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CONVERGENCE ANALYSIS OF THE VORTEX BLOB METHOD FOR THE b-EQUATION

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Abstract. In this paper, we prove the convergence of the vortex blob method for a family of nonlinear evolutionary partial differential equations (PDEs), the so-called b-equation. This kind of PDEs, including the Camassa-Holm equation and the Degasperis-Procesi equation, has many applications in diverse scientific fields. Our convergence analysis also provides a proof for the existence of the global weak solution to the b-equation when the initial data is a nonnegative Radon measure with compact support.

1. **Introduction.** We consider the Cauchy problems of the following family of nonlinear evolutionary PDEs (the so-called b-equation) in 1-dimensional case. The b-equation is given by

$$\partial_t m + \partial_x(um) + (b-1)m\partial_x u = 0,\ m = (1-\alpha^2\partial_{xx})u,\ x\in\mathbb{R},\ t>0, \tag{1}$$

with $b>1$ and subject to the initial condition

$$m(x,0) = m_0(x),\ x\in\mathbb{R}. \tag{2}$$

Here, functions $m(x,t)$ and $u(x,t)$ represent the momentum and velocity, respectively. The parameter b expresses the stretching factor. The velocity function $u(x,t)$ can also be written as a convolution of $m(x,t)$ with the kernel G

$$u(x,t) = G*m = \int_{\mathbb{R}} G(x-y)m(y,t)dy. \tag{3}$$

For the b-equation, the kernel is taken as $G(x) = \frac{1}{2\alpha}e^{-|x|/\alpha}$ with α representing the length scale of the kernel. The kernel $G(x)$ is the fundamental solution for

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Helmholtz operator $1-\alpha^2\partial_{xx}$, i.e., $(1-\alpha^2\partial_{xx})G=\delta$ with δ representing the Dirac δ distribution.

This kind of evolutionary equations is established in diverse scientific fields based on different choices of parameter b. When $b=2$, the associated b-equation is the so-called Camassa-Holm (C-H) equation, which was established by Camassa and Holm to model the unidirectional propagation of waves at free surface of a shallow layer of water ($u(x,t)$ representing the water's free surface above a flat bottom) [4]. The C-H equation was also independently derived by Dai [17] to model the nonlinear waves in cylindrical hyper-elastic rods with $u(x,t)$ representing the radial stretch relative to a pre-stressed state. In the case of $b=3$, the associated equation is named the Degasperis-Procesi (D-P) equation used to model the propagation of nonlinear dispersive waves [18]. In higher dimensional cases, the corresponding equation is called the Euler-Poincaré equation, which appears in the mathematical model of fully nonlinear shallow water waves [6, 24]. Beyond these, this equation has many further applications in computer vision [25] and computational anatomy [26].

Mathematical analysis and numerical analysis of the Cauchy problems for both the C-H equation and the D-P equation have been extensively studied in the literature. We refer [3, 9, 14, 32] for more details related to well-posedness results of the C-H equation and the D-P equation. In terms of numerical methods, one can find in [5, 10, 23, 30, 31, 33] of the traditional numerical methods, such as finite difference methods, finite element methods and spectral methods, for the C-H equation or the D-P equation.

It is well known that the b-equation (1)-(2) has solitary wave solutions of the form $u(x,t)=aG(x-ct)$ with speed $c=-aG(0)$, proportional to the amplitude of the solution. A remarkable characteristic of those solutions is the discontinuity in their first derivative at peaks. This kind of solution is named as peakon [11, 12, 13, 15, 19, 20, 27, 29]. Those peakons are the leading driver of the time evolution of the b-equation. Based on the N-peakon solution, the particle method was established to solve the b-equation numerically [5, 7, 8, 19, 24].

The N-peakon solution to the b-equation (1)-(2) is given by

$$m^N(x,t)=\sum_{i=1}^{N}p_i(t)\delta(x-x_i(t)), \tag{4}$$

$$u^N(x,t)=G*m^N(x,t)=\sum_{i=1}^{N}p_i(t)G(x-x_i(t)). \tag{5}$$

The unknowns $x_i(t),p_i(t)$ are determined by the following ODEs [8].

$$\frac{dx_i(t)}{dt}=u^N(x_i(t),t), \tag{6}$$

$$\frac{dp_i(t)}{dt}=-(b-1)p_i(t)\partial_x u^N(x_i(t),t), \tag{7}$$

subject to some initial data $x_i(0),\ p_i(0)$. Here, $x_i(t)$ represents the location of the $i-th$ particle and $p_i(t)$ represents the momentum of the $i-th$ particle. N denotes the total number of particles. This N-peakon solution is exactly the particle method for the b-equation with initial data $\{x_i(0)\}_{i=1}^N$, $\{p_i(0)\}_{i=1}^N$ chosen to approximate $m_0(x)$. We introduce a way to choose those initial data below. We assume that supp $\{m_0\}\subset[-L,L]$, $m_0\in\mathcal{M}_+(\mathbb{R})$ and denote its mass as $M_0:=\int_{\mathbb{R}}dm_0$. The computational interval $[-L, L]$ is divided into N non-overlapping sub-interval I_j by using the uniform grid with size $h = \frac{2L}{N}$. One possible way, as it was advised in [8], to determine $x_i(0),\ p_i(0)$, is that we choose $x_i(0)$ as

$$x_i(0) = -L + (i - \frac{1}{2})h; \quad p_i(0) = \int_{x_i(0)-\frac{h}{2}}^{x_i(0)+\frac{h}{2}} m_0 * \sigma_\epsilon dx, \quad i = 1, 2, \cdots, N. \tag{8}$$

Here, the nonnegative mollifier $\sigma_\epsilon(x)$ belongs to $C_0^\infty(\mathbb{R})$ with scale ϵ. With these choices, it was proved that m_0 is approximated by $m^N(x, 0) = \sum_{j=1}^N p_j(0)\delta(x - x_j(0))$ in the sense of measures [8]. Actually, for any test function $\phi(x) \in C_0^\infty(\mathbb{R})$, the following estimate holds [8]

$$|\int_{\mathbb{R}} \phi(x) dm_0 - \int_{\mathbb{R}} \phi(x) dm^N(x, 0)| \leq C\frac{1}{N}. \tag{9}$$

Since the ODE (7) involves the singular kernel $G^{'}$ with discontinuity at $x = 0$, the standard existence theorem can not be applied here. Fortunately, one crucial property of the b-equation is that the momentum of the N-peakon solution is positivity-preserving, i.e., if $p_i(0) \geq 0$, then $p_i(t) \geq 0$. For the case $b = 2,\ m_0 \geq 0$, there is a Lax-pair for the N-peakon solution and the ODEs (6)-(7) is complete integrable. As a result, there exists a global solution to this ODEs [5]. For the N-peakon solution, the space-time BV estimate was established for $u^N,\ u_x^N$ by using the positivity-preserving property in [8]. Then, by using compactness argument, the authors proved that there exists a global weak solution to the b-equation (1)-(2) under the initial condition $m_0 \geq 0$. The BV estimate also provided a slightly improvement on the regularity results [8]. In their paper, the existence of a global solution to (6)-(7) heavily depends on conservative quantities which ensure that the trajectories cannot cross over at any finite time.

