DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **22**, Number **2**, March **2017**

doi:10.3934/dcdsb.2017015

pp. **307–338**

UNIFORM L^∞ BOUNDEDNESS FOR A DEGENERATE PARABOLIC-PARABOLIC KELLER-SEGEL MODEL

Wenting Cong*

School of Mathematics
Jilin University
Changchun 130012, China
and
Department of Physics and Department of Mathematics
Duke University
Durham, NC 27708, USA

Jian-Guo Liu

Department of Physics and Department of Mathematics
Duke University
Durham, NC 27708, USA

(Communicated by Yuan Lou)

Abstract. This paper investigates the existence of a uniform in time L^∞ bounded weak entropy solution for the quasilinear parabolic-parabolic Keller-Segel model with the supercritical diffusion exponent $0 < m < 2 - \frac{2}{d}$ in the multi-dimensional space $\mathbb{R}^d$ under the condition that the $L^{\frac{d(2-m)}{2}}$ norm of initial data is smaller than a universal constant. Moreover, the weak entropy solution $u(x,t)$ satisfies mass conservation when $m > 1 - \frac{2}{d}$. We also prove the local existence of weak entropy solutions and a blow-up criterion for general $L^1 \cap L^\infty$ initial data.

1. **Introduction.** We study the following quasilinear parabolic-parabolic Keller-Segel model in $d \geq 3$:

$$\begin{cases} \partial_t u = \Delta u^m - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^d,\ t > 0, \\ \partial_t v = \Delta v - v + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = 0, & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where the diffusion exponent m is taken to be supercritical in this paper, i.e. $0 < m < 2 - \frac{2}{d}$.

The Keller-Segel model was firstly presented in 1970 to describe the chemotaxis of cellular slime molds [11][14]. $u(x,t)$ represents the cell density, and $v(x,t)$ represents the concentration of the chemical substance. In this model, cells are attracted by the chemical substance and also able to emit it. Without loss of generality, we suppose $v(x,0) = 0$ which is reasonable with the meaning that there is no chemical substance at the beginning, and then it is generated by cells.

2010 *Mathematics Subject Classification.* Primary: 35K65, 35K59, 92C17.

Key words and phrases. Chemotaxis, fast diffusion, critical space, semi-group theory, global existence.

The first author is supported by NSFC grant 11271154.

* Corresponding author: Wenting Cong.

For $1 < m < 2-\frac{2}{d}$, the associate free energy of problem (1) involves a conservative variational function u and a non-conservative variational function v,

$$\mathcal{F}\big(u(\cdot,t),v(\cdot,t)\big) = \frac{1}{m-1}\int_{\mathbb{R}^d} u^m \, dx - \int_{\mathbb{R}^d} uv \, dx + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla v|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^d} v^2 \, dx.$$

Model (1) can be recast into the following mixed conservative and non-conservative gradient flow

$$u_t = \nabla\cdot\left(u\nabla\frac{\delta\mathcal{F}}{\delta u}\right), \qquad v_t = -\frac{\delta\mathcal{F}}{\delta v}.$$

This mixed variational structure is known as the Le Châterlier Principle and it formally possesses the following entropy-dissipation equality

$$\frac{d}{dt}\mathcal{F}(t) + \int_{\mathbb{R}^d} u\left|\nabla\left(\frac{m}{m-1}u^{m-1}-v\right)\right|^2 dx + \int_{\mathbb{R}^d} |\partial_t v|^2 \, dx = 0.$$

In the original parabolic-parabolic Keller-Segel model ($m=1, d=2$), there exists a critical mass 8π for the initial data $u_0(x)$. If the initial mass $\int_{\mathbb{R}^2} u_0(x)dx = M < 8\pi$, there exists a global weak non-negative solution [5].

By a natural extension to the quasilinear parabolic-parabolic Keller-Segel model, the diffusion exponent m plays an important role. $0 < m < 1$ is called the fast diffusion and $m > 1$ is called the slow diffusion to describe the limiting behaviors of the diffusivity coefficient in the diffusion term $\Delta u^m = \nabla\cdot(mu^{m-1}\nabla u)$.

When $0 < m < 2-\frac{2}{d}$ which is called the supercritical case, the aggregation dominates the diffusion for the high density (large λ) which leads to the finite-time blow-up [3, 4, 9, 18], and the diffusion dominates the aggregation for the low density (small λ) which leads to the infinite-time spreading [1, 18, 20]. While $m > 2-\frac{2}{d}$ which is called the subcritical case, the aggregation dominates the diffusion for the low density (small λ) which prevents spreading, while the diffusion dominates the aggregation for the high density (large λ) which prevents blow-up [12, 19, 20].

The model (1) has been widely studied in the slow diffusion case. Sugiyama [19, 20] proved the global in time existence of weak solutions without any restriction on the size of the initial date for $m \geq 2$. Then Ishida and Yokota [12] improved the global existence result from $m \geq 2$ to $m > 2-\frac{2}{d}$. For the blow-up result in the slow diffusion case, Ishida and Yokota [13] proved that every radially symmetric energy solution with large negative initial energy blows up in either finite or infinite time when $1 \leq m < 2-\frac{2}{d}$. However, in the fast diffusion case, i.e. $0 < m < 1$, few work has been done for the parabolic-parabolic Keller-Segel model.

In the supercritical case $0 < m < 2-\frac{2}{d}$, there is an L^p space, where $p = \frac{d(2-m)}{2}$. The p is crucial when studying the existence and blow-up results of (1) and almost all the results are related to $\|u_0\|_{L^p(\mathbb{R}^d)}$. In fact, this critical L^p space is widely used in studying the parabolic-elliptic Keller-Segel models [1, 2, 20], especially $p = \frac{d}{2}$ for the original parabolic-parabolic Keller-Segel model ($m=1$) in $\mathbb{R}^d$ [7].

For $0 < m < 2-\frac{2}{d}$, if $\|u_0\|_{L^p(\mathbb{R}^d)} < C_{d,m}$, where $C_{d,m}$ is a universal constant depending on d and m, then we prove that there exists a global weak solution (u,v) with the properties that $u(x,t)$ preserves mass when $1-\frac{2}{d} < m < 2-\frac{2}{d}$, and extincts at a finite time when $0 < m < 1-\frac{2}{d}$. Furthermore, for $m > 1$, this weak solution is also a weak entropy solution satisfying energy inequality if the initial second moment is bounded and $u_0 \in L^m(\mathbb{R}^d)$. With the initial condition $u_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$, we can prove that the weak solution is bounded uniformly in time by using bootstrap iterative method(See [2], [16]). With no restriction of the

L^p norm on initial data, we prove the local existence of a weak entropy solution for $1 < m < 2 - \frac{2}{d}$. This result also provides a natural blow-up criterion that all $\|u\|_{L^q(\mathbb{R}^d)}$ blow up at exactly the same time for $q \in (p, +\infty)$.

The results concerning the finite-time blow-up for the solutions of the Keller-Segel model in multi-dimension have only been proved for its parabolic-elliptic type until Winkler made a breakthrough in [21] to introduce a new method in fully parabolic problem when $m = 1$. There is few paper containing the finite time blow-up result for the solutions when $m \neq 1$. This is still an open problem.

The paper is organized as follows. In Section 2, we define a weak solution and introduce some crucial inequalities about semigroup theory and some lemmas. In Section 3, we propose *a priori* estimates of a weak solution. In Section 4, we prove our main theorem about uniformly in time L^∞ bound of weak solutions using a bootstrap iterative method. In Section 5, we construct a regularized problem to prove the existence of a weak solution. Finally, in Section 6, we prove the local existence of weak entropy solutions and a blow-up criterion.

2. Preliminaries.

The generic constant will be denoted by C, even if it is different from line to line. At the beginning, we define a weak solution of (1).

Definition 2.1. (Weak solution) Let $u_0 \in L^1_+(\mathbb{R}^d)$ be the initial data and $T \in (0, \infty)$. Then (u, v) is a weak solution to (1) if it satisfies

(i) Regularity:

$$u \in L^\infty\left(0, T; L^1(\mathbb{R}^d)\right) \cap L^2\left(0, T; L^2(\mathbb{R}^d)\right), \ u^m \in L^1\left(0, T; L^1(\mathbb{R}^d)\right),$$

$$\partial_t u \in L^{\bar{p}}\left(0, T; W_{loc}^{-2, \frac{2(p+1)}{p+3}}(\mathbb{R}^d)\right), \quad \bar{p} = \min\left\{\frac{p+1}{m}, p+1\right\} > 1,$$

$$v \in L^\infty\left(0, T; H^1(\mathbb{R}^d)\right), \ \partial_t v \in L^2\left(0, T; W_{loc}^{-2,2}(\mathbb{R}^d)\right).$$

(ii) $\forall \ \psi(x) \in C_c^\infty(\mathbb{R}^d)$ and any $0 < t < \infty$,

$$\int_{\mathbb{R}^d} u(x,t)\psi(x)\ dx - \int_{\mathbb{R}^d} u_0(x)\psi(x)\ dx = \int_0^t \int_{\mathbb{R}^d} u^m(x,s)\Delta\psi(x)\ dxds + \int_0^t \int_{\mathbb{R}^d} u(x,s)\nabla v(x,s)\cdot\nabla\psi(x)\ dxds,$$

$$\int_{\mathbb{R}^d} v(x,t)\psi(x)\ dx = -\int_0^t \int_{\mathbb{R}^d} \nabla v(x,s)\cdot\nabla\psi(x)\ dxds - \int_0^t \int_{\mathbb{R}^d} v(x,s)\psi(x)\ dxds + \int_0^t \int_{\mathbb{R}^d} u(x,s)\psi(x)\ dxds.$$

We use semigroup theory in this paper. The following definition and estimates are standard(See [12, 17]). Consider the following Cauchy problem:

$$\begin{cases} \partial_t h = \Delta h - h + f, & x \in \mathbb{R}^d, t > 0, \\ h(x,0) = h_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{2}$$

Definition 2.2. Let $T > 0$, $p \geq 1$, $h_0 \in L^p(\mathbb{R}^d)$ and $f \in L^2\left(0, T; L^2(\mathbb{R}^d)\right)$. The function $h(x,t) \in C\left([0,T]; L^2(\mathbb{R}^d)\right)$ given by

$$h(x,t) = e^{-t}e^{t\Delta}h_0(x) + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}f(x,s)\ ds, \quad 0 \leq t \leq T, \tag{3}$$

is the unique *mild solution* of problem (2) on $[0,T]$. The heat semigroup operator $e^{t\Delta}$ is defined by

$$\left(e^{t\Delta}f\right)(x,t) := G(x,t) * f(x,t),$$

where $G(x,t)$ is the heat kernel by $G(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$.

Using Young's inequality of the convolution and property of Gamma function, we immediately obtain that

$$\left\|e^{t\Delta}f\right\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q(\mathbb{R}^d)},$$

$$\left\|\nabla e^{t\Delta}f\right\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|f\|_{L^q(\mathbb{R}^d)},$$

where C is a positive constant depending on p,q and d, for any $1 \leq q \leq p \leq +\infty$, $f \in L^q(\mathbb{R}^d)$ and all $t > 0$.

Let $1 \leq q \leq p \leq \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{d}$. Assuming $f \in L^\infty\left(0,\infty; L^q(\mathbb{R}^d)\right)$ and $h_0 \in L^p(\mathbb{R}^d)$, using two inequalities above and Bochner Theorem in [8, pp.650], we have for $t \in [0,\infty)$

$$\|h(\cdot,t)\|_{L^p} \leq \|h_0(\cdot)\|_{L^p} + C\cdot\Gamma\left(1-\left(\frac{1}{q}-\frac{1}{p}\right)\frac{d}{2}\right)\|f\|_{L^\infty(0,\infty;L^q)}, \tag{4}$$

$$\|\nabla h(\cdot,t)\|_{L^p} \leq Ct^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|h_0(\cdot)\|_{L^q} + C\cdot\Gamma\left(\frac{1}{2}-\left(\frac{1}{q}-\frac{1}{p}\right)\frac{d}{2}\right)\|f\|_{L^\infty(0,\infty;L^q)}, \tag{5}$$

where C is a positive constant depending on p, q and d.

Remark 1. It is well known that the mild solution defined above is also a weak solution. In fact, for any test function $\phi \in C_c^\infty\left([0,T)\times\mathbb{R}^d\right)$, multiply ϕ_t to both sides of (3) and integrate over $[0,T)\times\mathbb{R}^d$ to obtain

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d} h(x,t)\phi_t(x,t)\ dxdt = -\int_{\mathbb{R}^d} h_0(x)\phi(x,0)\ dx\\
&\quad -\int_0^T\int_{\mathbb{R}^d}\left[e^{-t}e^{t\Delta}h_0(x)\right]_t\phi(x,t)\ dxdt - \int_0^T\int_{\mathbb{R}^d} f(x,t)\phi(x,t)\ dxdt\\
&\quad -\int_0^T\int_{\mathbb{R}^d}\int_0^t\left[e^{-(t-s)}e^{(t-s)\Delta}f(x,s)\right]_t ds\phi(x,t)\ dxdt\\
&= -\int_{\mathbb{R}^d} h_0(x)\phi(x,0)\ dx - \int_0^T\int_{\mathbb{R}^d} f(x,t)\phi(x,t)\ dxdt\\
&\quad -\int_0^T\int_{\mathbb{R}^d}(\Delta-\mathrm{Id})h(x,t)\phi(x,t)\ dxdt\\
&= -\int_{\mathbb{R}^d} h_0(x)\phi(x,0)\ dx - \int_0^T\int_{\mathbb{R}^d} f(x,t)\phi(x,t)\ dxdt\\
&\quad +\int_0^T\int_{\mathbb{R}^d}\nabla h(x,t)\cdot\nabla\phi(x,t)\ dxdt + \int_0^T\int_{\mathbb{R}^d} h(x,t)\phi(x,t)\ dxdt,
\end{aligned} \tag{6}$$

where in the last equality, we use the regularity in (5).

Then recall the following well-known maximal L^p-regularity result for the heat kernel:

Lemma 2.3. *Let $1 < p < +\infty$ and $T > 0$. Then for each $f \in L^p\left(0,T;L^p(\mathbb{R}^d)\right)$, problem (2) has a unique solution $h(x,t)$ with $h_0(x) = 0$ in the $L^p\left(0,T;L^p(\mathbb{R}^d)\right)$ sense. Moreover, there exists a positive constant C_p such that*

$$\|\Delta h(x,t)\|_{L^p\left(0,T;L^p(\mathbb{R}^d)\right)} \le C_p\|f\|_{L^p\left(0,T;L^p(\mathbb{R}^d)\right)}, \tag{7}$$

for all $f \in L^p\big(0,T;L^p(\mathbb{R}^d)\big)$.

The lemma above is a special case of the famous maximal L^p-regularity Theorem which was proved by Hieber and Prüss in [10]. We can use the maximal L^p result in our paper since the space $\mathbb{R}^d$ and elliptical operator Δ satisfy the conditions of the Theorem 3.1 in [10], and we consider $v_0(x) = 0$. We also refer the readers to a thorough review on maximal L^p-regularity for parabolic equation [15].

The following four lemmas which are proved in [1] are useful for later estimations.

Lemma 2.4. *Let $0 < m \le 2 - \frac{2}{d}$, $p = \frac{d(2-m)}{2}$. Then for $q \ge p$*

$$\|u\|_{L^{q+1}(\mathbb{R}^d)}^{q+1} \le S_d^{-1}\left\|\nabla u^{\frac{m+q-1}{2}}\right\|_{L^2(\mathbb{R}^d)}^2 \|u\|_{L^p(\mathbb{R}^d)}^{2-m}, \tag{8}$$

where S_d is the sharp constant in Sobolev inequality for $d \ge 3$.
Moreover, for $q \ge r > p$, we have

$$\|u\|_{L^{q+1}}^{q+1} \le \frac{2mq}{C_q(m+q-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_{L^2}^2 + C(q,r,d)\left(\|u\|_{L^r}^r\right)^\delta, \tag{9}$$

where $\delta = 1 + \frac{1+q-r}{r-p} > 1$,

$$C(q,r,d) = \left[\frac{2mq\big(q-r+1+2(r-p)/d\big)}{S_d^{-1}C_q(q+m-1)^2(q-r+1)}\right]^{-\frac{d(q-r+1)}{2(r-p)}} \frac{2(r-p)}{d(q-r+1)+2(r-p)}.$$

Lemma 2.5. *Let $0 < m < 2 - \frac{2}{d}$, $p = \frac{d(2-m)}{2}$. Then for $q \ge p$ and $u \in L^1_+(\mathbb{R}^d)$, we have*

$$\left(\|u\|_{L^q(\mathbb{R}^d)}^q\right)^{1+\frac{m-1+\frac{2}{d}}{q-1}} \le S_d^{-1}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_{L^2(\mathbb{R}^d)}^2 \|u\|_{L^1(\mathbb{R}^d)}^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}. \tag{10}$$

Lemma 2.6. *Assume $y(t) \ge 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \le \gamma - \beta y(t)^a$ for $\gamma \ge 0, \beta > 0$ and $a > 0$. Then*

(i) for $a > 1$, $y(t)$ has the following hyper-contractive property:

$$y(t) \le \left(\frac{\gamma}{\beta}\right)^{\frac{1}{a}} + \left[\frac{1}{\beta(a-1)t}\right]^{\frac{1}{a-1}}, \quad t > 0,$$

(ii) for $a = 1$, $y(t)$ decays as

$$y(t) \le \frac{\gamma}{\beta} + y(0)e^{-\beta t},$$

(iii) for $a < 1$, $\gamma = 0$, $y(t)$ has the finite time extinction, which means that there exists a T_{ext} satisfying $0 < T_{ext} \le \frac{y^{1-a}(0)}{\beta(1-a)}$ such that $y(t) = 0$ for all $t > T_{ext}$.

