = \frac{g^2 - 4k^2c}{4k^2} \text{ when } g^2 - 4k^2c > 0.$$

Hence,

$$H(\phi_*, v_*) = \frac{c^2}{4} - \frac{c^2}{2} + \frac{g^2}{4k} - \frac{g^2}{2k} = -\frac{c^2}{4} - \frac{g^2}{4k}.$$

$$\begin{aligned} \frac{\partial^2 H}{\partial v \partial \phi}(\phi_*, v_*) &= \frac{g}{k} \sqrt{\frac{g^2 - 4k^2c}{4k^2}}, \quad \frac{\partial^2 H}{\partial \phi^2}(\phi_*, v_*) \\ &= \frac{g^2 + 4k^3}{2k^2}, \quad \frac{\partial^2 H}{\partial v^2}(\phi_*, v_*) = \frac{g^2 - 4k^2c}{2k^2}. \end{aligned}$$

Hence

$$\begin{aligned} \left(\frac{\partial^2 H}{\partial v \partial \phi}\right)^2 - \frac{\partial^2 H}{\partial \phi^2} \frac{\partial^2 H}{\partial v^2} &= \frac{g^2(g^2 - 4k^2c)}{4k^4} \\ &\quad - \frac{(g^2 + 4k^3)(g^2 - 4k^2c)}{4k^4} = -\frac{g^2 - 4k^2c}{k} \end{aligned}$$

Hence, when $k > 0$, we have

$$\left(\frac{\partial^2 H}{\partial v \partial \phi}\right)^2 - \frac{\partial^2 H}{\partial \phi^2} \frac{\partial^2 H}{\partial v^2} < 0.$$

By Lemma 3.1, (ϕ_*, v_*) is a center point. When $k < 0$, we have

$$\left(\frac{\partial^2 H}{\partial v \partial \phi}\right)^2 - \frac{\partial^2 H}{\partial \phi^2} \frac{\partial^2 H}{\partial v^2} > 0.$$

Lemma 3.1 tells us that (ϕ_*, v_*) is a saddle point. \square

Appendix B. Proof of Proposition 3.2

Proof. If $(k, g) = (0, 0)$, all the level sets are of the form $\phi^2 - v^2 = c$ and the claim is clearly true. Now, we assume $(k, g) \neq (0, 0)$. We use five steps to prove this proposition. In Step 1 and Step 2, we give some useful results. With these arguments, we prove proposition by Step 3, Step 4 and Step 5. In this proof, we always use $\tilde{\alpha}$ to represent the reflection of a curve α about $v = 0$.

Step 1.

$\Gamma_+ \cap \Sigma$ ($\Gamma_- \cap \Sigma$) if nonempty must contain an arc with positive length.

Due to Lemma 3.3, $\Gamma_+ \cap P_{\pm}$ are graphs of continuous functions. Because Σ is a connected component, when $\Gamma_+ \cap \Sigma \cap P_{\pm}$ is not empty, it must contain an arc with positive length or it is a single point.

Assume $\Gamma_+ \cap \Sigma \cap P_+ = \{(\phi_0, v_0)\}$ is a single point. Because Σ is a nontrivial connected set, we can see that $\Gamma_+ = \{(\phi_0, v_0)\}$. This implies that the square root in the definition of Γ_+ is defined only at ϕ_0 and at this time we have $\Gamma_- = \Gamma_+$, in which case Σ is trivial contradicting with the assumption. Hence, $\Gamma_+ \cap \Sigma$ ($\Gamma_- \cap \Sigma$) if nonempty must contain an arc with positive length.

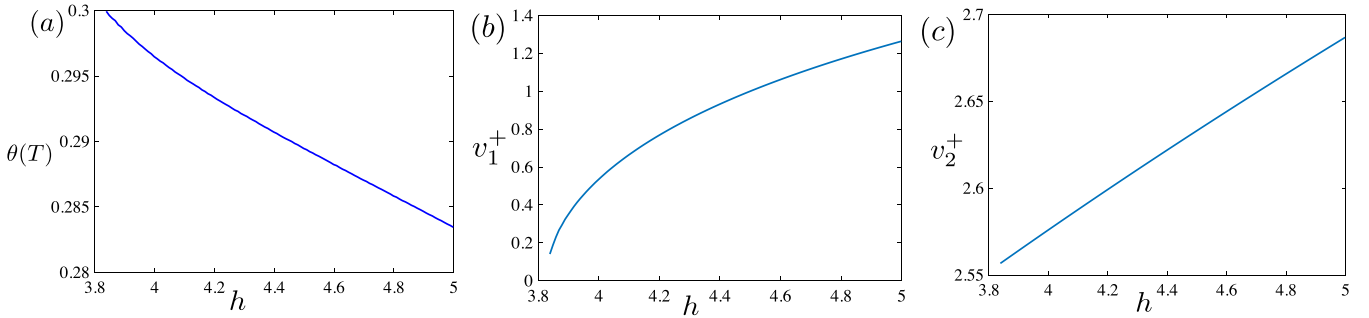


Fig. 10. (a): $\theta_h(T_h)$ versus h . Figures (b) and (c) show the jump of $\theta_h(T_h)$ at the end of Γ_h .