Those conservative quantities can be established for the b-equation [8]. For more general problems, it is hard to find such conservative quantities. This difficulty limits the applications of this method for more general problems. One way to avoid using conservative quantities for the existence proof is to replace the particle method by the vortex blob method. In the vortex blob method, the singular kernel $G^{'}$ is regularized and hence the existence of a global solution to the resulted ODEs is automatically obtained according to the standard ODEs theory. Therefore, this method can be applied to more general problems. It is worthy being pointed out that the particle methods, with or without regularization, are analogous to the vortex blob method and the point vortex method for the incompressible Euler equations [1, 16, 22, 28].

Beyond the theoretical benefit, the vortex blob method can also improve the stability and accuracy in computations. Compared with traditional numerical methods, the particle method has two main advantages: (i). It posses low numerical diffusion which allows one to capture a variety of nonlinear waves with high resolution; (ii). It is easy and accurate to handle the peakon solutions. In the case of classical solution under initial condition $m_0 \in C_0^2[-L, L]$, the optimal error estimates for both particle method and vortex blob method for the b-equation were established in [21].

In Proposition 3.1, we obtain the consistency error of order $O(\epsilon)$ uniform in N arising in the vortex blob method for the b-equation. Then, by combining the space-time BV estimate and the consistency error, we prove that the approximated solution of the vortex blob method converges to a global weak solution to the b-equation under the initial condition $m_0 \in \mathcal{M}_+(\mathbb{R})$ with finite mass.

To state our main results, we begin with the definition of the mollifier.

Definition 1.1. (i) Define the mollifier $0 \le \rho(x) = f(|x|) \in C^k(\mathbb{R}),\ k \ge 2$ satisfying

$$\int_{\mathbb{R}} \rho(x)dx = 1, \mathrm{supp}\{\rho\} \subset \{x \in \mathbb{R}: \ |x| < 1\}. \tag{10}$$

(ii) For each $\epsilon > 0$, set

$$\rho_\epsilon(x) := \frac{1}{\epsilon}\rho(\frac{x}{\epsilon}).$$

Then, we use ρ_ϵ to mollify the kernel $G(x),\ G^{'}(x)$ and denote

$$m^{N,\epsilon}(x,t) = \sum_{j=1}^{N} p_j^\epsilon(t)\rho_\epsilon(x - x_j^\epsilon(t)) \tag{11}$$

$$u^{N,\epsilon}(x,t) = \sum_{j=1}^{N} p_j^\epsilon(t)G^\epsilon(x - x_j^\epsilon(t)) \tag{12}$$

with notation $G^\epsilon = \rho_\epsilon * G$. The undetermined $x_i^\epsilon(t)$ and $p_i^\epsilon(t)$ satisfy the following ODEs

$$\frac{dx_i^\epsilon(t)}{dt} = u^{N,\epsilon}(x_i^\epsilon(t), t) \tag{13}$$

$$\frac{dp_i^\epsilon(t)}{dt} = -(b-1)p_i^\epsilon(t)\partial_x u^{N,\epsilon}(x_i^\epsilon(t), t) \tag{14}$$

subject to the same initial data with (8).

The existence and uniqueness of a global solution to this ODEs follows from the standard ODEs theory.

Remark 1. In the same way as that of [5, 8], we know that $p_i^\epsilon(t) \ge 0$ provided the initial condition $p_i^\epsilon(0) \ge 0$. It is clear that (13) implies that $x_i^\epsilon(t) > x_i^\epsilon(s)$ if $t > s$ due to the fact that $u^{N,\epsilon} > 0$. These properties will be used in the space-time BV estimates.

The following notations will be used in our analysis.

$$m_\epsilon^N(x,t) = \sum_{j=1}^{N} p_j^\epsilon(t)\delta(x - x_j^\epsilon(t)) \tag{15}$$

$$u_\epsilon^N(x,t) = \sum_{j=1}^{N} p_j^\epsilon(t)G(x - x_j^\epsilon(t)). \tag{16}$$

We will use the concept of space-time BV estimate to establish the compactness argument. Let us recall the space $BV(\mathbb{R})$ [2].

Definition 1.2. A function f belongs to BV if for any $\{x_i\} \subset \mathbb{R},\ x_i < x_{i+1}$, the following statement holds:

$$\sup_{\{x_i\}}\{\sum_i |f(x_i) - f(x_{i-1})|\} < +\infty.$$

If $f(x) \in BV(\mathbb{R})$, we denote by

$$\text{Tot.Var.}\{f\} = \sup_{\{x_i\}}\{\sum_i |f(x_i) - f(x_{i-1})|\}.$$

Then, we prove the following Proposition, which shows that the total variation with respect to x for any $t \geq 0$ and the maximum of $u^{N,\epsilon},\ u_x^{N,\epsilon}$ are uniformly bounded and $u \in \text{Lip}([0, +\infty);\ W^{1,1}(\mathbb{R}))$.

Proposition 1. *Assuming that the initial data $m_0 \in \mathcal{M}_+(\mathbb{R})$ has bounded mass M_0 and compact support and $u^{N,\epsilon}$ is given by (12), then, there exist constants $C,\ M,\ L$ independent of $N,\ \epsilon$ such that*

$$\textit{Tot. Var.}\{u^{N,\epsilon}(\cdot, t)\}, \quad \textit{Tot. Var.}\{u_x^{N,\epsilon}(\cdot, t)\} \leq C, \quad t \in [0, +\infty), \tag{17}$$

$$|u^{N,\epsilon}(x, t)|, \qquad |u_x^{N,\epsilon}(x, t)| \leq M, \quad (x,\ t) \in \mathbb{R} \times [0, +\infty) \tag{18}$$

and the following inequalities hold for all $t,\ s \in [0, +\infty)$

$$\int_{\mathbb{R}} |u^{N,\epsilon}(x, t) - u^{N,\epsilon}(x, s)|dx \leq L|t - s|; \quad \int_{\mathbb{R}} |u_x^{N,\epsilon}(x, t) - u_x^{N,\epsilon}(x, s)|dx \leq L|t - s|. \tag{19}$$

Consequently, by standard compactness argument (Theorem 2.4 [2]), there exist subsequences of $u^{N,\epsilon}$ and $u_x^{N,\epsilon}$ converging to some function u, u_x in $L^1_{loc}(\mathbb{R}\times[0, +\infty))$ as $\epsilon \to 0^+,\ N \to \infty$. The limit functions $u,\ u_x$ also satisfy (17)-(19). Finally, we obtain our main result.