Lemma 2.7. *Assume $f(t) \ge 0$ is a non-increasing function for $t > 0$, $y(t) \ge 0$ is a C^1 function for $t > 0$ and satisfies $y'(t) \le f(t) - \beta y(t)^a$ for some constants $a > 1$ and $\beta > 0$, then for any $t_0 > 0$ one has*

$$y(t) \le \left(\frac{f(t_0)}{\beta}\right)^{\frac{1}{a}} + \Big(\beta(a-1)(t-t_0)\Big)^{-\frac{1}{a-1}}, \quad \text{for } t > t_0.$$

With the additional condition that $y(0)$ is bounded, we have Lemma 2.8 which can be proved by contradiction arguments.

Lemma 2.8. *Assume $y(t) \geq 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \leq \gamma - \beta y(t)^a$ for $\gamma > 0$ and $\beta > 0$. If $y(0)$ is bounded, then*

$$y(t) \leq \max\left(y(0), \left(\frac{\gamma}{\beta}\right)^{\frac{1}{a}}\right), \quad t > 0,$$

for all $a > 0$.

3. *A priori* **estimates of weak solutions.** In this section, we prove Theorem 3.1 which is concerning *a priori* estimates of weak solutions for (1).

Theorem 3.1. *(A priori estimates) Let $d \geq 3$, $0 < m < 2 - \frac{2}{d}$ and $p = \frac{d(2-m)}{2}$. C_p is the positive constant in (7). Under the assumption that $u_0 \in L^1_+ \cap L^p(\mathbb{R}^d)$ and $\eta = C_{d,m}^{2-m} - \|u_0\|_{L^p(\mathbb{R}^d)}^{2-m} > 0$, where $C_{d,m}^{2-m} = \frac{4mp}{S_d^{-1}(m+p-1)^2 C_p}$ is a universal constant, let (u,v) be a non-negative weak solution of (1). Then $u \in L^\infty\left(\mathbb{R}_+; L^p(\mathbb{R}^d)\right)$, $u \in L^{p+1}\left(\mathbb{R}_+; L^{p+1}(\mathbb{R}^d)\right)$ and $\nabla u^{\frac{m+p-1}{2}} \in L^2\left(\mathbb{R}_+; L^2(\mathbb{R}^d)\right)$. Furthermore, the following a priori estimates hold true:*

(i) For $0 < m < 1 - \frac{2}{d}$, $\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)}$ has finite time extinction. The extinct time T_{ext} satisfies

$$0 < T_{ext} \leq T_0,$$

where T_0 depends on $d, m, \eta, \|u_0\|_{L^1(\mathbb{R}^d)}$ and $\|u_0\|_{L^p(\mathbb{R}^d)}$.

(ii) For $m = 1 - \frac{2}{d}$, $\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)}$ decays exponentially in time

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)} \leq \|u_0\|_{L^p(\mathbb{R}^d)} e^{-\frac{C_p(p-1)\eta}{p\|u_0\|_{L^1(\mathbb{R}^d)}^{1/(p-1)}} t}.$$

(iii) For $1 - \frac{2}{d} < m < 2 - \frac{2}{d}$, the solution $u(x,t)$ satisfies mass conservation and $\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)}$ decays in time

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)} \leq \frac{\|u_0\|_{L^p(\mathbb{R}^d)}}{\left[1 + C(d,m,\eta,\|u_0\|_{L^1},\|u_0\|_{L^p})t\right]^{\frac{p-1}{p(m-1+2/d)}}}.$$

And for any $1 \leq q \leq p$, $\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}$ decays in time

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leq \frac{\|u_0(\cdot)\|_{L^p(\mathbb{R}^d)}^{\frac{p(q-1)}{q(p-1)}} \|u_0(\cdot)\|_{L^1(\mathbb{R}^d)}^{\frac{p-q}{q(p-1)}}}{\left[1 + C(d,m,\eta,\|u_0\|_{L^1},\|u_0\|_{L^p})t\right]^{\frac{q-1}{q(m-1+2/d)}}}.$$

For any $p < q < \infty$, $u(x,t)$ has hyper-contractive property

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leq C\left(t^{-\frac{(p+\epsilon-1)(q-p+1)}{(m-1+2/d)\epsilon}\frac{q-1}{q+m-2+2/d}} + t^{-\frac{q-1}{m-1+2/d}}\right)^{\frac{1}{q}},$$

where ϵ satisfies $\frac{4m(p+\epsilon)}{S_d^{-1}(m+p+\epsilon-1)^2 C_{p+\epsilon}} - \|u_0\|_{L^p(\mathbb{R}^d)}^{2-m} \geq \frac{\eta}{2}$, and C is a constant depending on m, d, q, η and $\|u_0\|_{L^1(\mathbb{R}^d)}$.

Proof. **Step 1**. (L^p estimate for $0 < m < 2 - \frac{2}{d}$). Multiplying the first equation in model (1) by pu^{p-1} and integrating it over $\mathbb{R}^d$, we obtain

$$\frac{d}{dt}\|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} = -\frac{4mp(p-1)}{(m+p-1)^2}\left\|\nabla u^{\frac{m+p-1}{2}}(t)\right\|^2_{L^2(\mathbb{R}^d)} - (p-1)\int_{\mathbb{R}^d} u^p \Delta v \, dx. \quad (11)$$

Now we estimate the second term on the right hand side. Using Hölder's inequality, we have

$$\begin{aligned} -(p-1)\int_{\mathbb{R}^d} u^p \Delta v \; dx &\le (p-1)\int_{\mathbb{R}^d} u^p |\Delta v| \; dx \\ &\le (p-1)\|u(t)\|^p_{L^{p+1}(\mathbb{R}^d)}\|\Delta v(t)\|_{L^{p+1}(\mathbb{R}^d)}. \end{aligned} \quad (12)$$

Define

$$I(t) := (p-1)\|u(t)\|^p_{L^{p+1}(\mathbb{R}^d)}\|\Delta v(t)\|_{L^{p+1}(\mathbb{R}^d)}.$$

Then (11) turns to

$$\frac{d}{dt}\|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \le -\frac{4mp(p-1)}{(m+p-1)^2}\left\|\nabla u^{\frac{m+p-1}{2}}(t)\right\|^2_{L^2(\mathbb{R}^d)} + I(t). \quad (13)$$

Integrating (13) from 0 to t, it follows that

$$\begin{aligned} \|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \le &\|u_0\|^p_{L^p(\mathbb{R}^d)} - \frac{4mp(p-1)}{(m+p-1)^2}\int_0^t \left\|\nabla u^{\frac{m+p-1}{2}}(s)\right\|^2_{L^2(\mathbb{R}^d)} ds \\ &+ \int_0^t I(s) \; ds. \end{aligned} \quad (14)$$

Next, using Hölder's inequality and Lemma 2.3, we obtain

$$\begin{aligned} \int_0^t I(s) \; ds &\le (p-1)\left(\int_0^t \|u(s)\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} \; ds\right)^{\frac{p}{p+1}} \left(\int_0^t \|\Delta v(s)\|^{p+1}_{L^{p+1}(R^d)} \; ds\right)^{\frac{1}{p+1}} \\ &\le C_p(p-1)\int_0^t \|u(s)\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} \; ds, \end{aligned} \quad (15)$$

where C_p is the constant in Lemma 2.3. Substituting (15) into (14), we see that

$$\begin{aligned} \|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \le \|u_0\|^p_{L^p(\mathbb{R}^d)} &- \frac{4mp(p-1)}{(m+p-1)^2}\int_0^t \left\|\nabla u^{\frac{m+p-1}{2}}(s)\right\|^2_{L^2(\mathbb{R}^d)} ds \\ &+ C_p(p-1)\int_0^t \|u(s)\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} \; ds. \end{aligned} \quad (16)$$

From Lemma 2.4 with $q = p$, then (16) turns to

$$\begin{aligned} &\|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \le \|u_0\|^p_{L^p(\mathbb{R}^d)} \\ &\qquad - S_d^{-1}(p-1)C_p\int_0^t \left(C^{2-m}_{d,m} - \|u(s)\|^{2-m}_{L^p}\right)\left\|\nabla u^{\frac{m+p-1}{2}}\right\|^2_{L^2} ds, \end{aligned} \quad (17)$$

where

$$C^{2-m}_{d,m} = \frac{4mp}{S_d^{-1}(m+p-1)^2 C_p}. \quad (18)$$

By contradiction arguments, we can prove that for all $t > 0$,

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)} < \|u_0\|_{L^p(\mathbb{R}^d)} < C_{d,m}. \quad (19)$$

Therefore, combining (17) and (19), we obtain

$$\|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} + \frac{\eta(p-1)C_p}{S_d}\int_0^t \left\|\nabla u^{\frac{m+p-1}{2}}(s)\right\|^2_{L^2(\mathbb{R}^d)} ds \le \|u_0\|^p_{L^p(\mathbb{R}^d)} < C_{d,m},$$

i.e.

$$u \in L^\infty\left(\mathbb{R}_+; L^p(\mathbb{R}^d)\right),\ \nabla u^{\frac{m+p-1}{2}} \in L^2\left(\mathbb{R}_+; L^2(\mathbb{R}^d)\right).$$

In the same time, from Lemma 2.4, we have

$$u \in L^{p+1}\left(\mathbb{R}_+; L^{p+1}(\mathbb{R}^d)\right).$$

Step 2. (L^p decay estimates). From the fact $\|u(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)}$ and Lemma 2.5 with $q=p$, we have

$$\left\|\nabla u^{\frac{p+m-1}{2}}\right\|^2_{L^2(\mathbb{R}^d)} \ge \frac{\left(\|u\|^p_{L^p(\mathbb{R}^d)}\right)^{1+\frac{m-1+\frac{2}{d}}{p-1}}}{S_d^{-1}\|u_0\|^{\frac{1}{p-1}}_{L^1(\mathbb{R}^d)}}. \tag{20}$$

Substituting (20) into (17), we see that

$$\|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \le \|u_0\|^p_{L^p} - \frac{C_p(p-1)\eta}{\|u_0\|^{\frac{1}{p-1}}_{L^1(\mathbb{R}^d)}}\int_0^t \left(\|u(s)\|^p_{L^p(\mathbb{R}^d)}\right)^{1+\frac{m-1+\frac{2}{d}}{p-1}} ds. \tag{21}$$

Define

$$y(t) = \|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} - \|u_0\|^p_{L^p(\mathbb{R}^d)} + \frac{C_p(p-1)\eta}{\|u_0\|^{\frac{1}{p-1}}_{L^1(\mathbb{R}^d)}}\int_0^t \left(\|u(s)\|^p_{L^p(\mathbb{R}^d)}\right)^{1+\frac{m-1+\frac{2}{d}}{p-1}} ds.$$

For any small $\epsilon_0 > 0$, we have

$$\begin{aligned} y(t+\epsilon_0) =& \|u(\cdot,t+\epsilon_0)\|^p_{L^p(\mathbb{R}^d)} - \|u_0\|^p_{L^p(\mathbb{R}^d)} \\ &+ \frac{C_p(p-1)\eta}{\|u_0\|^{\frac{1}{p-1}}_{L^1(\mathbb{R}^d)}}\int_0^{t+\epsilon_0} \left(\|u(s)\|^p_{L^p(\mathbb{R}^d)}\right)^{1+\frac{m-1+\frac{2}{d}}{p-1}} ds. \end{aligned}$$

Then from two equations above, we obtain that

$$\begin{aligned} y(t+\epsilon_0) - y(t) =& \|u(\cdot,t+\epsilon_0)\|^p_{L^p(\mathbb{R}^d)} - \|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \\ &+ \frac{C_p(p-1)\eta}{\|u_0\|^{\frac{1}{p-1}}_{L^1(\mathbb{R}^d)}}\int_t^{t+\epsilon_0} \left(\|u(s)\|^p_{L^p(\mathbb{R}^d)}\right)^{1+\frac{m-1+\frac{2}{d}}{p-1}} ds. \end{aligned} \tag{22}$$

In the similar way of obtaining (21), integrating from t to $t+\epsilon_0$ instead of integrating from 0 to t, we see that

$$\|u(\cdot,t+\epsilon_0)\|^p_{L^p} - \|u(\cdot,t)\|^p_{L^p} + \frac{C_p(p-1)\eta}{\|u_0\|^{\frac{1}{p-1}}_{L^1(\mathbb{R}^d)}}\int_t^{t+\epsilon_0} \left(\|u(s)\|^p_{L^p}\right)^{1+\frac{m-1+\frac{2}{d}}{p-1}} ds \le 0.$$

It means that $y(t)$ is a non-increasing function in time, i.e.

$$\frac{d}{dt}\|u(\cdot,t)\|^p_{L^p(\mathbb{R}^d)} \le -\frac{C_p(p-1)\eta}{\|u_0\|^{\frac{1}{p-1}}_{L^1(\mathbb{R}^d)}}\left(\|u(\cdot,t)\|^p_{L^p}\right)^{1+\frac{m-1+\frac{2}{d}}{p-1}}. \tag{23}$$

Then we have the conclusion that

(a) for $1-\frac{2}{d}<m<2-\frac{2}{d}$, $\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)}$ decays in time

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)} \le \frac{\|u_0\|_{L^p(\mathbb{R}^d)}}{\left[1+C(d,m,\eta,\|u_0\|_{L^1},\|u_0\|_{L^p})t\right]^{\frac{p-1}{p(m-1+2/d)}}}, \tag{24}$$

where $C(d,m,\eta,\|u_0\|_{L^1},\|u_0\|_{L^p}) = \frac{C_p\eta(m-1+2/d)\left(\|u_0\|_{L^p}^p\right)^{\frac{m-1+2/d}{p-1}}}{\|u_0\|_{L^1}^{\frac{1}{p-1}}}$,

(b) for $m=1-\frac{2}{d}$, $\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)}$ decays exponentially in time

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)} \le \|u_0\|_{L^p(\mathbb{R}^d)} e^{-\frac{C_p(p-1)\eta}{p\|u_0\|_{L^1(\mathbb{R}^d)}^{1/(p-1)}}t},$$

(c) for $0<m<1-\frac{2}{d}$, $\|u(t)\|_{L^p(\mathbb{R}^d)}$ has finite time extinction. The extinct time T_{ext} satisfies $0<T_{\text{ext}}\le T_0$, where $T_0 = \frac{\|u_0\|_{L^p(\mathbb{R}^d)}^{-\frac{p(m-1+2/d)}{p-1}} \|u_0\|_{L^1(\mathbb{R}^d)}^{1/(p-1)}}{-C_p\eta(m-1+2/d)}$.

Step 3. (Hyper-contractive estimate for any $p<q<\infty$ with $1-\frac{2}{d}<m<2-\frac{2}{d}$). **$L^r$ estimate with $r:=p+\epsilon$ for ϵ small enough.**

Since $C_{d,m}^{2-m}-\|u_0\|_{L^p(\mathbb{R}^d)}^{2-m}=\eta$, there exists $\epsilon>0$ such that

$$\frac{4m(p+\epsilon)}{S_d^{-1}(m+p+\epsilon-1)^2C_{p+\epsilon}} - \|u_0\|_{L^p(\mathbb{R}^d)}^{2-m} \ge \frac{\eta}{2}. \tag{25}$$

In the similar way of obtaining (23), we obtain

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r \le -\beta\left(\|u(t)\|_{L^r(\mathbb{R}^d)}^r\right)^{1+\frac{m-1+2/d}{r-1}}, \quad \beta = \frac{\eta(r-1)C_r}{2\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{1}{r-1}(1+2\epsilon/d)}}.$$

Since $1-\frac{2}{d}<m<2-\frac{2}{d}$, from Lemma 2.6, we have

$$\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r \le C(d,m,\eta,r,\|u_0\|_{L^1})t^{-\frac{r-1}{m-1+2/d}}. \tag{26}$$

Hyper-contractive estimates of L^q norm for $q\ge r$.