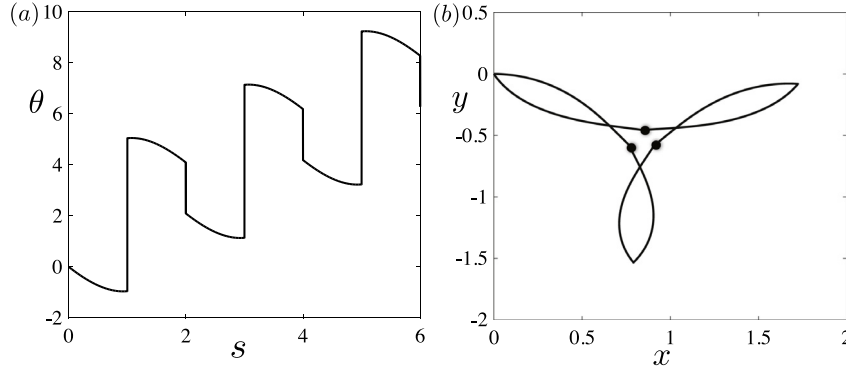


Fig. 11. Illustration of loops with peakons with Eq. (5.22). Figure (a) shows the graphs of $\theta(s)$ function, and Figure (b) shows the corresponding planar curves. The dots show the singular points at $s = 1, 3, 5$, while the other peakons correspond to $s = 0, 2, 4$.

Step 2.

If there is a point $A = (\phi_*, v_*) \in \Sigma$ such that there are more than three arcs emerging from A . The Implicit theorem tells us that A has to be a critical point of H (or System (3.6)).

If the critical point A is on $\phi^2 - v^2 = c$, we know $v_* \neq 0$. Otherwise, we have $\phi_*^2 = c$, which contradicts with our assumption. This point has to be the critical point of the form $\phi_* = \frac{g}{2k}$ for (3.6). By Proposition 3.1, this critical point is a center point when $k > 0$ and hence we must have $k < 0$ (For $k = 0$ and $g \neq 0$, this is not a critical point.) In this case, we have

$$\sqrt{c^2 - 4(k\phi_*^2 - g\phi_* - h)} = \sqrt{c^2 - 4k\left(\phi_* - \frac{g}{2k}\right)^2 + \frac{g^2}{k} + 4h} = 0.$$

Γ_{\pm} are defined both on the right and left of this critical point. As a result, on either side, there are two curves emerging from A .

If $A = (\phi_*, 0)$, since $\phi_*^2 \neq c$, Γ_+ and Γ_- do not meet here. There are and only are two curves emerging from A on each side of the line $\phi = \phi_*$.

Step 3.

If $\Gamma_+ \cap \Sigma = \emptyset$ or $\Gamma_- \cap \Sigma = \emptyset$, by Lemma 3.3, Σ must not intersect $\phi^2 - v^2 = c$ at $v \neq 0$. By the assumption there is no point $(\phi_0, 0)$ with $\phi_0^2 = c$ on it, Σ does not intersect the hyperbola on $v = 0$ as well. Moreover, $\Sigma \subset \Gamma_+$ or $\Sigma \subset \Gamma_-$. As a result, by Lemma 3.3, $\Sigma \cap P_+$ if nonempty is the graph of a function $v = f(\phi)$.

If there are no points on Σ as described in Step 2, from the third statement of Lemma 3.3 we have three cases for $f(\phi)$: case 1. two endpoints of f are at $v = 0$; case 2. one endpoint of f is on $v = 0$ the other end tends to infinity; case 3. both ends of f extend to infinity. Assume $\Sigma_+ = \Sigma \cap P_+$, which is given by the graph of function f . For case 1 and case 2, by our assumption about Σ , we know $\Sigma_+ \cup \tilde{\Sigma}_+ = \Sigma$. At this time Σ is a loop for case 1 while both ends tend to infinity for case 2. For case 3, we have $\Sigma = \Sigma_+$.

If there are points on Σ as described in Step 2, they must be on $v = 0$. Otherwise they are on $\phi^2 - v^2 = 0$ and at this time $\Gamma_+ \cap \Gamma_- \cap \Sigma \neq \emptyset$. We can use the lines $\phi = \phi_A$ (A is a critical point in Σ) to divide the whole plane into finite regions R_i . Consider the arc Σ_i of the graph of function f in R_i . The analysis is the same as the last paragraph.