Theorem 1.3. *Assume that the initial data $m_0 \in \mathcal{M}_+(\mathbb{R})$ has compact support and bounded mass M_0 and the numerical solution $(m^{N,\epsilon}, u^{N,\epsilon})$ of the vortex blob method is given by (11)-(12) with initial approximation*

$$m^{N,\epsilon}(x, 0) = \sum_{j=1}^{N} p_j^\epsilon(0)\rho_\epsilon(x - x_j^\epsilon(0))$$

where $p_j^\epsilon(0),\ x_j^\epsilon(0)$ are given by (8). Then, there exists subsequence of $u^{N,\epsilon}$ converging to a function u in $L^1_{loc}(\mathbb{R} \times [0, +\infty))$ as $\epsilon \to 0^+,\ N \to \infty$. This limit function u is the unique global weak solution of the b-equation (1)-(2) with the regularity

$$u(x, t) \in C^{(\frac{1}{p})}\ ([0, +\infty);\ W^{1,p}(\mathbb{R})), \quad \textit{for any } p \geq 1.$$

Furthermore, for any $T > 0$, we have

$$u(x, t) \in BV(\mathbb{R} \times [0, T)); \quad u_x \in BV(\mathbb{R} \times [0, T)),$$
$$m(x, t) = (1 - \alpha^2\partial_{xx})u(x, t)\ \in\ \mathcal{M}_+(\mathbb{R} \times [0, T))$$

and there exists subsequence of $m^{N,\epsilon}$ (also labelled as $m^{N,\epsilon}$) such that

$$m^{N,\epsilon}(x, t)\ \overset{*}{\rightharpoonup}\ m(x, t) \quad \textit{in} \quad \mathcal{M}_+(\mathbb{R} \times [0, T)) \quad (\textit{as} \quad \epsilon \to 0^+,\ N \to \infty).$$

2. **Uniform space-time BV estimates for $u^{N,\epsilon},\ u_x^{N,\epsilon}$.** For further analysis, we need more properties about the kernel G. It is easy to verify that the kernel $G(x) = \frac{1}{2\alpha}e^{-|x|/\alpha}$ satisfies

- $G(x)$ is even function and $G'(x)$ is odd function,
- $\|G\|_{L^\infty} = \frac{1}{2\alpha},\ \|G\|_{L^1} = \frac{1}{\alpha^2},\ \|G'\|_{L^\infty} = \frac{1}{2\alpha^2},\ \|G'\|_{L^1} = \frac{1}{\alpha^3}$,
- $\text{Tot.Var.}\{G\} = \frac{1}{\alpha},\ \ \text{Tot.Var.}\{G'\} = \frac{2}{\alpha^2}$.

For $G^\epsilon = \rho_\epsilon * G$, it can be verified directly that the following Lemma holds. We omit the proof here.

Lemma 2.1. *The following statements about $\rho_\epsilon * G$ hold:*

(1) $\rho_\epsilon * G$ *is even function and* $\partial_x(\rho_\epsilon * G)$ *is odd function,*

(2) $\rho_\epsilon * G,\ \partial_x(\rho_\epsilon * G) \in\ BV(\mathbb{R})$; *Tot.Var.*$\{\rho_\epsilon * G\} \le \frac{1}{\alpha}$, *Tot.Var.*$\{\partial_x(\rho_\epsilon * G)\} \le \frac{2}{\alpha^2}$,

(3) There exist constants $K_1,\ K_2$ *independent of* ϵ, *such that*

$$|G^\epsilon(a) - G^\epsilon(b)| \le K_1|a-b|, \quad \forall\ a,\ b; \tag{20}$$

$$|G_x^\epsilon(a) - G_x^\epsilon(b)| \le K_2|a-b|, \quad a\cdot b > 0,\ 0 < \epsilon < \min\{|a|,|b|\}. \tag{21}$$

For simplicity in notations, we omit the superscript N and denote $u^{N,\epsilon}(x,t)$, $m^{N,\epsilon}(x,t), m_\epsilon^N(x,t),\ u_\epsilon^N(x,t)$ as $u^\epsilon(x,t),\ m^\epsilon(x,t),\ m_\epsilon(x,t),\ u_\epsilon(x,t)$, respectively. The following Lemma provides an important invariant in our analysis.

Lemma 2.2. *Let* $p_i^\epsilon(t), x_i^\epsilon(t)$ *be the solution to (13)-(14). Then,* $\sum_{i=1}^N p_i^\epsilon(t)$ *is independent of time variable t, i.e.* $\frac{d}{dt}\sum_{i=1}^N p_i^\epsilon(t) = 0$.

Proof. since G_x^ϵ is odd function, a direct computation yields

$$\begin{aligned}\frac{d}{dt}\sum_{i=1}^N p_i^\epsilon(t) &= -\sum_{i=1}^N (b-1)u_x^\epsilon(x_i^\epsilon(t),t)p_i^\epsilon(t)\\ &= -(b-1)\sum_{i=1}^N\sum_{j=1}^N p_i^\epsilon(t)p_j^\epsilon(t)(G_x^\epsilon(x_i^\epsilon(t)-x_j^\epsilon(t)))\\ &= -(b-1)\sum_{i=1}^N (p_i^\epsilon(t))^2 G_x^\epsilon(0) = 0.\end{aligned}$$

This finishes the proof of this Lemma. □

Without any confusion, we simplify the notations $p_i^\epsilon(t),\ x_i^\epsilon(t)$ as $p_i(t),\ x_i(t)$, respectively. Then, it is clear that

$$\sum_{i=1}^N p_i(t) = \sum_{i=1}^N p_i(0) = \int_{\mathbb{R}} dm_0 = M_0.$$

Now, we prove Proposition 1.

Proof of Proposition 1. (1) In terms of space variable, it is easy to prove that the following estimates hold for any $t \ge 0,\ x \in \mathbb{R}$.

$$\text{Tot.Var.}\{u^\epsilon(\cdot.t)\} \le \sum_{j=1}^N p_j(t)\ \text{Tot.Var.}\{G^\epsilon(x)\} \le \sum_{j=1}^N p_j(0)\ \text{Tot.Var.}\{G(x)\} = \frac{1}{\alpha}M_0,$$

$$\text{Tot.Var.}\{u_x^\epsilon(\cdot.t)\} \le \sum_{j=1}^N p_j(t)\ \text{Tot.Var.}\{G_x^\epsilon(x)\} \le \sum_{j=1}^N p_j(0)\ \text{Tot.Var.}\{G_x(x)\} = \frac{2}{\alpha^2}M_0,$$

$$\|u^\epsilon(\cdot,t)\|_{L^1} \le M_0\|G\|_{L^1}, \quad \|u_x^\epsilon(\cdot,t)\|_{L^1} \le M_0\|G'\|_{L^1},$$

$$|u^\epsilon(x,t)| \le \sum_{i=1}^N p_i(t)\|G^\epsilon\|_{L^\infty} \le M_0\|G\|_{L^\infty};$$

$$|u_x^\epsilon(x,t)| \le \sum_{i=1}^N p_i(t)\|G_x^\epsilon\|_{L^\infty} \le M_0\|G'\|_{L^\infty}.$$

Therefore, the assertion (17) and (18) hold.