Combining (9) and (16) with $q=p$, we have

$$\begin{aligned}\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^q \le \|u_0\|_{L^q(\mathbb{R}^d)}^q &- \frac{2mq(q-1)}{(m+q-1)^2}\int_0^t \left\|\nabla u^{\frac{m+q-1}{2}}(s)\right\|_{L^2(\mathbb{R}^d)}^2 ds\\ &+ C(q,r,d)\int_0^t \left(\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)}^r\right)^\delta ds,\end{aligned} \tag{27}$$

where $\delta = 1+\frac{1+q-r}{r-p}$. Substituting (26) into (27), we obtain

$$\begin{aligned}\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^q \le &\|u_0\|_{L^q(\mathbb{R}^d)}^q - \frac{2mq(q-1)}{(m+q-1)^2}\int_0^t \left\|\nabla u^{\frac{m+q-1}{2}}(s)\right\|_{L^2(\mathbb{R}^d)}^2 ds\\ &+ C(d,m,\eta,q,\|u_0\|_{L^1})\int_0^t s^{-\frac{(r-1)\delta}{m-1+2/d}} ds.\end{aligned} \tag{28}$$

Then in the similar way of obtaining (23), (28) turns to

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^q}^q \le -\hat{\beta}\left(\|u\|_{L^q(\mathbb{R}^d)}^q\right)^{1+\frac{m-1+\frac{2}{d}}{q-1}} + C(d,m,\eta,q,\|u_0\|_{L^1})t^{-\frac{(r-1)(q-p+1)}{(m-1+2/d)(r-p)}}, \tag{29}$$

where $\hat{\beta} = \frac{2mq(q-1)}{(m+q-1)^2 S_d^{-1}\|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}}$. Using Lemma 2.7 and choosing $t_0 = \frac{t}{2}$, we obtain that for any $t > 0$

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^q \leq C\left(t^{-\frac{(p+\epsilon-1)(q-p+1)(q-1)}{\epsilon(m-1+2/d)(q+m-2+2/d)}} + t^{-\frac{q-1}{m-1+2/d}}\right), \tag{30}$$

where C is a constant depending on m, d, q, η and $\|u_0\|_{L^1(\mathbb{R}^d)}$, ϵ satisfies (25).

Step 4. (L^q decay estimate for any $1 \leq q \leq p$ with $1 - \frac{2}{d} < m < 2 - \frac{2}{d}$). For $1 - \frac{2}{d} < m < 2 - \frac{2}{d}$, by using interpolation inequality and (24), we obtain that for any $t > 0$,

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \leq \frac{\|u_0(\cdot)\|_{L^p(\mathbb{R}^d)}^{\frac{p(q-1)}{q(p-1)}} \|u_0(\cdot)\|_{L^1(\mathbb{R}^d)}^{\frac{p-q}{q(p-1)}}}{\left[1 + C(d,m,\eta,\|u_0\|_{L^1},\|u_0\|_{L^p})t\right]^{\frac{q-1}{q(m-1+2/d)}}}. \tag{31}$$

Step 5. (Mass conservation for $u(x,t)$ when $1 - \frac{2}{d} < m < 2 - \frac{2}{d}$).

We take a cut-off function $0 \leq \psi_1(x) \leq 1$, satisfying

$$\psi_1(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2, \end{cases}$$

where $\psi_1(x) \in C_c^\infty(\mathbb{R}^d)$.

Define $\psi_R(x) := \psi_1\left(\frac{x}{R}\right)$, then we know that $\lim_{R\to\infty} \psi_R(x) = 1$, $|\nabla\psi_R(x)| \leq \frac{C_1}{R}$ and $|\Delta\psi_R(x)| \leq \frac{C_2}{R^2}$ for $x \in \mathbb{R}^d$, where C_1 and C_2 are positive constants.

From the definition of weak solution for u and taking $\psi_R(x) \in C_c^\infty(\mathbb{R}^d)$ as test function, we have

$$\begin{aligned}\int_{\mathbb{R}^d} u(x,t)\psi_R(x)\,dx - \int_{\mathbb{R}^d} u_0(x)\psi_R(x)\,dx &= \int_0^t\int_{\mathbb{R}^d} u^m(x,s)\Delta\psi_R(x)\,dxds \\ &+ \int_0^t\int_{\mathbb{R}^d} u(x,s)\nabla v(x,s)\cdot\nabla\psi_R(x)\,dxds.\end{aligned} \tag{32}$$

For $1 - \frac{2}{d} < m < 1$, we can estimate the first term on RHS by using Hölder's inequality

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^d} u^m(x,s)\Delta\psi_R(x)\,dxds &\leq \frac{C_2}{R^2}\int_0^t\int_{B_{2R}} u^m(x,s)\,dxds \\ &\leq \frac{C}{R^{2-d(1-m)}}\int_0^t \|u(\cdot,s)\|_{L^1(B_{2R})}^m\,ds \\ &\leq \frac{C\left(\|u_0\|_{L^1(\mathbb{R}^d)}\right)}{R^{2-d(1-m)}}t.\end{aligned} \tag{33}$$

Using young's inequality, the second term on RHS of (32) goes to

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^d} u(x,s)\nabla v(x,s)\cdot\nabla\psi_R(x)\,dxds &\leq \frac{C_1}{R}\int_0^t\int_{B_{2R}} u(x,s)|\nabla v(x,s)|\,dxds \\ &\leq \frac{C}{R}\int_0^t\int_{B_{2R}} u^2(x,s)\,dxds + \frac{C}{R}\int_0^t\int_{B_{2R}} |\nabla v(x,s)|^2\,dxds.\end{aligned} \tag{34}$$

Recalling the second equation of (1) $v_t = \Delta v - v + u$, multiplying it by $-\Delta v$ and integrating from 0 to t and over $\mathbb{R}^d$, we have

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^d} |\nabla v(x,t)|^2 \, dx + \int_0^t \int_{\mathbb{R}^d} |\Delta v(x,s)|^2 \, dxds + \int_0^t \int_{\mathbb{R}^d} |\nabla v(x,s)|^2 \, dxds& \\
\le \int_0^t \int_{\mathbb{R}^d} |\Delta v(x,s)| u(x,s) \, dxds& \\
\le \int_0^t \int_{\mathbb{R}^d} |\Delta v(x,s)|^2 \, dxds + \int_0^t \int_{\mathbb{R}^d} u^2(x,s) \, dxds& \\
\le C \int_0^t \int_{\mathbb{R}^d} u^2(x,s) \, dxds,& \qquad (35)
\end{aligned}$$

where the last inequality can be obtained from (7).

From (34) and (35), by using interpolation inequality, Hölder's inequality and $u \in L^{p+1}\left(\mathbb{R}_+; L^{p+1}(\mathbb{R}^d)\right)$, we have

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} u(x,s)\nabla v(x,s)\cdot\nabla\psi_R(x) \, dxds &\le \frac{C}{R}\int_0^t \int_{B_{2R}} u^2(x,s) \, dxds \\
&\le \frac{C}{R}\int_0^t \|u(\cdot,s)\|_{L^1(B_{2R})}^{\frac{p-1}{p}} \|u(\cdot,s)\|_{L^{p+1}(B_{2R})}^{\frac{p+1}{p}} \, ds \\
&\le \frac{C\left(p, \|u_0\|_{L^1(\mathbb{R}^d)}\right)}{R}\left(\int_0^t \|u(\cdot,s)\|_{L^{p+1}(B_{2R})}^{p+1} \, ds\right)^{\frac{1}{p}} t^{\frac{p-1}{p}} \\
&\le \frac{C\left(p, \|u_0\|_{L^1(\mathbb{R}^d)}\right)}{R} t^{\frac{p-1}{p}}. \qquad (36)
\end{aligned}$$

Therefore, collecting (32), (33) and (36) together, it shows that

$$\left| \int_{\mathbb{R}^d} u(x,t)\psi_R(x) \, dx - \int_{\mathbb{R}^d} u_0(x)\psi_R(x) \, dx \right| \le \frac{C\left(\|u_0\|_{L^1}\right)}{R^{2-d(1-m)}} t + \frac{C\left(p, \|u_0\|_{L^1}\right)}{R} t^{\frac{p-1}{p}}.$$

Since $2 - d(1-m) > 0$ from $1 - \frac{2}{d} < m < 1$, we have

$$\int_{\mathbb{R}^d} u(x,t)dx = \int_{\mathbb{R}^d} u_0(x)dx, \text{ as } R \to \infty,$$

by the dominated convergence theorem.

For $1 \le m < 2 - \frac{2}{d}$, also using interpolation inequality and Hölder's inequality, we have the following estimate

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} u^m(x,s)\Delta\psi_R(x) \, dxds &\le \frac{C_2}{R^2}\int_0^t \int_{B_{2R}} u^m(x,s) \, dxds \\
&\le \frac{C_2}{R^2}\int_0^t \|u(\cdot,s)\|_{L^1(B_{2R})}^{\frac{p-m+1}{p}} \|u(\cdot,s)\|_{L^{p+1}(B_{2R})}^{\frac{(m-1)(p+1)}{p}} \, ds \\
&\le \frac{C\left(\|u_0\|_{L^1(\mathbb{R}^d)}\right)}{R^2}\left(\int_0^t \|u(\cdot,s)\|_{L^{p+1}(B_{2R})}^{p+1} \, ds\right)^{\frac{m-1}{p}} \left(\int_0^t 1 \, ds\right)^{\frac{p-m+1}{p}} \\
&\le \frac{C\left(m, p, \|u_0\|_{L^1(\mathbb{R}^d)}\right)}{R^2} t^{\frac{p-m+1}{p}}. \qquad (37)
\end{aligned}$$

Then from (36) and (37), we have

$$\left| \int_{\mathbb{R}^d} u(x,t)\psi_R(x)\ dx - \int_{\mathbb{R}^d} u_0(x)\psi_R(x)\ dx \right| \le \frac{C\left(p, \|u_0\|_{L^1(\mathbb{R}^d)}\right)}{R} t^{\frac{p-1}{p}} + \frac{C\left(m, p, \|u_0\|_{L^1(\mathbb{R}^d)}\right)}{R^2} t^{\frac{p-m+1}{p}}, \quad (38)$$

i.e. $\int_{\mathbb{R}^d} u(x,t)\ dx = \int_{\mathbb{R}^d} u_0(x)\ dx$, as $R \to \infty$. Therefore, for $1 - \frac{2}{d} < m < 2 - \frac{2}{d}$, we have mass conservation for u. □

4. **The uniformly in time L^∞ estimate of weak solutions.**

In this section, we prove our main theorem about uniformly in time L^∞ boundness of weak solution by using a bootstrap iterative method.

At the beginning of this section, we prove the following proposition concerning L^q norm estimates of the weak solution for $1 < q < \infty$.

Proposition 1. *Let $d \ge 3$, $0 < m < 2-\frac{2}{d}$ and $p = \frac{d(2-m)}{2}$. If $u_0 \in L^1_+(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for $1 < q < \infty$ and $\eta = C^{2-m}_{d,m} - \|u_0\|^{2-m}_{L^p(\mathbb{R}^d)} > 0$, where $C^{2-m}_{d,m} = \frac{4mp}{S_d^{-1}(m+p-1)^2 C_p}$ is a universal constant, let (u,v) be a non-negative weak solution of (1). Then $u(x,t)$ satisfies for any $t > 0$*

$$\|u(\cdot,t)\|^q_{L^q(\mathbb{R}^d)} \le C(p,q,\|u_0\|_{L^1})\|u_0\|^{\frac{p(q-1)}{p-1}}_{L^p(\mathbb{R}^d)}, \quad 1 < q \le p, \quad (39)$$

where C depends on p, q and $|u_0\|_{L^1(\mathbb{R}^d)}$,

$$\|u(\cdot,t)\|^q_{L^q(\mathbb{R}^d)} \le C^q_u, \quad p < q < \infty, \quad (40)$$

where C^q_u is a constant depending on $d, m, q, \|u_0\|_{L^1(\mathbb{R}^d)}$ and $\|u_0\|_{L^q(\mathbb{R}^d)}$, ϵ satisfies $\frac{4m(p+\epsilon)}{S_d^{-1}(m+p+\epsilon-1)^2 C_{p+\epsilon}} - \|u_0\|^{2-m}_{L^p(\mathbb{R}^d)} \ge \frac{\eta}{2}$. Furthermore, for any $t > 0$

$$\|v(\cdot,t)\|_{W^{1,\infty}(\mathbb{R}^d)} \le C^\infty_v, \quad (41)$$

where C^∞_v is a positive constant depending on C^{d+1}_u.

Proof. Actually, the proof of Proposition 1 is almost the same as the proof of Theorem 3.1, except for the different initial condition $u_0 \in L^1_+(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ for $1 < q < \infty$. Step 1 is L^q estimate for $u(x,t)$ and Step 2 is the uniform estimate for $v(x,t)$. We omit some details which are similar to the proof of Proposition 1 in [2].

Step 1. (L^q estimate for $u(x,t)$)We have obtained the uniform L^p estimate for $0 < m < 2 - \frac{2}{d}$ in (19)

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^d)} < \|u_0\|_{L^p(\mathbb{R}^d)} < C_{d,m}, \text{ for all } t > 0.$$

Then for $1 < q \le p$, using interpolation inequality, we have

$$\|u(\cdot,t)\|^q_{L^q(\mathbb{R}^d)} \le \|u_0\|^{\frac{p-q}{p-1}}_{L^1(\mathbb{R}^d)} \|u_0\|^{\frac{p(q-1)}{p-1}}_{L^p(\mathbb{R}^d)}, \quad (42)$$

which is (39) by taking $C(p,q,\|u_0\|_{L^1}) = \|u_0\|^{\frac{p-q}{p-1}}_{L^1(\mathbb{R}^d)}$. For $p < r \le q$, it is not hard to see that $\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} \le \|u_0\|_{L^r(\mathbb{R}^d)}$ for any $t > 0$. By the similar way of obtaining (29), we have

$$\frac{d}{dt}\|u(\cdot,t)\|^q_{L^q(\mathbb{R}^d)} \le -\tilde{\beta}\left(\|u\|^q_{L^q(\mathbb{R}^d)}\right)^{1+\frac{m-1+\frac{2}{d}}{q-1}} + C(q,r,d)\left(\|u_0\|^r_{L^r(\mathbb{R}^d)}\right)^\delta, \quad (43)$$

where $\delta = 1 + \frac{1+q-r}{r-p}$ and $\tilde{\beta} = \frac{2mq(q-1)}{(m+q-1)^2 S_d^{-1} \|u_0\|_{L^1(\mathbb{R}^d)}^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}}$. Using Lemma 2.8 and interpolation inequality, we can obtain

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^q \le \max\left\{\|u_0\|_{L^q(\mathbb{R}^d)}^q, C(d,m,q,\|u_0\|_{L^1})\left(\|u_0\|_{L^q(\mathbb{R}^d)}^q\right)^{\frac{(p+\epsilon-1)(q-p+1)}{\epsilon(q+m-2+2/d)}}\right\}$$
$$:= C_u^q,$$

where ϵ satisfies $\frac{4m(p+\epsilon)}{S_d^{-1}(m+p+\epsilon-1)^2 C_{p+\epsilon}} - \|u_0\|_{L^p(\mathbb{R}^d)}^{2-m} \ge \frac{\eta}{2}$.

Step 2. (Uniform $W^{1,\infty}$ estimate for $v(x,t)$). From (4) and (5) with $v_0(x) = 0$, choosing $p = \infty$ and $q = d+1$ to satisfy $1 \le q \le p \le \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{d}$, we obtain for any $t > 0$

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C(d)\|u\|_{L^\infty(0,\infty;L^{d+1}(\mathbb{R}^d))} \le C(d)C_u^{d+1},$$

$$\|\nabla v(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C(d)\|u\|_{L^\infty(0,\infty;L^{d+1}(\mathbb{R}^d))} \le C(d)C_u^{d+1},$$

i.e.

$$\|v(\cdot,t)\|_{W^{1,\infty}(\mathbb{R}^d)} \le C(d)C_u^{d+1} := C_v^\infty.$$

□

Next, we will prove the uniformly in time L^∞ boundness of $u(x,t)$ by using a bootstrap iterative technique [2, 16] with Proposition 1 and an additional initial condition $u_0 \in L^\infty(\mathbb{R}^d)$.