Step 4

Now, we assume $\Gamma_+ \cap \Sigma \neq \emptyset$ and $\Gamma_- \cap \Sigma \neq \emptyset$. By Step 1, both parts contain at least an arc with positive length.

Without loss of generality, we can assume $\Gamma_+ \cap \Sigma \cap P_+ \neq \emptyset$. By Lemma 3.3, this is a function graph $v = f_1(\phi)$ with the largest domain D . We claim first that D is an interval. Suppose otherwise, then there are two disjoint intervals I_1 and I_2 such that $v = f_1(\phi)$ has no graphs between them. Let I_1 be on the left of I_2 . Consider the right endpoint of I_1 , ϕ^* . If Γ_+ is defined for $\phi \in (\phi^*, \phi^* + \delta)$, then the graph in P_+ must be continuously extended from $f_1(\phi)$ by the expression of Γ_+ in (3.10). Hence, Γ_+ is not defined on $\phi \in (\phi^*, \phi^* + \delta)$ (for $\delta > 0$) and consequently, Γ_- is not defined either (the domain of Γ_- is contained in the domain of Γ_+). This means the graphs of Γ_+ over I_1 and I_2 cannot be in the same connected component Σ .

To be simple, let γ be the function graph of f_1 . If there are any points $A = (\phi_A, v_A)$ as described in Step 2 on γ , we use the lines $\phi = \phi_A$ to divide the whole planes into finite regions R_i (of course, if there are no such points, R_1 is the whole plane). Consider the arc $\gamma_i = \gamma \cap R_i$. Assume its end points are $B = (\phi_B, v_B)$ and $C = (\phi_C, v_C)$. γ_i must have one end that ends on $v = 0$ or $\phi^2 - v^2 = c$ with $v \neq 0$ (if both ends tend to infinity, we then have the case in Step 3). We use the following four cases to study the structure of γ_i .

1. The first case is that one end, B , of γ_i tends to infinity.
 - If the other end C satisfies $v_C = 0$, then C is not on $\phi^2 - v^2 = c$. This means Γ_+ and Γ_- do not meet there. By the symmetry of Σ , we know $\Sigma \cap R_i = \gamma_i \cup \tilde{\gamma}_i$. In this case, $\Sigma \cap R_i$ has both ends tend to infinity.

- If the other end C falls onto $\phi^2 - v^2 = c$, then we must have $v \neq 0$. Then, Γ_+ and Γ_- meet here and Γ_- has an arc α_i merging from C which is under γ_i in $R_i \cap P_+$.

If the other end of α_i ends on a point C' with $v_{C'} = 0$, then $\gamma_i \cup \alpha_i \cup \tilde{\gamma}_i \cup \tilde{\alpha}_i$ is a simple curve with both ends tending to infinity. Furthermore, if C' happens to be a critical point in Step 2, there are some arc of Γ_- beyond this critical point and the discussion for these arcs will be similar as Step 3.

If the other end of α_i tends to infinity without intersecting $v = 0$, then $\gamma_i \cup \alpha_i = \Sigma \cap R_i \cap P_+$ is a simple curve with both ends tending to infinity. $\Sigma \cap R_i \cap P_-$ if nonempty (this happens only if C' is the critical point on the hyperbola) will be the reflection of this simple curve, which is again a simple curve.

2. The second possibility is that both ends are on $v = 0$. In this case, by the discussion in Step 2, $\Gamma_- \cap \Sigma \cap R_i = \emptyset$. As a result, $\Sigma \cap R_i = \gamma_i \cup \tilde{\gamma}_i$ is a loop.
3. Another possibility is one (say B) on $v = 0$ and one (say C) on $\phi^2 - v^2 = c$ with $v \neq 0$. Then, there is an arc α_i in $\Gamma_- \cap \Sigma \cap R_i$ merging from C in R_i and under γ_i . The other end of α_i must terminate on a point C' with $v_{C'} = 0$. As a result, $\Sigma \cap R_i$ is a loop. Similarly, if C' happens to be a critical point in Step 2, there are some arc of Γ_- beyond this critical point and the discussion for these arcs will be similar as that in Step 3.
4. Lastly, both B and C are on $\phi^2 - v^2 = c$ with $v \neq 0$. We call the left endpoint B . In this case, $\Gamma_- \cap \Sigma \cap R_i$ exists both on the right of B and on the left of C , called α_i^- and α_i^+ .