(2) For the estimate (19), we first prove that

$$\int_{\mathbb{R}} |u^\epsilon(x,t) - u^\epsilon(x,s)|dx \leq L|t-s|$$

holds for some constant L. Without any loss of generality, we assume that $s < t$. Then, a direct estimate yields

$$\begin{aligned}\int_{\mathbb{R}} |u^\epsilon(x,t) - u^\epsilon(x,s)|dx &\leq \int_{\mathbb{R}} \sum_{j=1}^{N} |p_j(t)G^\epsilon(x - x_j(t)) - p_j(s)G^\epsilon(x - x_j(s))|dx \\ &\leq \int_{\mathbb{R}} \sum_{j=1}^{N} p_j(t)|G^\epsilon(x - x_j(t)) - G^\epsilon(x - x_j(s))|dx \\ &+ \int_{\mathbb{R}} \sum_{j=1}^{N} |p_j(t) - p_j(s)|G^\epsilon(x - x_j(s))dx.\end{aligned}$$

Then, by using the results from (Lemma 2.3 [2]) and the fact that $G^\epsilon \in BV(\mathbb{R})$ and $x_j(t) - x_j(s) > 0$ (see Remark 1), one has

$$\begin{aligned}&\int_{\mathbb{R}} |u^\epsilon(x,t) - u^\epsilon(x,s)|dx \leq \\ &\quad \text{Tot.Var.}\{G^\epsilon(x)\} \sum_{j=1}^{N} p_j(t)(x_j(t) - x_j(s)) + \|G^\epsilon\|_{L^1} \sum_{j=1}^{N} |p_j(t) - p_j(s)|. \quad (22)\end{aligned}$$

It is clear that

$$\begin{aligned}\|G^\epsilon\|_{L^1} = \int_{\mathbb{R}}\int_{\mathbb{R}} \rho_\epsilon(x-y)G(y)dydx &= \int_{\mathbb{R}} G(y)dy \int_{\mathbb{R}} \rho_\epsilon(x-y)dx \\ &= \int_{\mathbb{R}} G(y)dy = \|G\|_{L^1}.\end{aligned}$$

Now, we estimate $x_j(t) - x_j(s)$ and $p_j(t) - p_j(s)$. According to (13), one has

$$\begin{aligned}x_j(t) - x_j(s) = \int_s^t \frac{dx_j(\tau)}{d\tau} d\tau &= \int_s^t u^\epsilon(x_j(\tau), \tau)d\tau \quad (23)\\ &\leq \|G^\epsilon\|_{L^\infty} \int_s^t \sum_{j=1}^{N} p_j(\tau)d\tau \leq M_0\|G\|_{L^\infty}(t-s).\end{aligned}$$

Similarly, according to (14), we also have

$$|p_j(t) - p_j(s)| = |\int_s^t \frac{dp_j(\tau)}{d\tau} d\tau| \leq (b-1)M_0^2\|G'\|_{L^\infty}(t-s). \quad (24)$$

Substituting (23) and (24) into (22) , we have

$$\begin{aligned}&\int_{\mathbb{R}} |u^\epsilon(x,t) - u^\epsilon(x,s)|dx \\ &\leq \left\{\text{Tot.Var.}\{G^\epsilon(x)\}\|G\|_{L^\infty} + (b-1)\|G'\|_{L^\infty}\|G^\epsilon\|_{L^1}\right\} M_0^2(t-s) \\ &\leq \left\{\frac{1}{2\alpha^2} + (b-1)\frac{1}{2\alpha^2}\|G\|_{L^1}\right\} M_0^2(t-s).\end{aligned}$$

Similarly, for u^ϵ_x, by noticing that $G^\epsilon_x \in BV(\mathbb{R})$, we also have

$$\int_{\mathbb{R}} |u^\epsilon_x(x,t) - u^\epsilon_x(x,s)| \le \int_{\mathbb{R}} \sum_{j=1}^{N} p_j(t)|G^\epsilon_x(x,t) - G^\epsilon_x(x,s)| + |G^\epsilon_x(x,s)||p_j(t) - p_j(s)|dx$$

$$\le \{\text{Tot.Var.}\{G^\epsilon_x(x)\}\|G^\epsilon_x\|_{L^\infty} + (b-1)\|G^\epsilon_x\|_{L^\infty}\|G^\epsilon_x\|_{L^1}\} M_0^2|t-s|$$

$$\le \left\{\frac{1}{\alpha^4} + (b-1)\frac{1}{2\alpha^2}\|G'\|_{L^1}\right\} M_0^2|t-s|.$$

This proves the assertion (19). The proof of Theorem 1 is finished. □

Remark 2. Based on this theorem, we can assert that there exist subsequences of u^ϵ, u^ϵ_x (also denoted as u^ϵ, u^ϵ_x) converging to some function u, u_x in $L^1_{loc}(\mathbb{R}\times[0,+\infty))$ (Theorem 2.4, [2]). The limit functions $u(x,t), u_x(x,t)$ also satisfy (17) -(19). In other words, according to (Theorem 2.6 [2]), for any $T>0$, we have

$$u(x,t) \in BV(\mathbb{R}\times[0,T)), \quad u_x(x,t) \in BV(\mathbb{R}\times[0,T))$$

and as $\epsilon \to 0^+,\ N\to\infty$

$$u^\epsilon \to u, \quad u^\epsilon_x \to u_x \text{ in } L^1_{loc}(\mathbb{R}\times[0,+\infty)). \tag{25}$$

3. **Weak consistency.** In this section, we show that u^ϵ defined by (12) is weak consistent with the b-equation (1)-(2). We first introduce the definition of weak solution of the b-equation in terms of u. To this end, for $u \in C((0,T);H^1(\mathbb{R}))$ and $\phi(x,t) \in C_0^\infty(\mathbb{R}\times[0,T))$, we denote the functional

$$\begin{aligned}\mathcal{L}(u,\phi) := &\int_0^T\int_{\mathbb{R}} u(x,t)[\phi_t(x,t) - \alpha^2\phi_{txx}(x,t)]dxdt\\ &+\int_0^T\int_{\mathbb{R}} u^2(x,t)[\frac{b+1}{2}\phi_x(x,t) - \frac{\alpha^2}{2}\phi_{xxx}(x,t)]dxdt\\ &-\int_0^T\int_{\mathbb{R}} \frac{\alpha^2(b-3)}{2}u_x^2(x,t)\phi_x(x,t)dxdt.\end{aligned}$$

Then, the definition of the weak solution to the b-equation in terms of $u(x,t)$ is given as follows.

Definition 3.1. A function $u \in C([0,T);H^1(\mathbb{R})), m(x,t) = u(x,t) - \alpha^2 u_{xx}(x,t)$ is said to be a weak solution of the b-equation (1)-(2) if

$$\mathcal{L}(u,\phi) = -\int_{\mathbb{R}} m(x,0)\phi(x,0)dx$$

holds for all $\phi(x,t) \in C_0^\infty(\mathbb{R}\times[0,T))$. If $T=+\infty$, we call $u(x,t)$ as a global weak solution of the b-equation.

According to this definition, we will prove the weak consistency of (u^ϵ, m^ϵ) with (1)-(2). For simplicity in notations, we denote

$$\langle f(x,t), g(x,t)\rangle := \int_0^T\int_{\mathbb{R}} f(x,t)g(x,t)dxdt$$

throughout the rest of this paper. With this notation, we find that the following statement holds for the pair (u^ϵ, m_ϵ).