Theorem 4.1. *Let $d \ge 3$, $0 < m < 2-\frac{2}{d}$ and $p = \frac{d(2-m)}{2}$. If $u_0 \in L_+^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\eta = C_{d,m}^{2-m} - \|u_0\|_{L^p(\mathbb{R}^d)}^{2-m} > 0$, where $C_{d,m}^{2-m} = \frac{4mp}{S_d^{-1}(m+p-1)^2 C_p}$ is a universal constant, suppose (u,v) be a non-negative weak solution of (1). Then for any $t > 0$,*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C(m,d,K_0),$$

where $K_0 = \max\left\{1, \|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}\right\}$.

Proof. **Step 1**. (The L^{q_k} estimate). We denote

$$q_k = 3^k + \frac{d(2-m)}{2} + 1, \text{ for } k \ge 1.$$

Multiplying the first equation in (1) by $q_k u^{q_k-1}$ and integrating, we have

$$\begin{aligned}\frac{d}{dt}\|u(\cdot,t)\|_{L^{q_k}}^{q_k} &= -\frac{4mq_k(q_k-1)}{(m+q_k-1)^2}\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2}^2 + q_k(q_k-1)\int_{\mathbb{R}^d} u^{q_k-1}\nabla u\cdot\nabla v\, dx \\ &\le -\frac{4mq_k(q_k-1)}{(m+q_k-1)^2}\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2}^2 + q_k(q_k-1)C_v^\infty\int_{\mathbb{R}^d} u^{q_k-1}|\nabla u|\, dx, \qquad (44)\end{aligned}$$

where the inequality holds from (41). By using Young's inequality and interpolation inequality, we obtain

$$\begin{aligned}q_k(q_k-1)C_v^\infty\int_{\mathbb{R}^d} u^{q_k-1}|\nabla u|\, dx &= \frac{2q_k(q_k-1)C_v^\infty}{q_k+m-1}\int_{\mathbb{R}^d} u^{\frac{q_k-m+1}{2}}\left|\nabla u^{\frac{q_k+m-1}{2}}\right| dx \\ &\le \frac{2mq_k(q_k-1)}{(m+q_k-1)^2}\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2}^2 + \frac{q_k(q_k-1)(C_v^\infty)^2}{2m}\int_{\mathbb{R}^d} u^{q_k-m+1}\, dx\end{aligned}$$

$$\leq \frac{2mq_k(q_k-1)}{(m+q_k-1)^2}\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2}^2 + \frac{q_k(q_k-1)(C_v^\infty)^2}{2m}\|u_0\|_{L^1}^{\frac{m}{q_k}}\|u\|_{L^{q_k+1}(\mathbb{R}^d)}^{(q_k+1)\frac{q_k-m}{q_k}}$$
$$\leq \frac{2mq_k(q_k-1)}{(m+q_k-1)^2}\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2}^2 + \frac{(q_k-1)(C_v^\infty)^2}{2}\|u_0\|_{L^1}$$
$$+ \frac{(q_k-m)(q_k-1)(C_v^\infty)^2}{2m}\|u\|_{L^{q_k+1}(\mathbb{R}^d)}^{q_k+1}, \tag{45}$$

where inequalities hold since $1 < q_k - m + 1 < q_k + 1$ and $q_k > m$. Then substituting (45) into (44) yields to

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^{q_k}(\mathbb{R}^d)}^{q_k} \leq -2C_1\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2}^2 + \frac{(q_k-1)(C_v^\infty)^2}{2}\|u_0\|_{L^1(\mathbb{R}^d)}$$
$$+ \frac{(q_k-m)(q_k-1)(C_v^\infty)^2}{2m}\|u\|_{L^{q_k+1}(\mathbb{R}^d)}^{q_k+1}, \tag{46}$$

where $0 < C_1 \leq \frac{mq_k(q_k-1)}{(m+q_k-1)^2}$ is a fixed constant since $\frac{mq_k(q_k-1)}{(m+q_k-1)^2} \to m$ as $k \to \infty$. In order to change the form of (46) into what we want, firstly we try to estimate $\|u(\cdot,t)\|_{L^{q_k+1}}^{q_k+1}$ by using interpolation inequality and Sobolev inequality,

$$\|u(\cdot,t)\|_{L^{q_k+1}(\mathbb{R}^d)}^{q_k+1} \leq \|u(\cdot,t)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{(q_k+1)\theta}\|u(\cdot,t)\|_{L^{\frac{(m+q_k-1)d}{d-2}}(\mathbb{R}^d)}^{(q_k+1)(1-\theta)}$$
$$\leq S_d^{-\frac{(q_k+1)(1-\theta)}{m+q_k-1}}\|u(\cdot,t)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{(q_k+1)\theta}\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2(\mathbb{R}^d)}^{\frac{2(q_k+1)(1-\theta)}{m+q_k-1}}, \tag{47}$$

where

$$\theta = \frac{q_{k-1}(2q_k + md - 2d + 2)}{(q_k+1)[(m+q_k-1)d - q_{k-1}(d-2)]},$$

$$1-\theta = \frac{d(q_k - q_{k-1} + 1)(m+q_k-1)}{(q_k+1)[(m+q_k-1)d - q_{k-1}(d-2)]}.$$

We can see that $\frac{(q_k+1)(1-\theta)}{m+q_k-1} = \frac{d(q_k-q_{k-1}+1)}{d(q_k-q_{k-1}+1)+2q_{k-1}+md-2d} < 1$ since $q_{k-1} > \frac{d(2-m)}{2}$. Then using Young's inequality, we obtain

$$\frac{(q_k-m)(q_k-1)(C_v^\infty)^2}{2m}\|u(\cdot,t)\|_{L^{q_k+1}(\mathbb{R}^d)}^{q_k+1} \leq \frac{1}{b}{\delta_1}^b\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2(\mathbb{R}^d)}^2$$
$$+ \frac{1}{a}{\delta_1}^{-a}\left[\frac{(q_k-m)(q_k-1)(C_v^\infty)^2 S_d^{-\frac{(q_k+1)(1-\theta)}{m+q_k-1}}}{2m}\right]^a\|u(\cdot,t)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{a(q_k+1)\theta}$$
$$\leq C_1\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2}^2 + C_2(q_k){q_k}^{2a}\|u(\cdot,t)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{a(q_k+1)\theta}, \tag{48}$$

where

$$b = \frac{m+q_k-1}{(q_k+1)(1-\theta)} = \frac{d(q_k-q_{k-1}+1)+2q_{k-1}+md-2d}{d(q_k-q_{k-1}+1)} > 1,$$

$$a = \frac{b}{b-1} = \frac{d(q_k-q_{k-1}+1)+2q_{k-1}+md-2d}{2q_{k-1}+md-2d} > 1,$$

$$\delta_1 = (C_1 b)^{\frac{1}{b}},\ C_2(q_k) = \frac{1}{a2^a m^a}(C_1 b)^{-\frac{a}{b}}(C_v^\infty)^{2a} S_d^{-\frac{a(q_k+1)(1-\theta)}{m+q_k-1}}.$$

By some simple computations, we know that $a \to 1+d$, $b \to \frac{1+d}{d}$ as $k \to \infty$. Then $C_2(q_k)$ is uniformly bounded as $k \to \infty$. Substituting (48) into (46), we obtain

$$\begin{aligned} \frac{d}{dt}\|u(\cdot,t)\|_{L^{q_k}(\mathbb{R}^d)}^{q_k} \le & -C_1\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2}^2 + \frac{(q_k-1)(C_v^\infty)^2}{2}\|u_0\|_{L^1(\mathbb{R}^d)} \\ & + C_2(q_k){q_k}^{2a}\left(\|u(\cdot,s)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_{k-1}}\right)^{\gamma_1}, \end{aligned} \tag{49}$$

where

$$\gamma_1 = \frac{a\theta(q_k+1)}{q_{k-1}} = \frac{2q_k+md-2d+2}{2q_{k-1}+md-2d} < 3.$$

Secondly, we will estimate $\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2(\mathbb{R}^d)}^2$. From interpolation inequality, it shows that

$$\begin{aligned} \|u(\cdot,t)\|_{L^{q_k}(\mathbb{R}^d)}^{q_k} &\le \|u(\cdot,t)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_k\beta}\|u(\cdot,t)\|_{L^{\frac{(m+q_k-1)d}{d-2}}(\mathbb{R}^d)}^{q_k(1-\beta)} \\ &\le {S_d}^{-\frac{q_k(1-\beta)}{m+q_k-1}}\|u(\cdot,t)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_k\beta}\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2(\mathbb{R}^d)}^{\frac{2q_k(1-\beta)}{m+q_k-1}}, \end{aligned} \tag{50}$$

where

$$\beta = \frac{q_{k-1}(2q_k+md-d)}{q_k\big[(m+q_k-1)d-q_{k-1}(d-2)\big]},$$

$$1-\beta = \frac{d(q_k-q_{k-1})(m+q_k-1)}{q_k\big[(m+q_k-1)d-q_{k-1}(d-2)\big]}.$$

It is shown that $\frac{q_k(1-\beta)}{m+q_k-1} = \frac{d(q_k-q_{k-1})}{d(q_k-q_{k-1})+2q_{k-1}+md-d} < 1$ since $q_{k-1} > \frac{d(2-m)}{2}$. Using Young's inequality for (50), we have

$$\begin{aligned} \|u(\cdot,t)\|_{L^{q_k}(\mathbb{R}^d)}^{q_k} &\le \frac{1}{a'}{\delta_2}^{-a'}{S_d}^{-\frac{a'q_k(1-\beta)}{m+q_k-1}}\|u(\cdot,t)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{a'\beta q_k} + \frac{1}{b'}{\delta_2}^{b'}\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2(\mathbb{R}^d)}^2 \\ &:= C_1\left\|\nabla u^{\frac{m+q_k-1}{2}}\right\|_{L^2(\mathbb{R}^d)}^2 + C_3(q_k)\left(\|u(\cdot,t)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_{k-1}}\right)^{\gamma_2}, \end{aligned} \tag{51}$$

where

$$b' = \frac{m+q_k-1}{q_k(1-\beta)} = \frac{d(q_k-q_{k-1})+2q_{k-1}+md-d}{d(q_k-q_{k-1})} > 1,$$

$$a' = \frac{b'}{b'-1} = \frac{d(q_k-q_{k-1})+2q_{k-1}+md-d}{2q_{k-1}+md-d} > 1,$$

$$\delta_2 = (C_1b')^{\frac{1}{b'}},\ C_3(q_k) = \frac{1}{a'}(C_1b')^{-\frac{a'}{b'}}{S_d}^{-\frac{a'q_k(1-\beta)}{m+q_k-1}},$$

$$\gamma_2 = \frac{q_k\beta a'}{q_{k-1}} = \frac{2q_k+md-d}{2q_{k-1}+md-d} < 3.$$

We can check that $C_3(q_k)$ is uniformly bounded as $k \to \infty$. Substituting (51) into (49), we obtain

$$\begin{aligned} \frac{d}{dt}\|u(\cdot,t)\|_{L^{q_k}(\mathbb{R}^d)}^{q_k} \le & -\|u(\cdot,t)\|_{L^{q_k}(\mathbb{R}^d)}^{q_k} + C_4q_k\|u_0\|_{L^1(\mathbb{R}^d)} \\ & + C_2(q_k){q_k}^{2a}\left(\|u(\cdot,s)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_{k-1}}\right)^{\gamma_1} + C_3(q_k)\left(\|u(\cdot,t)\|_{L^{q_{k-1}}(\mathbb{R}^d)}^{q_{k-1}}\right)^{\gamma_2}, \end{aligned} \tag{52}$$

where $C_4 = \frac{(C_v^\infty)^2}{2}$. Since $C_2(q_k)$ and $C_3(q_k)$ are all uniformly bounded as $q_k \to \infty$, we can choose a constant $C_5 > 1$ which is an upper bound of $C_2(q_k)$, $C_3(q_k)$ and $C_4\|u_0\|_{L^1(\mathbb{R}^d)}$. Then by $q_k > 1$ and $a > 1$, we have L^{q_k} estimate

$$\frac{d}{dt}\|u(\cdot,t)\|_{L^{q_k}}^{q_k} \le -\|u(t)\|_{L^{q_k}}^{q_k} + C_5 {q_k}^{2a}\Big[\left(\|u(t)\|_{L^{q_{k-1}}}^{q_{k-1}}\right)^{\gamma_1} + \left(\|u(t)\|_{L^{q_{k-1}}}^{q_{k-1}}\right)^{\gamma_2} + 1\Big]. \quad (53)$$

Step 2. (Uniform L^∞ estimate). Let $y_k(t) = \|u(\cdot,t)\|_{L^{q_k}(\mathbb{R}^d)}^{q_k}$ and multiply e^t to both sides of (53)

$$\frac{d}{dt}\left(e^t y_k(t)\right) \le C_5 {q_k}^{2a}\left(y_{k-1}^{\gamma_1}(t) + y_{k-1}^{\gamma_2}(t) + 1\right)e^t \le 3C_5 {q_k}^{2a} \max\left\{1, \sup_{t\ge 0} y_{k-1}^3(t)\right\} e^t.$$

Solving this ODE, we obtain for $t \ge 0$

$$\begin{aligned} y_k(t) &\le e^{-t} y_k(0) + 3C_5 {q_k}^{2a} \max\left\{1, \sup_{t\ge 0} y_{k-1}^3(t)\right\}\left(1 - e^{-t}\right) \\ &\le 3C_5 {q_k}^{2a} \max\left\{1,\ y_k(0),\ \sup_{t\ge 0} y_{k-1}^3(t)\right\}. \end{aligned} \quad (54)$$

We have

$${q_k}^{2a} = \left(3^k + \frac{d(2-m)}{2} + 1\right)^{2a} \le C_0 3^{2ak}\left(\frac{d(2-m)}{2} + 1\right)^{2a}, \quad (55)$$

where C_0 is an appropriate positive constant. Combining (54) and (55) together, we can see

$$y_k(t) \le C_6\left(\frac{d(2-m)}{2} + 1\right)^{2a} 3^{2ak} \max\left\{1,\ y_k(0),\ \sup_{t\ge 0} y_{k-1}^3(t)\right\},$$

where $C_6 = 3C_0C_5$. Then after some iterative steps, we have

$$\begin{aligned} y_k(t) \le & \left(C_6\left(\frac{d(2-m)}{2} + 1\right)^{2a}\right)^{\frac{3^k-1}{2}} 3^{2a\left(\frac{3^{k+1}}{4} - \frac{k}{2} - \frac{3}{4}\right)} \\ & \cdot \max\left\{1,\ \sum_{i=0}^{k-1} y_{k-i}^{3^i}(0),\ \sup_{t\ge 0} y_0^{3^k}(t)\right\}. \end{aligned} \quad (56)$$

Denote $K_0 = \max\left\{1, \|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}\right\}$, then

$$y_k(0) = \|u_0\|_{L^{q_k}(\mathbb{R}^d)}^{q_k} \le \max\left\{\|u_0\|_{L^1(\mathbb{R}^d)}^{q_k}, \|u_0\|_{L^\infty(\mathbb{R}^d)}^{q_k}\right\} \le K_0^{q_k},$$

and

$$\max\left\{1,\ \sum_{i=0}^{k-1} y_{k-i}^{3^i}(0)\right\} \le kK_0^{q_k}.$$

Taking power $\frac{1}{q_k}$ to both sides of (56) and letting $k \to \infty$, we obtain

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C \max\left\{\sup_{t\ge 0} y_0(t), K_0\right\}, \quad (57)$$

where $C = 3^{\frac{3(d+1)}{2}} C_6^{\frac{1}{2}} \left(\frac{d(2-m)}{2}+1\right)^{d+1}$ since $a \to d+1$ as $k \to \infty$. Recalling (40) in Proposition 1, it shows that

$$y_0(t) = \|u(\cdot,t)\|_{L^{p+2}(\mathbb{R}^d)}^{p+2} \le C_u^{p+2}. \tag{58}$$

Then (57) turns to

$$\|u(\cdot,t)\|_{L^\infty} \le C(m,d,K_0).$$

□

5. **Global existence of weak entropy solutions.** In this section, we prove a theorem of the existence of a weak entropy solution by constructing a corresponding regularized problem.