- If $\alpha_i^- = \alpha_i^+$, we call this α_i . Then, $\gamma_i \cup \alpha_i = \Sigma \cap R_i \cap P_+$ is a simple curve, which is a loop. If $\Sigma \cap R_i \cap P_-$ is nonempty (this happens only if α_i touches $v = 0$ at a critical point), then, it is the reflection of the loop in P_+ .
- If $\alpha_i^- \neq \alpha_i^+$, then, they must both touch $v = 0$. As a result, $\gamma_i \cup \alpha_i^- \cup \alpha_i^+ = \beta$ and $\beta \cup \tilde{\beta}$ is a simple curve and forms a loop. Further, if these points on $v = 0$ happen to be critical points in Step 2, there are some arc of Γ_- beyond this critical point and the discussion for these arcs will be similar as that in Step 3.

This then finishes the discussion of the structures of Σ .

Step 5

The last part of the proposition is obvious by Lemma 3.3 and the expressions of Γ_{\pm} . \square

Appendix C. Peakon weak solutions for the CH equation (1.7)

The definition of weak solution to the CH equation (1.7) is given similarly as the mCH equation. $u \in C([0, T]; H^1(\mathbb{T}_\ell))$ is a weak solution to (1.7) subject to $u(x, 0) = u_0(x)$ if the following holds for any $\varphi(x, t) \in C_c^\infty([0, T] \times \mathbb{T}_\ell)$

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}_\ell} u \phi_t dx dt - \int_0^T \int_{\mathbb{T}_\ell} u \phi_{xxt} dx dt + \frac{3}{2} \int_0^T \int_{\mathbb{T}_\ell} u^2 \phi_x dx dt \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{T}_\ell} u_x^2 \phi_x dx dt \\ & - \int_0^T \int_{\mathbb{T}_\ell} \frac{1}{2} u^2 \phi_{xxx} dx dt = - \int_{\mathbb{T}_\ell} u \phi(x, 0) dx \\ & + \int_{\mathbb{T}_\ell} u_0 \phi_{xxx}(x, 0) dx. \end{aligned} \tag{C.1}$$

As shown in [16], the N -peakon solutions are of the form

$$u(x, t) = \sum_{i=1}^N p_i(t) G(x - x_i(t)).$$

Using the weak formulation, we find that the jump conditions given by $x_i(t)$ and $p_i(t)$ satisfy the following ODE system

$$\begin{cases} \frac{dx_i(t)}{dt} = u(x_i(t), t), \\ \frac{dp_i(t)}{dt} = -p_i(t) \frac{1}{2} [u_x(x_i(t)+) + u_x(x_i(t)-)]. \end{cases} \tag{C.2}$$

Here $p_i(t)$ depends on time and this is different with the N -peakon weak solutions to the mCH equation given in Proposition 4.2. Notice that for any $x \in \mathbb{R}$, we have

$$-\frac{1}{2} [G_x(x+) + G_x(x-)] = \text{sgn}(x)G(x).$$

Hence, (C.2) can be changed into

$$\begin{cases} \frac{dx_i(t)}{dt} = \sum_{j=1}^N p_j(t) G(x_i(t) - x_j(t)), \\ \frac{dp_i(t)}{dt} = \sum_{j=1}^N p_i(t) p_j(t) \text{sgn}(x_i(t) - x_j(t)) G(x_i(t) - x_j(t)). \end{cases} \tag{C.3}$$

This is a Hamiltonian system and the Hamiltonian function is given by

$$\mathcal{H}_0(t) = \frac{1}{2} \sum_{i,j=1}^N p_i(t) p_j(t) G(x_i(t) - x_j(t)).$$

When $\ell = +\infty$, $p_i(t)$ and $x_i(t)$ satisfy the following Hamiltonian system of ODEs:

$$\begin{cases} \frac{d}{dt} x_i(t) = \frac{1}{2} \sum_{j=1}^N p_j(t) e^{-|x_i(t) - x_j(t)|}, \quad i = 1, \dots, N, \\ \frac{d}{dt} p_i(t) = \frac{1}{2} \sum_{j=1}^N p_i(t) p_j(t) \text{sgn}(x_i(t) - x_j(t)) e^{-|x_i(t) - x_j(t)|}, \\ \quad i = 1, \dots, N. \end{cases} \tag{C.4}$$

One can refer to [27–29] for a rigorous analysis of the Hamiltonian system (C.4).

If $N = 1$, from (C.3) we have

$$p_1(t) \equiv p \quad \text{and} \quad \frac{d}{dt} x_1(t) = \frac{p}{2} \coth\left(\frac{\ell}{2}\right) = c. \tag{C.5}$$

The one peakon weak solution is given by

$$u(x, t) = pG(x - ct). \tag{C.6}$$

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