Lemma 3.2. *Let u^ϵ, m_ϵ be defined by (12), (15), respectively. Then, the pair (u^ϵ, m_ϵ) satisfies*

$$\langle m_\epsilon, \phi_t\rangle + \langle u^\epsilon m_\epsilon, \phi_x\rangle - (b-1)\langle u^\epsilon_x m_\epsilon, \phi\rangle = -\sum_{j=1}^N p_j(0)\phi(x_j(0), 0).$$

Proof. For any test function $\phi(x,t) \in C_0^\infty(\mathbb{R}\times[0,T))$, according to (4) and (12), one has

$$\begin{aligned}
&\langle m_\epsilon, \phi_t\rangle + \langle u^\epsilon m_\epsilon, \phi_x\rangle - (b-1)\langle u^\epsilon_x m_\epsilon, \phi\rangle \\
&= \sum_{j=1}^N \int_0^T p_j(t)\partial_t\phi(x,t)|_{x=x_j(t)}dt + \sum_{j=1}^N \int_0^T p_j(t)u^\epsilon(x_j(t),t)\phi_x(x_j(t),t)dt \\
&- \sum_{j=1}^N (b-1)\int_0^T p_j(t)u^\epsilon_x(x_j(t),t)\phi(x_j(t),t)dt \\
&= \sum_{j=1}^N \int_0^T p_j(t)\left[\frac{d\phi(x_j(t),t)}{dt} - \phi_x(x_j(t),t)\frac{dx_j(t)}{dt}\right]dt \\
&+ \sum_{j=1}^N \int_0^T p_j(t)u^\epsilon(x_j(t),t)\phi_x(x_j(t),t)dt \\
&- \sum_{j=1}^N (b-1)\int_0^T p_j(t)u^\epsilon_x(x_j(t),t)\phi(x_j(t),t)dt.
\end{aligned}$$

Then, using integration by parts and the initial condition (9), we have

$$\begin{aligned}
&\langle m_\epsilon, \phi_t\rangle + \langle u^\epsilon m_\epsilon, \phi_x\rangle - (b-1)\langle u^\epsilon_x m_\epsilon, \phi\rangle \\
&= -\sum_{j=1}^N p_j(0)\phi(x_j(0),0) - \sum_{j=1}^N \int_0^T p_j(t)\phi_x(x_j(t),t)\left[\frac{dx_j(t)}{dt} - u^\epsilon(x_j(t),t)\right]dt \\
&- \sum_{j=1}^N \int_0^T \phi(x_j(t),t)\left[\frac{dp_j(t)}{dt} + (b-1)p_j(t)u^\epsilon_x(x_j(t),t)\right]dt.
\end{aligned}$$

According to (13)-(14), the assertion holds. □

To estimate $\mathcal{L}(u^\epsilon,\phi)$, we change its form into the representation in terms of u^ϵ and m^ϵ. To this end, we compute

$$\begin{aligned}
\mathcal{L}(u^\epsilon,\phi) = \langle u^\epsilon, \phi_t - \alpha^2\phi_{txx}\rangle + \langle (u^\epsilon)^2, \phi_x\rangle + \alpha^2\langle (u^\epsilon_x)^2, \phi_x\rangle - \frac{\alpha^2}{2}\langle (u^\epsilon)^2, \phi_{xxx}\rangle \\
+\frac{b-1}{2}\langle (u^\epsilon)^2, \phi_x\rangle - \frac{b-1}{2}\alpha^2\langle (u^\epsilon_x)^2, \phi_x\rangle.
\end{aligned}$$

Using integration by parts, one has

$$\begin{aligned}
\mathcal{L}(u^\epsilon,\phi) &= \langle u^\epsilon, \phi_t\rangle + \langle u^\epsilon, -\alpha^2\phi_{txx}\rangle + \langle (u^\epsilon)^2, \phi_x\rangle - \alpha^2\langle \partial_{xx}u^\epsilon, u^\epsilon\phi_x\rangle \\
&\quad -\frac{b-1}{2}\langle \partial_x(u^\epsilon)^2, \phi\rangle + \frac{b-1}{2}\alpha^2\langle \partial_x(u^\epsilon_x)^2, \phi\rangle. \\
&= \langle u^\epsilon, \phi_t\rangle - \alpha^2\langle u^\epsilon_{xx}, \phi_t\rangle + \langle (u^\epsilon)^2, \phi_x\rangle - \alpha^2\langle \partial_{xx}u^\epsilon, u^\epsilon\phi_x\rangle \\
&\quad -\frac{b-1}{2}\langle \partial_x(u^\epsilon)^2, \phi\rangle + \frac{b-1}{2}\alpha^2\langle \partial_x(u^\epsilon_x)^2, \phi\rangle. \qquad (26)
\end{aligned}$$

Substituting $m^\epsilon = (I - \alpha^2\partial_{xx})u^\epsilon$ into (26), one has

$$\mathcal{L}(u^\epsilon, \phi) = \langle m^\epsilon, \phi_t\rangle + \langle u^\epsilon m^\epsilon, \phi_x\rangle - (b-1)\langle u^\epsilon_x m^\epsilon, \phi\rangle. \tag{27}$$

Therefore, we arrive at

$$\begin{aligned}\mathcal{L}(u^\epsilon, \phi) + \int_{\mathbb{R}} m^\epsilon(x,0)\phi(x,0)dx &= \int_{\mathbb{R}} m^\epsilon(x,0)\phi(x,0)dx - \sum_{j=1}^N p_j(0)\phi(x_j(0),0) \\ &+\langle m^\epsilon - m_\epsilon, \phi_t\rangle + \langle u^\epsilon(m^\epsilon - m_\epsilon), \phi_x\rangle - (b-1)\langle u^\epsilon_x(m^\epsilon - m_\epsilon), \phi\rangle.\end{aligned} \tag{28}$$

We denote the RHS of (28) as E_ϕ.

Proposition 2. *Assuming that the mollifier satisfies the momentum condition* $\int_R \rho(x)x^\alpha dx = 0,\ 1 \le \alpha \le k,\ k \ge 1,$ *then we have*

$$|E_\phi| \le C\epsilon. \tag{29}$$

The constant C is a generic constant independent of N, ϵ.

Proof. The first term $\int_{\mathbb{R}} m^\epsilon(x,0)\phi(x,0)dx - \sum_{j=1}^N p_j(0)\phi(x_j(0),0)$ can be estimated directly as follows.

$$\begin{aligned}|\int_{\mathbb{R}} m^\epsilon(x,0)\phi(x,0)dx - \sum_{j=1}^N p_j(0)\phi(x_j(0),0)| &= |\int (m^\epsilon(x,0) - m_\epsilon(x,0))\phi(x,0)dx| \\ &\le \sum_{j=1}^N p_j(0)|\int \rho_\epsilon(x - x_j(0))\phi(x,0)dx - \phi(x_j(0),0)|.\end{aligned}$$

On the other hand, for any y, it is clear that the following estimate holds.

$$|\int \rho_\epsilon(x-y)\phi(x,0)dx - \phi(y,0)| \le C\epsilon^{k+1}. \tag{30}$$

Hence, we have

$$|\int_{\mathbb{R}} m^\epsilon(x,0)\phi(x,0)dx - \sum_{j=1}^N p_j(0)\phi(x_j(0),0)| \le CM_0\epsilon^{k+1}. \tag{31}$$

We now turn to estimating

$$\langle m^\epsilon - m_\epsilon, \phi_t\rangle + \langle u^\epsilon m^\epsilon - u^\epsilon m_\epsilon, \phi_x\rangle - (b-1)\langle u^\epsilon_x m^\epsilon - u^\epsilon_x m_\epsilon, \phi\rangle =: I_1 + I_2 + I_3.$$

For the term I_1, a direct computation shows that

$$|I_1| \le \int_0^T \sum_{j=1}^N p_j(t)|\int \phi_t(x,t)\rho_\epsilon(x - x_j)dx - \phi_t(x_j,t)|dt.$$

Then, using the estimate (30), one has

$$|I_1| \le C\int_0^T \sum_{j=1}^N p_j(t)\epsilon^{k+1}dt = CTM_0\epsilon^{k+1}. \tag{32}$$