Theorem 5.1. *Let $d \ge 3$, $0 < m < 2 - \frac{2}{d}$ and $p = \frac{d(2-m)}{2}$. Assume $u_0 \in L_+^1 \cap L^p(\mathbb{R}^d)$ and $\eta = C_{d,m}^{2-m} - \|u_0\|_{L^p(\mathbb{R}^d)}^{2-m} > 0$, where $C_{d,m}^{2-m} = \frac{4mp}{S_d^{-1}(m+p-1)^2 C_p}$ is a universal constant. Then there exists a non-negative global weak solution (u,v) of (1), such that all the a priori estimates in Theorem 3.1 hold true. Furthermore, for $1 < m < 2 - \frac{2}{d}$, if the initial data satisfies $\int_{\mathbb{R}^d} |x|^2 u_0(x)dx < \infty$, and $\|u_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \le C$, then*

(i) the second moments $\int_{\mathbb{R}^d} |x|^2 u(x,t)\, dx$ and $\int_{\mathbb{R}^d} |x|^2 v(x,t)\, dx$ are bounded for any $0 \le t < \infty$,

(i) the free energy of (1) is

$$\mathcal{F}\big(u(\cdot,t),v(\cdot,t)\big) = \frac{1}{m-1}\int_{\mathbb{R}^d} u^m dx - \int_{\mathbb{R}^d} uv dx + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} v^2 dx,$$

which is non-increasing in time,

(ii) with an extra assumption that $u_0 \in L^m(\mathbb{R}^d)$ when $\frac{2d}{d+2} < m < 2 - \frac{2}{d}$, for all $1 < m < 2 - \frac{2}{d}$, the weak solution of (1) also satisfies energy inequality

$$\mathcal{F}(t) + \int_0^t \int_{\mathbb{R}^d} u\left|\nabla\left(\frac{m}{m-1}u^{m-1} - v\right)\right|^2 dxds + \int_0^t \int_{\mathbb{R}^d} |\partial_t v|^2\, dxds \le \mathcal{F}(0),$$

a.e. $t > 0$.

Proof. We separate the proof of Theorem 5.1 into nine steps. In Step 1, we construct the regularized problem of (1) and show that all the *a priori* estimates in Theorem 3.1 hold true. In Step 2-5, by applying Aubin-Lions-Dubinskiĭ Lemma, we prove that the non-negative weak solution of regularized problem (59) converges strongly to a non-negative weak solution of (1) in a bounded region which shows the existence of a non-negative weak solution of (1) in $\mathbb{R}^d$. Then in Step 6, with a little improvement of initial data, we extend the strong convergence to the whole space $\mathbb{R}^d$ through the proof of the second moments are finite when $1 < m < 2-\frac{2}{d}$. In Step 7 and 8, we show the convergence of the free energy and the lower semi-continuity of the dissipation term. Furthermore, In Step 9, we prove that the global weak solution satisfies energy inequality.

Step 1. (Regularized problem and *a priori* estimates). We consider the regularized problem of (1) for $\epsilon > 0$,

$$\begin{cases} \partial_t u_\epsilon = \Delta u_\epsilon^m + \epsilon \Delta u_\epsilon - \nabla\cdot(u_\epsilon \nabla v_\epsilon), \quad x \in \mathbb{R}^d, t > 0, \\ \partial_t v_\epsilon = \Delta v_\epsilon - v_\epsilon + u_\epsilon, \quad x \in \mathbb{R}^d, t > 0, \\ u_\epsilon(x,0) = u_{0\epsilon}(x),\ v_\epsilon(x,0) = 0, \quad x \in \mathbb{R}^d, \end{cases} \tag{59}$$

where $d \geq 3$, $0 < m < 2 - \frac{2}{d}$. The initial data $u_{0\epsilon}(x) \in C^\infty(\mathbb{R}^d)$ is a sequence of approximation for $u_0(x)$, which satisfies that there exists $\delta > 0$ such that for all $0 < \epsilon < \delta$,

$$\begin{gathered} u_{0\epsilon}(x) > 0, x \in \mathbb{R}^d, \\ u_{0\epsilon}(x) \in L^r(\mathbb{R}^d), \text{ for all } r \geq 1, \\ \|u_{0\epsilon}(\cdot)\|_{L^1(\mathbb{R}^d)} = \|u_0(\cdot)\|_{L^1(\mathbb{R}^d)}, \\ \|u_{0\epsilon}(\cdot)\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \leq C, \\ \int_{\mathbb{R}^d} |x|^2 u_{0\epsilon}\, dx \to \int_{\mathbb{R}^d} |x|^2 u_0\, dx, \text{ as } \epsilon \to 0. \end{gathered}$$

For the existence of a strong solution of problem (59), we refer to [20, Section 3]. Our existence result of regularized problem can be obtained by almost the same way of proving Theorem 7 in [20], except for some small details. Then the regularized problem has a global strong solution (u_ϵ, v_ϵ) with $u_\epsilon \in W^{2,1}_{d+3}(\mathbb{R}^d \times \mathbb{R}_+)$. Since $W^{2,1}_{d+3}(\mathbb{R}^d \times \mathbb{R}_+)$ is a subset of $L^\infty\left(\mathbb{R}_+; L^r(\mathbb{R}^d)\right) \cap L^{r+1}\left(\mathbb{R}_+; L^{r+1}(\mathbb{R}^d)\right)$ for all $r \geq 1$, we have

$$u_\epsilon \in L^\infty\left(\mathbb{R}_+; L^r(\mathbb{R}^d)\right) \cap L^{r+1}\left(\mathbb{R}_+; L^{r+1}(\mathbb{R}^d)\right).$$

Then we will prove that all the *a priori* estimates in Theorem 3.1 hold true for our regularized problem. Multiplying the first equation of (59) by $pu_\epsilon^{p-1}\psi_R(x)$ and integrating over $\mathbb{R}^d \times (0,t)$, where $\psi_R(x)$ is the cut-off function defined before, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} u_\epsilon^p(x,t)\psi_R(x)\, dx + \frac{4mp(p-1)}{(m+p-1)^2}\int_0^t\int_{\mathbb{R}^d} \left|\nabla u_\epsilon^{\frac{m+p-1}{2}}\right|^2 \psi_R(x)\, dxds \\ &\qquad + \frac{4\epsilon(p-1)}{p}\int_0^t\int_{\mathbb{R}^d} \left|\nabla u_\epsilon^{\frac{p}{2}}\right|^2\psi_R(x)\, dxds \\ &\qquad = \int_{\mathbb{R}^d} u_{0\epsilon}^p(x)\psi_R(x)\, dx - (p-1)\int_0^t\int_{\mathbb{R}^d} u_\epsilon^p \Delta v_\epsilon \psi_R(x)\, dxds \\ &\qquad + \frac{mp}{m+p-1}\int_0^t\int_{\mathbb{R}^d} u_\epsilon^{m+p-1}\Delta\psi_R(x)\, dxds \\ &\qquad + \int_0^t\int_{\mathbb{R}^d} u_\epsilon^p \nabla v_\epsilon\cdot\nabla\psi_R(x)\, dxds + \epsilon\int_0^t\int_{\mathbb{R}^d} u_\epsilon^p\Delta\psi_R(x)\, dxds. \end{aligned} \tag{60}$$

In order to estimate the right hand side of (60), we should have estimates of v_ϵ at first.

Multiplying $\partial_t v_\epsilon = \Delta v_\epsilon - v_\epsilon + u_\epsilon$ by $-\Delta v_\epsilon$ and integrating over $\mathbb{R}^d$ and from 0 to t, we have

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}^d} |\nabla v_\epsilon(x,t)|^2\, dx + \int_0^t\int_{\mathbb{R}^d} |\Delta v_\epsilon(x,s)|^2\, dxds + \int_0^t\int_{\mathbb{R}^d} |\nabla v_\epsilon(x,s)|^2\, dxds \\ &\qquad \leq \int_0^t\int_{\mathbb{R}^d} |\Delta v_\epsilon(x,s)| u_\epsilon(x,s)\, dxds \\ &\qquad \leq \int_0^t\int_{\mathbb{R}^d} |\Delta v_\epsilon(x,s)|^2\, dxds + \int_0^t\int_{\mathbb{R}^d} u_\epsilon^2(x,s)\, dxds \\ &\qquad \leq (C_p+1)\int_0^t\int_{\mathbb{R}^d} u_\epsilon^2(x,s)\, dxds. \end{aligned} \tag{61}$$

In the same way, multiplying $\partial_t v_\epsilon = \Delta v_\epsilon - v_\epsilon + u_\epsilon$ by v_ϵ and integrating over $\mathbb{R}^d$ and from 0 to t, we have

$$\frac{1}{2}\int_{\mathbb{R}^d} v_\epsilon^2(x,t)\,dx + \int_0^t\int_{\mathbb{R}^d} |\nabla v_\epsilon(x,s)|^2\,dxds + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} v_\epsilon^2(x,s)\,dxds \le \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u_\epsilon^2(x,s)\,dxds. \tag{62}$$

Combining (61) with (62), we see that

$$v_\epsilon \in L^\infty\left(\mathbb{R}_+; H^1(\mathbb{R}^d)\right) \cap L^2\left(\mathbb{R}_+; H^2(\mathbb{R}^d)\right),$$

since $u_\epsilon \in L^2\left(\mathbb{R}_+; L^2(\mathbb{R}^d)\right)$. Then using Hölder's inequality, we obtain

$$\begin{aligned} -(p-1)\int_0^t\int_{\mathbb{R}^d} u_\epsilon^p \Delta v_\epsilon\,dxds &\le (p-1)\int_0^t \|u_\epsilon\|_{L^{2p}(\mathbb{R}^d)}^p \|\Delta v_\epsilon\|_{L^2(\mathbb{R}^d)}\,ds \\ &\le (p-1)\left(\int_0^t \|u_\epsilon\|_{L^{2p}(\mathbb{R}^d)}^{2p}\,ds\right)^{\frac{1}{2}} \|\Delta v_\epsilon\|_{L^2(\mathbb{R}_+;L^2(\mathbb{R}^d))} \\ &\le C(\epsilon), \end{aligned}$$

which means that we can use the dominated convergence theorem for this term as $R \to \infty$ for any small ϵ.

Next, we prove that last three terms on the right hand side of (60) go to 0 as $R \to \infty$. Firstly, from $u_\epsilon \in L^\infty\left(\mathbb{R}_+; L^r(\mathbb{R}^d)\right)$, for any $t > 0$ and small ϵ, we have

$$\int_0^t\int_{\mathbb{R}^d} u_\epsilon^{m+p-1}\Delta\psi_R(x)\,dxds \le \frac{C}{R^2}\int_0^t\int_{B_{2R}} u_\epsilon^{m+p-1}\,dxds \le \frac{C(t,\epsilon)}{R^2},$$

since $m+p-1 \ge 1$.

Secondly, from $u_\epsilon \in L^\infty\left(\mathbb{R}_+; L^r(\mathbb{R}^d)\right)$ and $v_\epsilon \in L^2\left(\mathbb{R}_+; H^2(\mathbb{R}^d)\right)$, we have

$$\int_0^t\int_{\mathbb{R}^d} u_\epsilon^p \nabla v_\epsilon \cdot \nabla\psi_R(x)\,dxds \le \frac{C(\epsilon)}{R},$$

$$\int_0^t\int_{\mathbb{R}^d} u_\epsilon^p \Delta\psi_R(x)\,dxds \le \frac{C(t,\epsilon)}{R^2}.$$

Using the dominated convergence theorem, when $R \to \infty$, (60) turns to

$$\int_{\mathbb{R}^d} u_\epsilon^p(x,t)\,dx - \int_{\mathbb{R}^d} u_{0\epsilon}^p(x)\,dx + \frac{4mp(p-1)}{(m+p-1)^2}\int_0^t\int_{\mathbb{R}^d}\left|\nabla u_\epsilon^{\frac{m+p-1}{2}}\right|^2 dxds \le -(p-1)\int_0^t\int_{\mathbb{R}^d} u_\epsilon^p \Delta v_\epsilon\,dxds, \tag{63}$$

which is same to (11) by the method of obtaining (23). From all above, we have the conclusion that all the *a priori* estimates in Theorem 3.1 hold true for the solution of the regularized problem. Then we have following estimates,

$$\|u_\epsilon\|_{L^\infty\left(\mathbb{R}_+; L^1_+ \cap L^p(\mathbb{R}^d)\right)} \le C, \tag{64}$$

$$\|u_\epsilon\|_{L^{p+1}\left(\mathbb{R}_+; L^{p+1}(\mathbb{R}^d)\right)} \le C, \tag{65}$$

$$\left\|\nabla u_\epsilon^{\frac{m+r-1}{2}}\right\|_{L^2\left(\mathbb{R}_+; L^2(\mathbb{R}^d)\right)} \le C, \quad 1 < r \le p. \tag{66}$$

Letting $r = 3 - m - \frac{2}{d}$, we know that $1 < r \le p$ since $0 < m < 2 - \frac{2}{d}$. From (66), by using interpolation inequality and Sobolev inequality, we have

$$\begin{aligned}\int_0^\infty \|u_\epsilon(\cdot,t)\|_{L^2(\mathbb{R}^d)}^2 \, dt &\le \int_0^\infty \|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^d)}^{\frac{(m+r-3)d+4}{(m+r-2)d+2}} \|u_\epsilon(\cdot,t)\|_{L^{\frac{(m+r-1)d}{d-2}}(\mathbb{R}^d)}^{\frac{(m+r-1)d}{(m+r-2)d+2}} \, dt \\ &\le S_d^{-1} \int_0^\infty \|u_\epsilon(\cdot,t)\|_{L^1(\mathbb{R}^d)}^{\frac{(m+r-3)d+4}{(m+r-2)d+2}} \left\| \nabla u_\epsilon^{\frac{m+r-1}{2}} \right\|_{L^2(\mathbb{R}^d)}^2 \, dt \le C,\end{aligned}$$

i.e.

$$\|u_\epsilon\|_{L^2\left(\mathbb{R}_+;L^2(\mathbb{R}^d)\right)} \le C. \tag{67}$$

Then we have uniform estimates for v_ϵ

$$\|v_\epsilon\|_{L^\infty\left(\mathbb{R}_+;H^1(\mathbb{R}^d)\right)} \le C, \tag{68}$$

$$\|v_\epsilon\|_{L^2\left(\mathbb{R}_+;H^2(\mathbb{R}^d)\right)} \le C. \tag{69}$$

Step 2. (Time regularity of u_ϵ). In this step, we estimate $\partial_t u_\epsilon$ in any bounded domain in order to use Aubin-Lions-Dubinskiĭ Lemma. For any test function $\varphi(x)$ which satisfies $\varphi \in W^{2,\frac{2(p+1)}{p-1}}(\Omega)$, $\|\varphi\|_{W^{2,\frac{2(p+1)}{p-1}}(\Omega)} \le 1$, we have

$$\begin{aligned}|\langle \partial_t u_\epsilon, \varphi\rangle| &= |\langle \Delta u_\epsilon^m, \varphi\rangle + \epsilon\langle \Delta u_\epsilon, \varphi\rangle - \langle \nabla\cdot(u_\epsilon \nabla v_\epsilon), \varphi\rangle| \\ &\le \|u_\epsilon^m\|_{L^{\frac{2(p+1)}{p+3}}(\Omega)} + \epsilon\|u_\epsilon\|_{L^{\frac{2(p+1)}{p+3}}(\Omega)} + \|u_\epsilon \nabla v_\epsilon\|_{L^{\frac{2(p+1)}{p+3}}(\Omega)} \\ &\le C(\Omega)\left(\|u_\epsilon\|_{L^{p+1}(\Omega)}^m + \epsilon\|u_\epsilon\|_{L^{p+1}(\Omega)} + \|u_\epsilon \nabla v_\epsilon\|_{L^{\frac{2(p+1)}{p+3}}(\Omega)} \right),\end{aligned} \tag{70}$$

where the last inequality holds since $\frac{2m(p+1)}{p+3} \le p+1$ and $\frac{2(p+1)}{p+3} \le p+1$ from $0 < m < 2 - \frac{2}{d}$. Choosing $\bar{p} = \min\left\{\frac{p+1}{m}, p+1\right\} > 1$, for any $T > 0$, we obtain

$$\begin{aligned}\int_0^T \|\partial_t u_\epsilon\|_{W^{-2,\frac{2(p+1)}{p+3}}(\Omega)}^{\bar{p}} \, dt &\le C(\Omega)\bigg(\int_0^T \|u_\epsilon\|_{L^{p+1}(\Omega)}^{m\bar{p}} \, dt + \epsilon \int_0^T \|u_\epsilon\|_{L^{p+1}(\Omega)}^{\bar{p}} \, dt \\ &\quad + \int_0^T \|u_\epsilon \nabla v_\epsilon\|_{L^{\frac{2(p+1)}{p+3}}(\Omega)}^{\bar{p}} \, dt \bigg) \\ &\le C(\Omega,T)(1+\epsilon) + C(\Omega)\int_0^T \|u_\epsilon\|_{L^{p+1}(\Omega)}^{\bar{p}} \|\nabla v_\epsilon\|_{L^2(\Omega)}^{\bar{p}} \, dt \\ &\le 2C(\Omega,T).\end{aligned} \tag{71}$$

Then we have $\|\partial_t u_\epsilon\|_{L^{\bar{p}}\left(0,T;W^{-2,\frac{2(p+1)}{p+3}}(\Omega)\right)} \le C$.

Step 3. (Application of Aubin-Lions-Dubinskiĭ Lemma). Before using Aubin-Lions-Dubinskiĭ Lemma, we introduce the definition of *Seminormed non-negative cone* in a Banach space which can be found in [6].

Definition 5.2. Let B be a Banach space, $M_+ \subset B$ satisfies

(1) $Cu \in M_+$, for all $u \in M_+$ and $C \ge 0$,
(2) there exists a function $[\cdot]\colon M_+ \to [0,\infty)$ such that $[u] = 0$ if and only if $u = 0$,
(3) $[Cu] = C[u]$, for all $C \ge 0$,

then M_+ is a Seminormed non-negative cone in B.

Now by choosing $B = L^{p+1}(\Omega)$, we construct

$$M_+(\Omega) := \left\{ u : [u] = \left\| \nabla u^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{\frac{2}{m+p-1}} + \|u\|_{L^1(\Omega)} + \|u\|_{L^{p+1}(\Omega)} \right\},$$

which is a Seminormed non-negative cone in $L^{p+1}(\Omega)$ that can be checked. Then we will prove $M_+(\Omega) \hookrightarrow\hookrightarrow L^{p+1}(\Omega)$, i.e. for any bounded sequence $\{u_\epsilon\}$ in $M_+(\Omega)$, there exists a subsequence converging in $L^{p+1}(\Omega)$.

Since $H^1(\Omega) \hookrightarrow\hookrightarrow L^{\frac{2(p+1)}{m+p-1}}(\Omega)$, from $\frac{2(p+1)}{m+p-1} \le \frac{2d}{d-2}$, we can find a subsequence $\left\{ u_\epsilon^{\frac{m+p-1}{2}} \right\}$ in $H^1(\Omega)$ without relabeling such that

$$u_\epsilon^{\frac{m+p-1}{2}} \to u^{\frac{m+p-1}{2}}, \quad \text{in } L^{\frac{2(p+1)}{m+p-1}}(\Omega) \quad \text{as } \epsilon \to 0.$$

For $m+p-1 \ge 2$, we have

$$\begin{aligned}
\int_\Omega |u_\epsilon - u|^{p+1}\, dx &= \int_\Omega \left| u_\epsilon^{\frac{m+p-1}{2}\frac{2}{m+p-1}} - u^{\frac{m+p-1}{2}\frac{2}{m+p-1}} \right|^{p+1} dx \\
&\le \int_\Omega \left| u_\epsilon^{\frac{m+p-1}{2}} - u^{\frac{m+p-1}{2}} \right|^{(p+1)\frac{2}{m+p-1}} dx \to 0, \quad \text{as } \epsilon \to 0.
\end{aligned}$$

For $m+p-1 < 2$, using Hölder's inequality, one has

$$\begin{aligned}
\int_\Omega |u_\epsilon - u|^{p+1}\, dx &= \int_\Omega \left| u_\epsilon^{\frac{m+p-1}{2}\frac{2}{m+p-1}} - u^{\frac{m+p-1}{2}\frac{2}{m+p-1}} \right|^{p+1} dx \\
&\le \int_\Omega \left| u_\epsilon^{\frac{m+p-1}{2}} - u^{\frac{m+p-1}{2}} \right|^{p+1} \left| u_\epsilon^{\frac{m+p-1}{2}} + u^{\frac{m+p-1}{2}} \right|^{\frac{(p+1)(3-m-p)}{m+p-1}} dx \\
&\le \left\| u_\epsilon^{\frac{m+p-1}{2}} - u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2(p+1)}{m+p-1}}(\Omega)}^{p+1} \left\| u_\epsilon^{\frac{m+p-1}{2}} + u^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2(p+1)}{m+p-1}}(\Omega)}^{\frac{(p+1)(3-m-p)}{m+p-1}} \\
&\longrightarrow 0, \text{ as } \epsilon \to 0.
\end{aligned}$$

From above, for all $0 < m < 2 - \frac{2}{d}$, $M_+(\Omega) \hookrightarrow\hookrightarrow L^{p+1}(\Omega)$.

Until now, we have already obtained

$$\|u_\epsilon\|_{L^{m+p-1}\left(0,T;M_+(\Omega)\right)} \le C,$$

$$\|u_\epsilon\|_{L^{m+p-1}\left(0,T;L^{p+1}(\Omega)\right)} \le C,$$

$$\|\partial_t u_\epsilon\|_{L^{\bar{p}}\left(0,T;W^{-2,\frac{2(p+1)}{p+3}}(\Omega)\right)} \le C,$$

and

$$M_+(\Omega) \hookrightarrow\hookrightarrow L^{p+1}(\Omega) \hookrightarrow W^{-2,\frac{2(p+1)}{p+3}}(\Omega).$$

By Aubin-Lions-Dubinskiĭ Lemma, there exists a subsequence of $\{u_\epsilon\}$ without relabeling such that

$$u_\epsilon \to u \quad \text{in } L^{m+p-1}\left(0,T;L^{p+1}(\Omega)\right). \tag{72}$$

Let $\{B_k\}_{k=1}^\infty \in \mathbb{R}^d$ be a sequence of balls centered at 0 with radius R_k, and $R_k \to \infty$ as $k \to \infty$. By a standard diagonal argument, there exists a subsequence $\{u_\epsilon\}$ without relabeling, such that the following uniformly strong convergence holds true

$$u_\epsilon \to u \quad \text{in } L^{m+p-1}\left(0,T;L^{p+1}(B_k)\right), \forall k. \tag{73}$$

Step 4. (Strong convergence of v_ϵ). From the second equation of (1), using (67) and (69), for any test function $\varphi(x)$ which satisfies $\varphi \in W^{2,2}(\Omega)$ and $\|\varphi\|_{W^{2,2}(\Omega)} \le 1$, we have

$$\begin{aligned}|\langle \partial_t \nabla v_\epsilon, \varphi\rangle| &\le |\langle \nabla v_\epsilon, \Delta\varphi\rangle| + |\langle v_\epsilon, \nabla\varphi\rangle| + |\langle u_\epsilon, \nabla\varphi\rangle| \\ &\le \|\nabla v_\epsilon\|_{L^2(\Omega)} + \|v_\epsilon\|_{L^2(\Omega)} + \|u_\epsilon\|_{L^2(\Omega)}. \end{aligned} \tag{74}$$

Then for any $T > 0$, we obtain

$$\begin{aligned}\int_0^T \|\partial_t \nabla v_\epsilon\|^2_{W^{-2,2}(\Omega)}\ dt \le & C\int_0^T \|\nabla v_\epsilon\|^2_{L^2(\Omega)} dt + C\int_0^T \|v_\epsilon\|^2_{L^2(\Omega)}\ dt \\ & + C\int_0^T \|u_\epsilon\|^2_{L^2(\Omega)}\ dt \le C,\end{aligned}$$

i.e. $\|\partial_t \nabla v_\epsilon\|_{L^2\left(0,T;W^{-2,2}(\Omega)\right)} \le C$. Since $\|\nabla v_\epsilon\|_{L^2\left(0,T;H^1(\Omega)\right)} \le C$, by using Aubin-Lions Lemma, there exists a subsequence of $\{v_\epsilon\}$ without relabeling such that

$$\nabla v_\epsilon \to \nabla v \quad \text{in } L^2\left(0,T;L^2(\Omega)\right), \tag{75}$$

$$v_\epsilon \to v \quad \text{in } L^2\left(0,T;H^1(\Omega)\right). \tag{76}$$

Also let $\{B_k\}_{k=1}^\infty \in \mathbb{R}^d$ be a sequence of balls centered at 0 with radius R_k, and $R_k \to \infty$ as $k \to \infty$, one has that

$$v_\epsilon \to v \quad \text{in } L^2\left(0,T;H^1(B_k)\right), \forall k. \tag{77}$$

Step 5. (Existence of a global weak solution). Next, we will prove that (u, v) is a weak solution of problem (1). The weak formulation for u_ϵ is that for any test function $\psi(x) \in C_c^\infty(\mathbb{R}^d)$ and any $0 < t < \infty$,

$$\begin{aligned}&\int_{\mathbb{R}^d} u_\epsilon(x,t)\psi(x)\ dx - \int_{\mathbb{R}^d} u_{0\epsilon}(x)\psi(x)\ dx = \int_0^t\int_{\mathbb{R}^d} u_\epsilon^m(x,s)\Delta\psi(x)\ dxds \\ &+ \epsilon\int_0^t\int_{\mathbb{R}^d} u_\epsilon(x,s)\Delta\psi(x)\ dxds + \int_0^t\int_{\mathbb{R}^d} u_\epsilon(x,s)\nabla v_\epsilon(x,s)\cdot\nabla\psi(x)\ dxds. \end{aligned} \tag{78}$$

Firstly, we try to prove that

$$u_\epsilon^m \to u^m \quad \text{in } L^1\left(0,T;L^1(\Omega)\right),$$

by using strong convergence (72). For $0 < m \le 1$, using Hölder's inequality, we have

$$\begin{aligned}\int_0^T\int_\Omega |u_\epsilon^m - u^m|\ dxds &\le \int_0^T\int_\Omega |u_\epsilon - u|^m\ dxds \\ &\le C(\Omega,T)\|u_\epsilon - u\|^m_{L^{m+p-1}\left(0,T;L^{p+1}(\Omega)\right)} \to 0, \quad \text{as } \epsilon \to 0. \end{aligned} \tag{79}$$

For $1 < m < 2 - \frac{2}{d}$, also using Hölder's inequality, we obtain

$$\begin{aligned}\int_0^T\int_\Omega |u_\epsilon^m - u^m|\ dxds &\le \int_0^T\int_\Omega |u_\epsilon - u||u_\epsilon + u|^{m-1}\ dxds \\ &\le C(\Omega,T)\|u_\epsilon - u\|_{L^{m+p-1}\left(0,T;L^{p+1}(\Omega)\right)}\|u_\epsilon + u\|^{\frac{m-1}{m+p-1}}_{L^{m+p-1}\left(0,T;L^{p+1}(\Omega)\right)} \\ &\to 0, \quad \text{for } \epsilon \to 0. \end{aligned} \tag{80}$$

From (79) and (80), we have proved that

$$u_\epsilon^m \to u^m \quad \text{in } L^1\left(0,T;L^1(\Omega)\right). \tag{81}$$

Next, we have

$$
\begin{aligned}
&\int_0^T \int_\Omega |u_\epsilon \nabla v_\epsilon - u\nabla v| \ dxds \\
&\le \int_0^T \int_\Omega |u_\epsilon \nabla v_\epsilon - u\nabla v_\epsilon| \ dxds + \int_0^T \int_\Omega |u\nabla v_\epsilon - u\nabla v| \ dxds \\
&\le C(\Omega,T)\|u_\epsilon - u\|_{L^{m+p-1}\left(0,T;L^{p+1}(\Omega)\right)} \|\nabla v_\epsilon\|_{L^\infty\left(0,T;L^2(\Omega)\right)} \\
&\quad + \|u\|_{L^2\left(0,T;L^2(\Omega)\right)} \|\nabla v_\epsilon - \nabla v\|_{L^2\left(0,T;L^2(\Omega)\right)} \to 0, \quad \text{as } \epsilon \to 0, \qquad (82)
\end{aligned}
$$

since

$$\|\nabla v_\epsilon\|_{L^\infty\left(0,T;L^2(\Omega)\right)} \le C,$$

$$\|u\|_{L^{p+1}\left(0,T;L^{p+1}(\Omega)\right)} \le C,$$

$$\nabla v_\epsilon \to \nabla v \quad \text{in } L^2\left(0,T;L^2(\Omega)\right).$$

Then (82) turns that

$$u_\epsilon \nabla v_\epsilon \to u\nabla v \quad \text{in } L^1\left(0,T;L^1(\Omega)\right). \qquad (83)$$

Owing to (81) and (83), passing limit $\epsilon \to 0$, one has that for any $0 < t < \infty$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} u(x,t)\psi(x) \ dx - \int_{\mathbb{R}^d} u_0(x)\psi(x) \ dx &= \int_0^t \int_{\mathbb{R}^d} u^m(x,s)\Delta\psi(x) \ dxds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} u(x,s)\nabla v(x,s)\cdot\nabla\psi(x) \ dxds. \qquad (84)
\end{aligned}
$$

The weak formulation for v_ϵ is that for any test function $\psi(x) \in C_c^\infty(\mathbb{R}^d)$ and any $0 < t < \infty$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} v_\epsilon(x,t)\psi(x) \ dx &= \int_0^t \int_{\mathbb{R}^d} v_\epsilon(x,s)\Delta\psi(x) \ dxds - \int_0^t \int_{\mathbb{R}^d} v_\epsilon(x,s)\psi(x) \ dxds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} u_\epsilon(x,s)\psi(x) \ dxds. \qquad (85)
\end{aligned}
$$

From strong convergences we have obtained for u_ϵ and v_ϵ, it is easy to see that

$$\int_0^T \int_\Omega |v_\epsilon - v| \ dxds \to 0, \quad \text{as } \epsilon \to 0, \qquad (86)$$

$$\int_0^T \int_\Omega |u_\epsilon - u| \ dxds \to 0, \quad \text{as } \epsilon \to 0. \qquad (87)$$

Then passing limit $\epsilon \to 0$, one has that for any $0 < t < \infty$,

$$
\begin{aligned}
\int_{\mathbb{R}^d} v(x,t)\psi(x) \ dx &= \int_0^t \int_{\mathbb{R}^d} v(x,s)\Delta\psi(x) \ dxds - \int_0^t \int_{\mathbb{R}^d} v(x,s)\psi(x) \ dxds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} u(x,s)\psi(x) \ dxds. \qquad (88)
\end{aligned}
$$

Now we have the conclusion that (u, v) is a global weak solution of (1).

Step 6. (Strong convergence in $\mathbb{R}^d$ for the weak solution). For $1 < m < 2 - \frac{2}{d}$, we estimate the second moments of u_ϵ and v_ϵ at first. From (59), one has that

$$\begin{aligned}\frac{d}{dt} m_2\big(u_\epsilon(\cdot,t)\big) &= \int_{\mathbb{R}^d} |x|^2 \partial_t u_\epsilon \, dx = \int_{\mathbb{R}^d} |x|^2 \big(\Delta u_\epsilon^m + \epsilon \Delta u_\epsilon - \nabla \cdot (u_\epsilon \nabla v_\epsilon)\big) \, dx \\ &\le 2d \int_{\mathbb{R}^d} u_\epsilon^m dx + 2d\epsilon \int_{\mathbb{R}^d} u_\epsilon dx + 2 \int_{\mathbb{R}^d} u_\epsilon x \cdot \nabla v_\epsilon \, dx \\ &\le 2d \|u_\epsilon\|_{L^m(\mathbb{R}^d)}^m + 2d\epsilon \|u_\epsilon\|_{L^1(\mathbb{R}^d)} + \int_{\mathbb{R}^d} u_\epsilon |\nabla v_\epsilon|^2 \, dx + m_2. \qquad (89)\end{aligned}$$

Then using Gronwall's inequality, (89) turns to

$$\begin{aligned} m_2\big(u_\epsilon(\cdot,t)\big) \le e^t m_2(u_{0\epsilon}) &+ 2de^t \int_0^t \|u_\epsilon\|_{L^m(\mathbb{R}^d)}^m \, ds + 2d\epsilon e^t \int_0^t \|u_\epsilon\|_{L^1(\mathbb{R}^d)} \, ds \\ &+ e^t \int_0^t \int_{\mathbb{R}^d} u_\epsilon |\nabla v_\epsilon|^2 \, dxds, \qquad (90)\end{aligned}$$

since $e^{-t} < 1$ from $t > 0$. By using interpolation inequality for $1 < m < p+1$, we can obtain that

$$\int_0^t \|u_\epsilon\|_{L^m(\mathbb{R}^d)}^m \, ds \le C(T) \int_0^t \|u_\epsilon\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} \, ds \le C(T), \qquad (91)$$

for any $t \in (0, T]$.