For the second term I_2, we have

$$I_2 = \int_0^T \sum_{j=1}^N p_j(t) \left\{ \int u^\epsilon(x)\rho_\epsilon(x-x_j)\phi_x(x)dx - u^\epsilon(x_j)\phi_x(x_j) \right\} dt$$
$$= \int_0^T \sum_{j=1}^N p_j(t) \left\{ \int \left[u^\epsilon(x)\phi_x(x) - u^\epsilon(x_j)\phi_x(x_j)\right] \rho_\epsilon(x-x_j)dx \right\} dt$$
$$= \int_0^T \sum_{j=1}^N p_j(t) \left\{ \int \left[u^\epsilon(x)(\phi_x(x) - \phi_x(x_j))\right] \rho_\epsilon(x-x_j)dx \right\} dt$$
$$+ \int_0^T \sum_{j=1}^N p_j(t) \left\{ \int \left[(u^\epsilon(x) - u^\epsilon(x_j))\phi_x(x_j)\right] \rho_\epsilon(x-x_j)dx \right\} dt =: J_1 + J_2.$$

For J_1, one has

$$|J_1| \le CM_0T\|u^\epsilon\|_{L^\infty}\epsilon \le CM_0^2T\|G\|_{L^\infty}\epsilon.$$

For J_2, we first estimate $u^\epsilon(x) - u^\epsilon(x_j)$. Using the property (20) and noticing that $|x - x_j| \le \epsilon$, one has

$$|u^\epsilon(x) - u^\epsilon(x_j)| \le \sum_{k=1}^N p_k(t)|G^\epsilon(x-x_k) - G^\epsilon(x_j - x_k)| \le K_1M_0\epsilon.$$

Therefore, we have

$$|J_2| \le CTM_0^2\epsilon$$

and

$$|I_2| \le CTM_0^2\epsilon. \tag{33}$$

Finally, we estimate I_3. In the same way with that of the estimate of I_2, we split I_3 as follows (we omit the constant $-(b-1)$).

$$I_3 = \int_0^T \sum_{j=1}^N p_j(t) \left\{ \int \left[u_x^\epsilon(x)(\phi(x) - \phi(x_j))\right] \rho_\epsilon(x-x_j)dx \right\} dt$$
$$+ \int_0^T \sum_{j=1}^N p_j(t) \left\{ \int \left[(u_x^\epsilon(x) - u_x^\epsilon(x_j))\phi(x_j)\right] \rho_\epsilon(x-x_j)dx \right\} dt =: J_3 + J_4.$$

It is easy to prove that

$$|J_3| \le CM_0T\|u_x^\epsilon\|_{L^\infty}\epsilon \le CM_0^2T\|G'\|_{L^\infty}\epsilon. \tag{34}$$

For J_4, we use (12) and split it as the following two parts in order to use the property (21).

$$\begin{aligned}
J_4 &= \int_0^T \sum_{j,k=1}^N p_j(t)p_k(t) \int \left[G_x^\epsilon(x-x_k) - G_x^\epsilon(x_j-x_k)\right]\phi(x_j)\rho_\epsilon(x-x_j)dxdt \\
&= \int_0^T \sum_{j,k=1}^N p_j(t)p_k(t) \int \left[G_x^\epsilon(x+x_j-x_k) - G_x^\epsilon(x_j-x_k)\right]\phi(x_j)\rho_\epsilon(x)dxdt \\
&= \int_0^T \sum_{|x_j-x_k|>2\epsilon} p_j(t)p_k(t) \int \left[G_x^\epsilon(x+x_j-x_k) - G_x^\epsilon(x_j-x_k)\right]\phi(x_j)\rho_\epsilon(x)dxdt \\
&\quad + \int_0^T \sum_{|x_j-x_k|\le 2\epsilon} p_j(t)p_k(t) \int \left[G_x^\epsilon(x+x_j-x_k) - G_x^\epsilon(x_j-x_k)\right]\phi(x_j)\rho_\epsilon(x)dxdt \\
&=: J_5 + J_6.
\end{aligned}$$

For the estimate of J_5, since $|x_j - x_k| > 2\epsilon$ and $|x| < \epsilon$, we have $\epsilon < \min\{|x + x_j - x_k|,\ |x_j - x_k|\}$ and $(x + x_j - x_k)\cdot(x_j - x_k) > 0$. According to the property (21), one has

$$|J_5| \le CM_0^2 TK_2\epsilon. \tag{35}$$

For J_6, it is obvious that if $x_j = x_k$, then

$$\int \left[G_x^\epsilon(x+x_j-x_k) - G_x^\epsilon(x_j-x_k)\right]\phi(x_j)\rho_\epsilon(x)dx = 0$$

because that G_x^ϵ is odd function. On the other hand, a direct computation shows that

$$\begin{aligned}
&\sum_{0<|x_j-x_k|\le 2\epsilon} p_j(t)p_k(t) \int \left[G_x^\epsilon(x+x_j-x_k) - G_x^\epsilon(x_j-x_k)\right]\phi(x_j)\rho_\epsilon(x)dx \\
&\quad = \frac{1}{2}\sum_{0<|x_j-x_k|\le 2\epsilon} p_j(t)p_k(t)\Big\{ \int \left[G_x^\epsilon(x+x_j-x_k) - G_x^\epsilon(x_j-x_k)\right]\phi(x_j)\rho_\epsilon(x)dx \\
&\qquad\qquad + \int \left[G_x^\epsilon(x+x_k-x_j) - G_x^\epsilon(x_k-x_j)\right]\phi(x_k)\rho_\epsilon(x)dx\Big\}.
\end{aligned}$$

We denote $x_j - x_k = a$ and assume that $a > 0$ without any loss of generality. Then, for the above integral, we have

$$\begin{aligned}
&\int \left[G_x^\epsilon(x+a) - G_x^\epsilon(a)\right]\phi(x_j)\rho_\epsilon(x)dx + \int \left[G_x^\epsilon(x-a) - G_x^\epsilon(-a)\right]\phi(x_k)\rho_\epsilon(x)dx \\
&\qquad = \int \left[G_x^\epsilon(x+a) - G_x^\epsilon(a)\right](\phi(x_j)-\phi(x_k))\rho_\epsilon(x)dx \\
&\qquad\quad + \int \left[G_x^\epsilon(x-a) + G_x^\epsilon(x+a)\right]\phi(x_k)\rho_\epsilon(x)dx
\end{aligned}$$

where we have used the fact that G_x^ϵ is odd function. It is obvious that the first term of the RHS of the above identity is bounded $C\|G'\|_{L^\infty}\epsilon$. To estimate the second term, by using the fact that G_x^ϵ is odd function again, a direct computation shows that

$$\int [G_x^\epsilon(x-a) + G_x^\epsilon(x+a)]\,\rho_\epsilon(x)dx$$
$$= \int_0^\epsilon [G_x^\epsilon(-x-a) + G_x^\epsilon(-x+a) + G_x^\epsilon(x-a) + G_x^\epsilon(x+a)]\,\rho_\epsilon(x)dx = 0.$$

Therefore, we have

$$|J_6| \le CTM_0^2\epsilon. \tag{36}$$

Collecting the estimates (34)(35)(36), one has

$$|I_3| \le CTM_0^2\epsilon. \tag{37}$$

Combining (31)(32)(33)(37), we arrive at

$$|\mathcal{L}(u^\epsilon, \phi)| \le C\epsilon.$$

This completes the proof of the weak consistency of (u^ϵ, m^ϵ) with the b-equation (1)-(2). □

4. **Convergence analysis and proof of the main Theorem 1.3.**

With the weak consistency and compactness at hand, we now prove the convergence result in Theorem 1.3.