Next we estimate $\int_0^t \int_{\mathbb{R}^d} u_\epsilon |\nabla v_\epsilon|^2 \, dxds$ in (90). Since $\|u_{0\epsilon}\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \le C$, from (40) in Proposition 1, we have

$$\|u_\epsilon\|_{L^\infty\left(0,t;L^{\frac{d}{2}}(\mathbb{R}^d)\right)} \le C. \qquad (92)$$

From Sobolev inequality and (69), one has that

$$\int_0^t \|\nabla v_\epsilon\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2 \, ds \le \frac{1}{S_d} \int_0^t \|\Delta v_\epsilon\|_{L^2(\mathbb{R}^d)}^2 \, ds \le C. \qquad (93)$$

Combining two estimates above and using Hölder's inequality, we obtain

$$\int_0^t \int_{\mathbb{R}^d} u_\epsilon |\nabla v_\epsilon|^2 \, dxds \le \int_0^t \|u_\epsilon\|_{L^{\frac{d}{2}}(\mathbb{R}^d)} \|\nabla v_\epsilon\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}^2 \, ds \le C(T). \qquad (94)$$

Until now, we have $m_2\left(u_\epsilon(\cdot,t)\right) \le C(T)$ for any $0 < t \le T$.

From the second equation of (59), it shows that

$$\frac{d}{dt} \int_{\mathbb{R}^d} v_\epsilon \, dx \le - \int_{\mathbb{R}^d} v_\epsilon \, dx + \int_{\mathbb{R}^d} u_{0\epsilon} \, dx. \qquad (95)$$

By using Gronwall's inequality, we have

$$\int_{\mathbb{R}^d} v_\epsilon \, dx \le \int_{\mathbb{R}^d} u_{0\epsilon} \, dx = \|u_0\|_{L^1(\mathbb{R}^d)}. \qquad (96)$$

Then for $m_2\big(v_\epsilon(\cdot,t)\big)$, one has that

$$\frac{d}{dt} m_2\big(v_\epsilon(\cdot,t)\big) \le 2d \int_{\mathbb{R}^d} v_\epsilon \, dx + m_2\big(u_\epsilon(\cdot,t)\big) \le C(T), \qquad (97)$$

i.e. $m_2\big(v_\epsilon(\cdot,t)\big) \le C(T)$ for any $0 < t \le T$.

By using $m_2\big(u_\epsilon(\cdot,t)\big) \le C(T)$ and $m_2\big(v_\epsilon(\cdot,t)\big) \le C(T)$, we obtain that for any $1 \le r_1 < p+1$, $1 \le r_2 < 2$

$$\begin{aligned}\int_0^T \|u_\epsilon\|_{L^{r_1}(|x|>R)}^{m+p-1}\,dt &\le \int_0^T \|u_\epsilon\|_{L^1(|x|>R)}^{(m+p-1)(1-\theta_1)}\|u_\epsilon\|_{L^{p+1}(|x|>R)}^{(m+p-1)\theta_1}\,dt\\ &\le \frac{1}{R^{2(m+p-1)(1-\theta_1)}}\int_0^T \big[m_2\big(u_\epsilon(\cdot,t)\big)\big]^{(m+p-1)(1-\theta_1)}\|u_\epsilon\|_{L^{p+1}(|x|>R)}^{(m+p-1)\theta_1}\,dt\\ &\le \frac{C(T)}{R^{2(m+p-1)(1-\theta_1)}} \to 0,\ \text{as } R\to\infty, \end{aligned} \tag{98}$$

where $\frac{1}{r_1} = \frac{1-\theta_1}{1} + \frac{\theta_1}{p+1}$, and

$$\begin{aligned}\int_0^T \|v_\epsilon\|_{L^{r_2}(|x|>R)}^{2}\,dt &\le \int_0^T \|v_\epsilon\|_{L^1(|x|>R)}^{2(1-\theta_2)}\|v_\epsilon\|_{L^{2}(|x|>R)}^{2\theta_2}\,dt\\ &\le \frac{1}{R^{4(1-\theta_2)}}\int_0^T \big[m_2\big(v_\epsilon(\cdot,t)\big)\big]^{2(1-\theta_2)}\|v_\epsilon\|_{L^{2}(|x|>R)}^{2\theta_2}\,dt\\ &\le \frac{C(T)}{R^{4(1-\theta_2)}} \to 0,\ \text{as } R\to\infty, \end{aligned} \tag{99}$$

where $\frac{1}{r_2} = \frac{1-\theta_2}{1} + \frac{\theta_2}{2}$. By weak semi-continuity of $L^{m+p-1}\big(0,T;L^{r_1}(|x|>R)\big)$ and $L^2\big(0,T;L^{r_2}(|x|>R)\big)$, we have

$$\int_0^T \|u\|_{L^{r_1}(|x|>R)}^{m+p-1}\,dt \le \liminf_{\epsilon\to0}\int_0^T \|u_\epsilon\|_{L^{r_1}(|x|>R)}^{m+p-1}\,dt \to 0, \quad \text{as } R\to\infty,$$
$$\int_0^T \|v\|_{L^{r_2}(|x|>R)}^{2}\,dt \le \liminf_{\epsilon\to0}\int_0^T \|v_\epsilon\|_{L^{r_2}(|x|>R)}^{2}\,dt \to 0, \quad \text{as } R\to\infty.$$

From (73), (77) and Hölder's inequality, one has that

$$\int_0^T \|u_\epsilon - u\|_{L^{r_1}(|x|\le R)}^{m+p-1}\,dt \to 0, \quad \text{as } \epsilon\to0, R\to\infty,$$
$$\int_0^T \|v_\epsilon - v\|_{L^{r_2}(|x|\le R)}^{2}\,dt \to 0, \quad \text{as } \epsilon\to0, R\to\infty,$$

and

$$\begin{aligned}&\int_0^T \|u_\epsilon - u\|_{L^{r_1}(\mathbb{R}^d)}^{m+p-1}\,dt = \int_0^T \Big(\|u_\epsilon - u\|_{L^{r_1}(|x|\le R)} + \|u_\epsilon - u\|_{L^{r_1}(|x|>R)}\Big)^{m+p-1}\,dt\\ &\le C\left[\int_0^T \|u_\epsilon - u\|_{L^{r_1}(|x|\le R)}^{m+p-1}\,dt + \int_0^T \|u_\epsilon\|_{L^{r_1}(|x|>R)}^{m+p-1}\,dt + \int_0^T \|u\|_{L^{r_1}(|x|>R)}^{m+p-1}\,dt\right]\\ &\to 0,\ \text{as } \epsilon\to0, R\to\infty, \end{aligned} \tag{100}$$

$$\begin{aligned}&\int_0^T \|v_\epsilon - v\|_{L^{r_2}(\mathbb{R}^d)}^{2}\,dt = \int_0^T \Big(\|v_\epsilon - v\|_{L^{r_2}(|x|\le R)} + \|v_\epsilon - v\|_{L^{r_2}(|x|>R)}\Big)^{2}\,dt\\ &\le C\left[\int_0^T \|v_\epsilon - v\|_{L^{r_2}(|x|\le R)}^{2}\,dt + \int_0^T \|v_\epsilon\|_{L^{r_2}(|x|>R)}^{2}dt + \int_0^T \|v\|_{L^{r_2}(|x|>R)}^{2}\,dt\right]\\ &\to 0,\ \text{as } \epsilon\to0, R\to\infty. \end{aligned} \tag{101}$$

Thus we have the following strong convergence in $\mathbb{R}^d$ for the weak solution

$$u_\epsilon \to u \text{ in } L^{m+p-1}\left(0,T;L^{r_1}(\mathbb{R}^d)\right), 1 \le r_1 < p+1, \tag{102}$$

$$v_\epsilon \to v \text{ in } L^2\left(0,T;L^{r_2}(\mathbb{R}^d)\right), 1 \le r_2 < 2. \tag{103}$$

Step 7. (Convergence of the free energy for $m > 1$). The free energy of the regularized problem is

$$\mathcal{F}\big(u_\epsilon(\cdot,t),v_\epsilon(\cdot,t)\big) = \frac{1}{m-1}\int_{\mathbb{R}^d} u_\epsilon^m \, dx - \int_{\mathbb{R}^d} u_\epsilon v_\epsilon \, dx + \frac{1}{2}\int_{\mathbb{R}^d} |\nabla v_\epsilon|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^d} v_\epsilon^2 \, dx. \tag{104}$$

In this step, we want to prove that as $\epsilon \to 0$,

$$\mathcal{F}\big(u_\epsilon(\cdot,t),v_\epsilon(\cdot,t)\big) \to \mathcal{F}\big(u(\cdot,t),v(\cdot,t)\big), \quad \text{a.e. in } (0,T).$$

Firstly, using the similar way of obtaining (80) and (82), we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |u_\epsilon^m - u^m| \, dxdt \\
&\le C(T)\left(\int_0^T \|u_\epsilon - u\|_{L^r(\mathbb{R}^d)}^{m+p-1} \, dt\right)^{\frac{1}{m+p-1}} \left(\int_0^T \|u_\epsilon\|_{L^{r'(m-1)}(\mathbb{R}^d)}^{m+p-1} \, dt\right)^{\frac{m-1}{m+p-1}} \\
&\to 0, \text{ as } \epsilon \to 0,
\end{aligned} \tag{105}$$

where

$$\frac{1}{r} + \frac{1}{r'} = 1,$$

$$1 < \frac{p+1}{p-m+2} < r < \frac{2}{3-m} < p+1,$$

$$2 < r'(m-1) < p+1,$$

and

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} |u_\epsilon v_\epsilon - uv| \, dxdt \\
&\le C(T)\left(\int_0^T \|u_\epsilon - u\|_{L^{s_1}(\mathbb{R}^d)}^{m+p-1} \, dt\right)^{\frac{1}{m+p-1}} \left(\int_0^T \|v_\epsilon\|_{L^{s_1'}(\mathbb{R}^d)}^{2} \, dt\right)^{\frac{1}{2}} \\
&+ C(T)\left(\int_0^T \|u\|_{L^{s_2}(\mathbb{R}^d)}^{m+p-1} \, dt\right)^{\frac{1}{m+p-1}} \left(\int_0^T \|v_\epsilon - v\|_{L^{s_2'}(\mathbb{R}^d)}^{2} \, dt\right)^{\frac{1}{2}} \\
&\to 0, \text{as } \epsilon \to 0,
\end{aligned} \tag{106}$$

where

$$\frac{1}{s_1} + \frac{1}{s_1'} = 1, \ \frac{1}{s_2} + \frac{1}{s_2'} = 1,$$

$$2 < s_1, s_2 < p+1, \ 1 < s_1', s_2' < 2.$$

Secondly, we estimate $\int_0^T \int_{\mathbb{R}^d} \left| |\nabla v_\epsilon|^2 - |\nabla v|^2 \right| \, dxdt$ and $\int_0^T \int_{\mathbb{R}^d} \left| {v_\epsilon}^2 - v^2 \right| \, dxdt$ together. We just give the detail of estimating $\int_0^T \int_{\mathbb{R}^d} \left| |\nabla v_\epsilon|^2 - |\nabla v|^2 \right| \, dxdt$, since the other one can be obtained in the similar way. From (7), (68) and (102) it shows that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}\left||\nabla v_\epsilon|^2-|\nabla v|^2\right|\,dxdt &\le C\int_0^T\|\nabla v_\epsilon-\nabla v\|_{L^2}\,dt\\ &\le C\int_0^T\|\Delta v_\epsilon-\Delta v\|_{L^{\frac{2d}{d+2}}}\,dt\\ &\le C(T)\Big(\int_0^T\|u_\epsilon-u\|_{L^{\frac{2d}{d+2}}}^{m+p-1}\,dt\Big)^{\frac{1}{m+p-1}}\\ &\to 0,\quad \text{as } \epsilon\to 0,\end{aligned}$$

where $m+p-1>\frac{2d}{d+2}$ since $1<m<2-\frac{2}{d}$. From estimates above, we have that as $\epsilon\to 0$,

$$\mathcal{F}\big(u_\epsilon(\cdot,t),v_\epsilon(\cdot,t)\big)\to\mathcal{F}\big(u(\cdot,t),v(\cdot,t)\big),\quad \text{a.e. in } (0,T).$$

Step 8. (Lower Semi-continuity of the dissipation term for $m>1$). With the extra assumption $u_0\in L^m(\mathbb{R}^d)$ when $\frac{2d}{d+2}<m<2-\frac{2}{d}$, we know that $u_0\in L^1_+\cap L^p\cap L^m(\mathbb{R}^d)$ for all $1<m<2-\frac{2}{d}$ and $\|u_{0\epsilon}\|_{L^m(\mathbb{R}^d)}\le C$. By denoting $q:=\max\{m,p\}$ and using the similar method in Step 1 of Theorem 3.1, we have for any $T>0$

$$\left\|\nabla u_\epsilon^{\frac{m+r-1}{2}}\right\|_{L^2\left(0,T;L^2(\mathbb{R}^d)\right)}\le C,\quad \text{for } 1<r\le q. \tag{107}$$

The dissipation term satisfies

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u_\epsilon^{m-\frac12}-\sqrt{u_\epsilon}\nabla v_\epsilon\right|^2\,dxdt+\int_0^T\int_{\mathbb{R}^d}|\partial_t v_\epsilon|^2\,dxdt\\ \le\ &2\int_0^T\int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u_\epsilon^{m-\frac12}\right|^2\,dxdt+2\int_0^T\int_{\mathbb{R}^d}u_\epsilon|\nabla v_\epsilon|^2\,dxdt\\ &+\int_0^T\int_{\mathbb{R}^d}|\partial_t v_\epsilon|^2\,dxdt.\end{aligned}$$

From (107) by taking $r=m$ and (94), we have for any $T>0$

$$\int_0^T\int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u_\epsilon^{m-\frac12}\right|^2\,dxdt\le C,$$

$$\int_0^T\int_{\mathbb{R}^d}u_\epsilon|\nabla v_\epsilon|^2\,dxdt\le C.$$

Then the first term in dissipation is uniformly bounded, i.e.

$$\int_0^T\int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u_\epsilon^{m-\frac12}-\sqrt{u_\epsilon}\nabla v_\epsilon\right|^2\,dxdt\le C.$$

Furthermore, there exists a subsequence of $\frac{2m}{2m-1}\nabla u_\epsilon^{m-\frac12}-\sqrt{u_\epsilon}\nabla v_\epsilon$ without relabeling which weakly converges to f in $L^2\left(0,T;L^2(\mathbb{R}^d)\right)$. By the lower semi-continuity of L^2 norm, we obtain for any $T>0$,

$$\|f\|_{L^2\left(0,T;L^2(\mathbb{R}^d)\right)}\le\liminf_{\epsilon\to 0}\left\|\frac{2m}{2m-1}\nabla u_\epsilon^{m-\frac12}-\sqrt{u_\epsilon}\nabla v_\epsilon\right\|_{L^2\left(0,T;L^2(\mathbb{R}^d)\right)}\le C.$$

Now we will prove that the weak limit $f=\frac{2m}{2m-1}\nabla u^{m-\frac12}-\sqrt{u}\nabla v$.