Proof of Theorem 1.3. We omit the superscript N of $u^{N,\epsilon},\ m^{N,\epsilon}$ in the proof. As it is shown in Proposition 1, there exists u which is the limit of some subsequence of u^ϵ in $L^1_{loc}(\mathbb{R}\times(0,+\infty))$(still denoted as u^ϵ for simplicity in notations). Then, the associated sequence u_x^ϵ also has a subsequence (also labelled as u_x^ϵ) converging to some function u_x in $L^1_{loc}(\mathbb{R}\times(0,+\infty))$. For these subsequences, according to (Theorem 2.4 [2]), one has

(i) As $\epsilon \to 0^+,\ N \to \infty$,

$$u^\epsilon \to u, \quad u_x^\epsilon \to u_x \text{ in } L^1_{loc}(\mathbb{R}\times[0,+\infty)). \tag{38}$$

(ii) For any $T > 0$, the limit functions $u,\ u_x$ satisfy (Theorem 2.6 [2])

$$u(x,t) \in BV(\mathbb{R}\times[0,T)), \quad u_x(x,t) \in BV(\mathbb{R}\times[0,T)).$$

(iii) u_x is the derivative of u with respect to x in the sense of distribution and $m^\epsilon = (1-\alpha^2\partial_{xx})u^\epsilon$ is also a subsequence of the original m^ϵ.

We split the proof as three parts and first prove that u is the unique global weak solution of the b-equation (1)-(2) in Step 1. Then, Step 2 is devoted to the proof of $u \in \mathrm{C}^{(\frac{1}{p})}([0,+\infty); W^{1,p}(\mathbb{R})),\ p \ge 1$. In the final Step, we prove that $m^\epsilon(x,t) \overset{*}{\rightharpoonup} m(x,t)$ in $\mathcal{M}_+(\mathbb{R}\times[0,T))$ for any $T > 0$.

Step 1. We prove that u is the unique global weak solution of (1)-(2). For any given test function $\phi \in C_0^\infty(\mathbb{R}\times[0,+\infty))$, we assume that $\mathrm{supp}\{\phi(x,t)\} \subset \mathbb{R}\times[0,T)$ for some $T > 0$. We recall that

$$\mathcal{L}(u^\epsilon, \phi) + \int_{\mathbb{R}} m^\epsilon(x,0)\phi(x,0)dx \ \to 0\ (\epsilon \to 0^+,\ N \to \infty)$$

holds for any function $\phi \in C_0^\infty(\mathbb{R}\times[0,T))$ and

$$\mathcal{L}(u^\epsilon, \phi) = \int_0^T \int_{\mathbb{R}} u^\epsilon(x,t)[\phi_t(x,t) - \alpha^2\phi_{txx}(x,t)]dxdt$$
$$+ \int_0^T \int_{\mathbb{R}} (u^\epsilon)^2(x,t)[\frac{b+1}{2}\phi_x(x,t) - \frac{\alpha^2}{2}\phi_{xxx}(x,t)]dxdt$$

$$-\int_0^T\int_{\mathbb{R}}\frac{\alpha^2(b-3)}{2}(u_x^\epsilon)^2(x,t)\phi_x(x,t)dxdt.$$

By using the fact that $u^\epsilon, u \in BV(\mathbb{R}\times[0,T))$ and (38), we have

$$\begin{aligned}&|\int_0^T\int_{\mathbb{R}}((u^\epsilon(x,t))^2-u^2(x,t))\phi(x,t)dxdt|\\&\quad=|\int_0^T\int_{\mathbb{R}}(u^\epsilon(x,t)-u(x,t))(u^\epsilon(x,t)+u(x,t))\phi(x,t)dxdt|\\&\quad\le\|\phi\|_{L^\infty}(\|u^\epsilon\|_{L^\infty}+\|u\|_{L^\infty})\int\int_{(x,t)\,\in\,\mathrm{supp}\{\phi\}}|u^\epsilon(x,t)-u(x,t)|dxdt\\&\quad\to 0\ (\epsilon\to 0^+,\ N\to\infty).\end{aligned}$$

Similarly, we also have

$$\int_0^T\int_{\mathbb{R}}((u_x^\epsilon(x,t))^2-u_x^2(x,t))\phi(x,t)dxdt\to 0\ (\epsilon\to 0^+,\ N\to\infty).$$

Therefore, as $\epsilon\to 0^+,\ N\to\infty$, we obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}}u^\epsilon(x,t)[\phi_t(x,t)-\alpha^2\phi_{txx}(x,t)]dxdt\\&\quad\to\int_0^T\int_{\mathbb{R}}u(x,t)[\phi_t(x,t)-\alpha^2\phi_{txx}(x,t)]dxdt,\\&\int_0^T\int_{\mathbb{R}}(u^\epsilon)^2(x,t)[\frac{b+1}{2}\phi_x(x,t)-\frac{\alpha^2}{2}\phi_{xxx}(x,t)]dxdt\\&\quad\to\int_0^T\int_{\mathbb{R}}u^2(x,t)[\frac{b+1}{2}\phi_x(x,t)-\frac{\alpha^2}{2}\phi_{xxx}(x,t)]dxdt,\\&\int_0^T\int_{\mathbb{R}}\frac{\alpha^2(b-3)}{2}(u_x^\epsilon)^2(x,t)\phi_x(x,t)dxdt\to\int_0^T\int_{\mathbb{R}}\frac{\alpha^2(b-3)}{2}u_x^2(x,t)\phi_x(x,t)dxdt.\end{aligned}$$

Finally we prove

$$\int_{\mathbb{R}}(m^\epsilon(x,0)-m_0(x))\phi(x,0)dx\to 0\quad(\epsilon\to 0^+,\ N\to\infty).$$

By using (9) and (29), we have

$$\begin{aligned}&|\int_{\mathbb{R}}\phi(x)dm_0-\int_{\mathbb{R}}\phi(x)dm_0^\epsilon|\\&\quad\le|\int_{\mathbb{R}}\phi(x)dm_0-\int_{\mathbb{R}}\phi(x)dm_0^N|+|\int_{\mathbb{R}}\phi(x)dm_0^N-\int_{\mathbb{R}}\phi(x)dm_0^\epsilon|\le C(\frac{1}{N}+\epsilon).\end{aligned}$$

The generic constants C is independent of $\epsilon,\ N$. Therefore, one has

$$\int_{\mathbb{R}}\phi(x,0)m^\epsilon(x,0)dx\to\int_{\mathbb{R}}\phi(x,0)m(x,0)dx\ (as\ \epsilon\to 0^+,\ N\to+\infty).$$