For any test function $\psi \in C_c^\infty\left([0,T)\times\mathbb{R}^d\right)$ which is dense in $H^1\left([0,T)\times\mathbb{R}^d\right)$, it turns to prove

$$\int_0^T\int_{\mathbb{R}^d}\left(\frac{2m}{2m-1}u_\epsilon^{m-\frac{1}{2}}\nabla\psi+\sqrt{u_\epsilon}\nabla v_\epsilon\psi\right)\,dxdt \to \int_0^T\int_{\mathbb{R}^d}\left(\frac{2m}{2m-1}u^{m-\frac{1}{2}}\nabla\psi+\sqrt{u}\nabla v\psi\right)\,dxdt. \tag{108}$$

From (81) by taking $m-\frac{1}{2}$ instead of m which is reasonable since we consider $1<m<2-\frac{2}{d}$ here, we have

$$u_\epsilon^{m-\frac{1}{2}}\to u^{m-\frac{1}{2}},\quad \text{in } L^1\left(0,T;L^1(\Omega)\right),$$

i.e.

$$\int_0^T\int_{\mathbb{R}^d}\frac{2m}{2m-1}u_\epsilon^{m-\frac{1}{2}}\nabla\psi\,dxdt\to\int_0^T\int_{\mathbb{R}^d}\frac{2m}{2m-1}u^{m-\frac{1}{2}}\nabla\psi\,dxdt. \tag{109}$$

Next from (75) and (81), we obtain

$$\int_0^T\int_\Omega\left|\sqrt{u_\epsilon}\nabla v_\epsilon-\sqrt{u}\nabla v\right|\,dxdt\le\|u_\epsilon\|^{\frac{1}{2}}_{L^1\left(0,T;L^1(\Omega)\right)}\|\nabla v_\epsilon-\nabla v\|_{L^2\left(0,T;L^2(\Omega)\right)} + \|u_\epsilon-u\|^{\frac{1}{2}}_{L^1\left(0,T;L^1(\Omega)\right)}\|\nabla v\|_{L^2\left(0,T;L^2(\Omega)\right)}\longrightarrow 0,\quad \text{as } \epsilon\to 0,$$

i.e.

$$\int_0^T\int_{\mathbb{R}^d}\sqrt{u_\epsilon}\nabla v_\epsilon\psi dxdt\to\int_0^T\int_{\mathbb{R}^d}\sqrt{u}\nabla v\psi dxdt. \tag{110}$$

Combining (109) and (110), we have proved (108), i.e. $f=\frac{2m}{2m-1}\nabla u^{m-\frac{1}{2}}-\sqrt{u}\nabla v$. Then for any $T>0$, we obtain lower semi-continuity of the first term in dissipation

$$\int_0^T\int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u^{m-\frac{1}{2}}-\sqrt{u}\nabla v\right|^2dxdt \le\liminf_{\epsilon\to 0}\int_0^T\int_{\mathbb{R}^d}\left|\frac{2m}{2m-1}\nabla u_\epsilon^{m-\frac{1}{2}}-\sqrt{u_\epsilon}\nabla v_\epsilon\right|^2dxdt. \tag{111}$$

Next we will use the same method to prove the lower semi-continuity of the second term in dissipation. From the second equation of (1), using (67) and (69), we have

$$\int_0^T\|\partial_t v_\epsilon\|^2_{L^2(\mathbb{R}^d)}\,dt\le C\int_0^T\|\Delta v_\epsilon\|^2_{L^2}\,dt+C\int_0^T\|v_\epsilon\|^2_{L^2}\,dt+C\int_0^T\|u_\epsilon\|^2_{L^2}\,dt \le C.$$

Then there exists a subsequence of $\partial_t v_\epsilon$ without relabeling which weakly converges to g in $L^2\left(0,T;L^2(\mathbb{R}^d)\right)$. Also by the lower semi-continuity of L^2 norm, we obtain that for any $T>0$

$$\|g\|_{L^2\left(0,T;L^2(\mathbb{R}^d)\right)}\le\liminf_{\epsilon\to 0}\|\partial_t v_\epsilon\|_{L^2\left(0,T;L^2(\mathbb{R}^d)\right)}.$$

We will prove $g=\partial_t v$. Choosing any test function $\psi\in C_c^\infty\left([0,T)\times\mathbb{R}^d\right)$, we have

$$\int_0^T\int_{\mathbb{R}^d}v_\epsilon\partial_t\psi\,dxdt\to\int_0^T\int_{\mathbb{R}^d}v\partial_t\psi\,dxdt,$$

directly from (86). Then it turns that

$$\int_0^T \int_{\mathbb{R}^d} |\partial_t v|^2 \, dxdt \le \liminf_{\epsilon\to 0} \int_0^T \int_{\mathbb{R}^d} |\partial_t v_\epsilon|^2 \, dxdt. \tag{112}$$

From (111) and (112), the dissipation term satisfies for any $T > 0$

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \left| \frac{2m}{2m-1} \nabla u^{m-\frac{1}{2}} - \sqrt{u} \nabla v \right|^2 dxdt + \int_0^T \int_{\mathbb{R}^d} |\partial_t v|^2 \, dxdt \\
&\le \liminf_{\epsilon\to 0} \left(\int_0^T \int_{\mathbb{R}^d} \left| \frac{2m}{2m-1} \nabla u_\epsilon^{m-\frac{1}{2}} - \sqrt{u_\epsilon} \nabla v_\epsilon \right|^2 dxdt + \int_0^T \int_{\mathbb{R}^d} |\partial_t v_\epsilon|^2 \, dxdt \right).
\end{aligned}$$

Step 9. (Weak entropy solution with the energy inequality for $1 < m < 2 - \frac{2}{d}$). Multiplying the first equation in (59) by $\frac{m}{m-1} u_\epsilon^{m-1} - v_\epsilon$ and integrating over $\mathbb{R}^d$ shows that

$$\begin{aligned}
\frac{1}{m-1} \frac{d}{dt} \int_{\mathbb{R}^d} u_\epsilon^m \, dx - \int_{\mathbb{R}^d} v_\epsilon \partial_t u_\epsilon \, dx + \int_{\mathbb{R}^d} u_\epsilon \left| \nabla \left(\frac{m}{m-1} u_\epsilon^{m-1} - v_\epsilon \right) \right|^2 dx \\
+ \frac{4\epsilon}{m} \int_{\mathbb{R}^d} \left| \nabla u_\epsilon^{\frac{m}{2}} \right|^2 dx = \epsilon \int_{\mathbb{R}^d} \nabla u_\epsilon \cdot \nabla v_\epsilon \, dx.
\end{aligned} \tag{113}$$

Multiplying the second equation in (59) by $\partial_t v_\epsilon$ and integrating over $\mathbb{R}^d$ turns that

$$\int_{\mathbb{R}^d} |\partial_t v_\epsilon|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |\nabla v_\epsilon|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} {v_\epsilon}^2 \, dx - \int_{\mathbb{R}^d} u_\epsilon \partial_t v_\epsilon \, dx = 0. \tag{114}$$

Then from two equations above, integrating from 0 to t, we have

$$\begin{aligned}
\mathcal{F}\big(u_\epsilon(t), v_\epsilon(t)\big) + \int_0^t \int_{\mathbb{R}^d} u_\epsilon \left| \nabla \left(\frac{m}{m-1} u_\epsilon^{m-1} - v_\epsilon \right) \right|^2 dxds + \int_0^t \int_{\mathbb{R}^d} |\partial_t v_\epsilon|^2 \, dxds \\
\le \mathcal{F}(0) + \epsilon \int_0^t \int_{\mathbb{R}^d} \nabla u_\epsilon \cdot \nabla v_\epsilon \, dxds.
\end{aligned} \tag{115}$$

From (67) and (69), one has that for any $t > 0$

$$\int_0^t \int_{\mathbb{R}^d} \nabla u_\epsilon \cdot \nabla v_\epsilon \, dxds \le \|u_\epsilon\|_{L^2\left(0,t;L^2(\mathbb{R}^d)\right)} \|\Delta v_\epsilon\|_{L^2\left(0,t;L^2(\mathbb{R}^d)\right)} \le C.$$

Then combining the convergence of the free energy and the lower semi-continuity of dissipation term, by letting $\epsilon \to 0$, there exists a global weak entropy solution which satisfies the energy inequality

$$\begin{aligned}
\mathcal{F}\big(u(t), v(t)\big) + \int_0^t \int_{\mathbb{R}^d} u \left| \nabla \left(\frac{m}{m-1} u^{m-1} - v \right) \right|^2 dxds + \int_0^t \int_{\mathbb{R}^d} |\partial_t v|^2 \, dxds \\
\le \mathcal{F}(0), \quad \text{a.e. } t > 0.
\end{aligned}$$

□

6. **Local existence of a weak entropy solution and a blow-up criterion.** In this section, we prove that for $u_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$, a weak entropy solution of (1) exists locally without any restriction for the size of initial data. Furthermore, we also prove that if a weak solution blows up in finite time, then all L^q-norms of the weak solution blow up at the same time for $q \in (p, +\infty)$.

Theorem 6.1. *(Local existence of a weak entropy solution) Let $d \geq 3$, $1 < m < 2 - \frac{2}{d}$ and $p = \frac{d(2-m)}{2}$. Assume $u_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$ and the initial second moment $\int_{\mathbb{R}^d} |x|^2 u_0(x)dx < \infty$. Then there are $T > 0$, such that (1) has a weak entropy solution in $0 < t < T$ with properties*

$$\int_{\mathbb{R}^d} u(x,t)\ dx = \int_{\mathbb{R}^d} u_0(x)\ dx, \quad \textit{for all } t \in (0,T)\,,$$

$$\int_{\mathbb{R}^d} v(x,t)\ dx \leq \int_{\mathbb{R}^d} u_0(x)\ dx, \quad \textit{for all } t \in (0,T)\,.$$

Proof. Take any fixed $q > p$. Using the same way of obtaining (16) and taking $q = r > p$ in (9), we have

$$\begin{aligned}
\frac{d}{dt}\|u(\cdot,t)\|^q_{L^q(\mathbb{R}^d)} + \frac{4mq(q-1)}{(q+m-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|^2_{L^2(\mathbb{R}^d)} &\leq (q-1)C_q\|u\|^{q+1}_{L^{q+1}(\mathbb{R}^d)} \\
&\leq -\frac{2mq(q-1)}{(q+m-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|^2_{L^2(\mathbb{R}^d)} + C(q,d)\left(\|u\|^q_{L^q(\mathbb{R}^d)}\right)^{1+\frac{1}{q-p}}
\end{aligned}$$

i.e.

$$\frac{d}{dt}\|u(\cdot,t)\|^q_{L^q(\mathbb{R}^d)} \leq C(q,d)\left(\|u\|^q_{L^q(\mathbb{R}^d)}\right)^{1+\frac{1}{q-p}}. \tag{116}$$

Solving the inequality (116) shows that

$$\|u(\cdot,t)\|^q_{L^q(\mathbb{R}^d)} \leq \left[\frac{\frac{q-p}{C(q,d)}}{\frac{q-p}{C(q,d)}\left(\|u_0\|^q_{L^q(\mathbb{R}^d)}\right)^{\frac{1}{p-q}} - t}\right]^{q-p}. \tag{117}$$

Denoting $T_q := \frac{q-p}{C(q,d)}\left(\|u_0\|^q_{L^q(\mathbb{R}^d)}\right)^{\frac{1}{p-q}}$, then for any fixed q, we choose $0 < T < T_q$. Next by the same way of proving Theorem 5.1, there exists a local in time weak entropy solution with properties

$$\int_{\mathbb{R}^d} u(x,t)\ dx = \int_{\mathbb{R}^d} u_0(x)\ dx, \quad \text{for all } t \in (0,T)\,,$$

$$\int_{\mathbb{R}^d} v(x,t)\ dx \leq \int_{\mathbb{R}^d} u_0(x)\ dx, \quad \text{for all } t \in (0,T)\,,$$

where the second one is obtained by (96). □

Proposition 2. *(Blow-up criterion) Under the same assumptions as Theorem 6.1 and $r = p + \epsilon$ where ϵ is small enough, let $T^r_{\max}$ be the largest L^r-norm existence time of a weak solution, i.e.*

$$\|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} < \infty, \quad \textit{for all } 0 < t < T^r_{\max},$$

$$\limsup_{t \to T^r_{\max}} \|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} = \infty,$$

and $T^q_{\max}$ be the largest L^q-norm existence time of a weak solution for $q \geq r > p$. Then if $T^q_{\max} < \infty$ for any q,

$$T^q_{\max} = T^r_{\max}, \quad \textit{for all } q \geq r.$$

Proof. Since $\|u(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)}$, by using interpolation inequality, we know that for $q \ge r$, $T^q_{\max} \le T^r_{\max}$. If $T^q_{\max} < T^r_{\max}$ for any $q \ge r$, then we will have contradiction arguments. $T^q_{\max} < T^r_{\max}$ implies

$$\limsup_{t\to T^q_{\max}} \|u(\cdot,t)\|_{L^r(\mathbb{R}^d)} =: A < \infty.$$

Then using the similar way of obtaining (116) and taking $q \ge r > p$, we have

$$\frac{d}{dt}\|u(\cdot,t)\|^q_{L^q(\mathbb{R}^d)} \le C(q,r,d)\Big(\|u\|^r_{L^r(\mathbb{R}^d)}\Big)^{1+\frac{1+q-r}{r-p}} \le C(q,r,d,A), \tag{118}$$

i.e.

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le C\left(q,r,A,\|u_0\|_{L^q(\mathbb{R}^d)},T^q_{\max}\right), \text{ for } t\in(0,T^q_{\max}),$$

which contradicts with

$$\limsup_{t\to T^q_{\max}} \|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} = \infty.$$

Thus we have the conclusion that $T^q_{\max} = T^r_{\max}$ for all $q \ge r > p$, i.e. L^q-norms blow up at the same time. □

Acknowledgments. The work of J.-G. Liu is partially supported by KI-Net NSF RNMS grant No. 1107291 and NSF grant DMS 1514826. Wenting Cong is partially supported by National Natural Science Foundation of China grant No. 11271154 and wishes to express her gratitude to the China Scholarship Council for the scholarship and Professor Wenjie Gao for his support and encouragement.

REFERENCES

[1] S. Bian and J.-G. Liu, Dynamic and steady states for multi-dimensional Keller-Segel model with diffusion exponent $m > 0$, *Comm. Math. Phys.*, **323** (2013), 1017–1070.

[2] S. Bian, J.-G. Liu and C. Zou, Ultra-contractivity for Keller-Segel model with diffusion exponent $m > 1 - 2/d$, *Kinet. Relat. Models*, **7** (2014), 9–28.

[3] A. Blanchet, J. Carrillo and N. Masmoudi, Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$, *Comm. Pure Appl. Math.*, **61** (2008), 1449–1481.

[4] M. Brenner, P. Constantin, L. Kadanoff, A. Schenkel and S. Venkataramani, Diffusion, attraction and collapse, *Nonlinearity*, **12** (1999), 1071–1098.

[5] V. Calvez and L. Corrias, The parabolic-parabolic Keller-Segel model in $\mathbb{R}^2$, *Commun. Math. Sci.*, **6** (2008), 417–447.

[6] X. Chen, A. Jüngel and J.-G. Liu, A note on Aubin-Lions-Dubinskiĭ lemmas, *Acta Appl. Math.*, **133** (2014), 33–43.

[7] L. Corrias and B. Perthame, Critical space for the parabolic-parabolic Keller–Segel model in $\mathbb{R}^d$, *C. R. Math. Acad. Sci. Paris*, **342** (2006), 745–750.

[8] L. Evans, *Partial Differential Equations*, Second edition, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.

[9] M. A. Herrero, E. Medina and J. J. L. Velázquez, Finite-time aggregation into a single point in a reaction-diffusion system, *Nonlinearity*, **10** (1997), 1739–1754.

[10] M. Hieber and J. Prüss, Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations, *Comm. Partial Differential Equations*, **22** (1997), 1647–1669.

[11] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I, *Jahresber. Deutsch. Math.-Verein.*, **105** (2003), 103–165.

[12] S. Ishida and T. Yokota, Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic–parabolic type, *J. Differential Equations*, **252** (2012), 2469–2491.

[13] S. Ishida and T. Yokota, Blow-up in finite or infinite time for quasilinear degenerate Keller-Segel systems of parabolic-parabolic type, *Discrete Contin. Dyn. Syst. Ser. B*, **18** (2013), 2569–2596.

[14] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theoret. Biol.*, **26** (1970), 399–415.

[15] P. Kunstmann and L. Weis, Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, *Lecture Notes in Math.*, **1855** (2004), 65–311.

[16] J.-G. Liu and J. Wang, A note on L^∞-bound and uniqueness to a degenerate Keller-Segel model, *Acta Appl. Math.*, **142** (2016), 173–188.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[18] B. Perthame, *Transport Equations in Biology*, Birkhäuser Verlag, Basel, 2007.

[19] Y. Sugiyama, Time global existence and asymptotic behavior of solutions to degenerate quasi-linear parabolic systems of chemotaxis, *Differential Integral Equations*, **20** (2007), 133–180.

[20] Y. Sugiyama and H. Kunii, Global existence and decay properties for a degenerate Keller-Segel model with a power factor in drift term, *J. Differential Equations*, **227** (2006), 333–364.

[21] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pures Appl.*, **100** (2013), 748–767.

Received February 2016; revised October 2016.

E-mail address: congwt13@mails.jlu.edu.cn
E-mail address: jliu@phy.duke.edu