Collecting the limits above, we arrive at, for any test function $\phi\in C_0^\infty(\mathbb{R}\times[0,+\infty))$,

$$\begin{aligned}&\int_{\mathbb{R}}m(x,0)\phi(x,0)dx+\int_0^{+\infty}\int_{\mathbb{R}}u(x,t)[\phi_t(x,t)-\alpha^2\phi_{txx}(x,t)]dxdt\\&\quad+\int_0^{+\infty}\int_{\mathbb{R}}u^2(x,t)[\frac{b+1}{2}\phi_x(x,t)-\frac{\alpha^2}{2}\phi_{xxx}(x,t)]dxdt\end{aligned}$$

$$-\int_0^{+\infty}\int_{\mathbb{R}}\frac{\alpha^2(b-3)}{2}u_x^2(x,t)\phi_x(x,t)dxdt=0.$$

This proves that u is a global weak solution to the b-equation. It is clear that the uniqueness holds by rewriting (1) as the conservative form [8, 14]

$$u_t+uu_x+G^{'}*\left[\frac{b}{2}u^2+\frac{3-b}{2}u_x^2\right]=0.$$

Step 2. We prove that $u\in C^{(\frac{1}{p})}([0,+\infty);W^{1,p}(\mathbb{R})),\ p\ge 1$. According to Proposition 1 and (Theorem 2.4 [2]), u, u_x satisfy

$$\int_{\mathbb{R}}|u(x,t)-u(x,s)|dx\le L|t-s|,\ \int_{\mathbb{R}}|u_x(x,t)-u_x(x,s)|dx\le L|t-s|,\ t,\ s\in[0,+\infty).$$

Moreover, for any $s,\ t\in[0,+\infty)$, we have

$$\begin{aligned}&\big|\|u(\cdot,t)\|^p_{W^{1,p}(\mathbb{R})}-\|u(\cdot,s)\|^p_{W^{1,p}(\mathbb{R})}\big|\\ &\le\int_{\mathbb{R}}|u^p(x,t)-u^p(x,s)|+|u_x^p(x,t)-u_x^p(x,s)|dx\\ &\le C(p,\ \|u\|_{L^\infty})\|u(\cdot,t)-u(\cdot,s)\|_{L^1(\mathbb{R})}+C_1(p,\ \|u_x\|_{L^\infty})\|u_x(\cdot,t)-u_x(\cdot,s)\|_{L^1(\mathbb{R})}\\ &\le C|t-s|.\end{aligned}\tag{39}$$

Therefore, by using the inequality

$$|a^m-b^m|\le|a-b|^m,\quad 0<m\le1,$$

we have

$$\begin{aligned}\big|\|u(\cdot,t)\|_{W^{1,p}(\mathbb{R})}-\|u(\cdot,s)\|_{W^{1,p}(\mathbb{R})}\big|&=\big|\|u(\cdot,t)\|^{p\cdot\frac{1}{p}}_{W^{1,p}(\mathbb{R})}-\|u(\cdot,s)\|^{p\cdot\frac{1}{p}}_{W^{1,p}(\mathbb{R})}\big|\\ &\le\big|\|u(\cdot,t)\|^p_{W^{1,p}(\mathbb{R})}-\|u(\cdot,s)\|^p_{W^{1,p}(\mathbb{R})}\big|^{\frac{1}{p}}.\end{aligned}$$

Then, by using (39), we arrive at

$$\big|\|u(\cdot,t)\|_{W^{1,p}(\mathbb{R})}-\|u(\cdot,s)\|_{W^{1,p}(\mathbb{R})}\big|\le C|t-s|^{\frac{1}{p}}.$$

Thus, $u\in C^{(\frac{1}{p})}([0,+\infty);W^{1,p}(\mathbb{R}))$.

Step 3. To prove that $m^\epsilon(x,t)\overset{*}{\rightharpoonup}m(x,t)$ in $\mathcal{M}_+(\mathbb{R}\times[0,T))$, according to the definition of the BV functions with two variables (Chapter 5, [34] or the proof of Theorem 2.6 [2]), we know that

$$m(x,t)=(1-\alpha^2\partial_{xx})u(x,t)\in\mathcal{M}(\mathbb{R}\times[0,T)).$$

Furthermore, for any test function $\phi(x,t)\in C_0^1(\mathbb{R}\times[0,T))$ satisfying $\phi(x,t)\ge0$, we have

$$\int_0^T\int_{\mathbb{R}}\phi(x,t)dm^\epsilon=\sum_{j=1}^N\int_0^T\int_{\mathbb{R}}p_j(t)\rho_\epsilon(x-x_j(t))\phi(x,t)dxdt\ge0.$$

This shows that $m^\epsilon\in\mathcal{M}_+(\mathbb{R}\times[0,T))$.

Now we prove $m^\epsilon(x,t)\overset{*}{\rightharpoonup}m(x,t)$ in $\mathcal{M}_+(\mathbb{R}\times[0,T))$. For any test function $\phi(x,t)\in C_0^1(\mathbb{R}\times[0,T))$, integrating by parts and using the relationship $m^\epsilon=(1-\alpha^2\partial_{xx})u^\epsilon$, one has

$$\int_0^T\int_{\mathbb{R}}\phi(x,t)dm^\epsilon(x,t)=\int_0^T\int_{\mathbb{R}}u^\epsilon(x,t)\phi(x,t)dxdt-\alpha^2\int_0^T\int_{\mathbb{R}}u^\epsilon_{xx}(x,t)\phi(x,t)dxdt$$

$$= \int_0^T \int_{\mathbb{R}} u^\epsilon(x,t)\phi(x,t)dxdt + \alpha^2 \int_0^T \int_{\mathbb{R}} u_x^\epsilon(x,t)\phi_x(x,t)dxdt.$$

Taking $\epsilon \to 0^+$, $N \to \infty$, the RHS of the above equality converges to

$$\int_0^T \int_{\mathbb{R}} u(x,t)\phi(x,t)dxdt + \alpha^2 \int_0^T \int_{\mathbb{R}} u_x(x,t)\phi_x(x,t)dxdt = \int_0^T \int_{\mathbb{R}} \phi(x,t)dm(x,t).$$

Hence, $m^\epsilon(x,t) \overset{*}{\rightharpoonup} m(x,t)$ in $\mathcal{M}_+(\mathbb{R} \times [0,T))$. This ends the proof. □

Remark 3. If we fix N and let $\epsilon \to 0^+$, a limit can also be obtained to converge to the solution of the point vortex method. Combining the convergence result of this numerical solution [8], the convergence of the vortex blob method is also constructed.

Remark 4. If we fix ϵ and let $N \to \infty$, a limit can be obtained to converge to the original problem with mollified initial condition. The convergence result can also be established by comparing with A. Constantin's work [14].

5. **Conclusions.** In this paper, we proved the convergence of the vortex blob method for the b-equation. The motivation of using the vortex blob method is to overcome some drawbacks of the point vortex method applied to the b-equation, which needs to find some conservative quantities to ensure the existence of the global solution to the associated ODEs. In the vortex blob method, the resulted ODEs is regularized and hence has global solution by standard ODEs theory. Then, with weak consistency and BV estimates, we obtained the convergence result of the vortex blob method for the b-equation. Our analysis also provides a way to prove the existence of global weak solution for more general problems. To this extent, we have only provided a theoretical study of the convergence of the vortex method applied to the b-equation. In some future work, we will concentrate on numerical experiments to illustrate the performance of the vortex blob method applied to the b-equation and its application to the two-component C-H equation.

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