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Well-Posedness and Singular Limit of a Semilinear Hyperbolic Relaxation System with a Two-Scale Discontinuous Relaxation Rate

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Abstract

Nonlinear hyperbolic systems with relaxations may encounter different scales of relaxation time, which is a prototype multiscale phenomenon that arises in many applications. In such a problem the relaxation time is of $O(1)$ in part of the domain and very small in the remaining domain in which the solution can be approximated by the zero relaxation limit which can be solved numerically much more efficiently. For the Jin–Xin relaxation system in such a two-scale setting, we establish its well-posedness and singular limit as the (smaller) relaxation time goes to zero. The limit is a multiscale coupling problem which couples the original Jin–Xin system on the domain when the relaxation time is $O(1)$ with its relaxation limit in the other domain through interface conditions which can be derived by matched interface layer analysis. As a result, we also establish the well-posedness and regularity (such as boundedness in sup norm with bounded total variation and L^1-contraction) of the coupling problem, thus providing a rigorous mathematical foundation, in the general nonlinear setting, to the multiscale domain decomposition method for this two-scale problem originally proposed in Jin et al. in *Math. Comp.* **82**, 749–779, 2013.

1. Introduction

Consider the hyperbolic relaxation system proposed by Jin and Xin [16]

$$\begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0, & (1.1a) \\ \partial_t v^\varepsilon + a^2 \partial_x u^\varepsilon = -\lambda(x,\varepsilon)(v^\varepsilon - f(u^\varepsilon)), & (1.1b) \end{cases}$$

for some small relaxation parameter ε. Here the relaxation rate λ depends on the space variable x in the following way:

$$\lambda(x,\varepsilon) = \begin{cases} 1, & x < 0, \\ \frac{1}{\varepsilon}, & x > 0, \quad 0 < \varepsilon \ll 1. \end{cases} \tag{1.2}$$

This is a prototype example of multiscale problems in which one encounters a drastic change of time or spatial scales within one problem. One famous example is the space shuttle reentry problem in which the space shuttle encounters transition from rarefied gas (described by the Boltzmann equation) to dense gas (described by fluid equations) [4,9,18]. Other examples are transport equations between differential materials or media [1,8,10,13], and other multiscale problems [11]. While a complete mathematical theory for the coupling between the (mesoscopic) Boltzmann equation and the (macroscopic) fluid dynamics equations remains one of the major challenges in PDEs and mathematical physics, a thorough mathematical analysis of the problem in question sheds some light in this direction, and is of significant interest in understanding some of the multiscale computational methods.

In the relaxation parameter $\lambda(x,\varepsilon)$, uniform in x with $\lambda(x,\varepsilon) = 1/\varepsilon$ for all $x \in \mathbf{R}$ and under the well-known sub-characteristic condition:

$$|f'(u)| < a, \tag{1.3}$$

for all the u under consideration, it has been proved [3,7,17,19,23,24] that the solution $\{(u^\varepsilon, v^\varepsilon)\}_{\varepsilon>0}$ of (1.1) converges as ε goes to zero to a limit (u, v) with $v = f(u)$ where u is the entropy weak solution to the nonlinear scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \ t > 0. \tag{1.4}$$

Since (1.4) is the macroscopic equation that does not depend on the small parameter ε, it is computationally more efficient to solve this equation rather than the original system (1.1) in the domain $x > 0$. However, one needs to couple this equation with the original system (1.1) for $x < 0$ through some interface condition at $x = 0$ that provides the transmission of data from one scale to the other, or from one domain to the other.

In [15], such a domain coupling method was provided as the following: When $f'(u) < 0$ for all u in between $u(t, 0^-)$ and $u(t, 0^+)$,

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \qquad x > 0, \ t > 0, & \text{(1.5a)} \\ v(t,x) = f(u(t,x)), & \text{(1.5b)} \\ u(0,x) = u_0(x), & \text{(1.5c)} \end{cases}$$

$$\begin{cases} \partial_t u + \partial_x v = 0, \qquad x < 0, \ t > 0, & \text{(1.6a)} \\ \partial_t v + a^2 \partial_x u = f(u) - v, & \text{(1.6b)} \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & \text{(1.6c)} \\ v(t,0^-) = v(t,0^+) = f(u(t,0^+)); & \text{(1.6d)} \end{cases}$$

when $f'(u) > 0$ for all u in between $u(t, 0^-)$ and $u(t, 0^+)$

$$\begin{cases} \partial_t u + \partial_x v = 0, \qquad\qquad x < 0,\ t > 0, & (1.7a) \\ \partial_t v + a^2 \partial_x u = f(u) - v, & (1.7b) \\ u(0, x) = u_0(x), v(0, x) = v_0(x), & (1.7c) \\ v(t, 0^-) = f(u(t, 0^-)), & (1.7d) \end{cases}$$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \qquad x > 0,\ t > 0, & (1.8a) \\ v(t, x) = f(u(t, x)), & (1.8b) \\ u(0, x) = u_0(x), & (1.8c) \\ u(t, 0^+) = u(t, 0^-). & (1.8d) \end{cases}$$

Based on the condition that the flux v is continuous across the interface, here the interface condition is constructed through a matched interface layer analysis, which depends on the sign of $f'(u)$ at the interface. Specifically, when $f'(u) < 0$, there will be an interface layer in u but not in v around $x = 0^+$; then one should solve the conservation law in the right domain first and then transfer the value of $v(t, 0)$ to the left, see (1.6d). On the other hand, when $f'(u) > 0$, there is no interface layer in u or v to the leading order, one just uses $v(t, 0) = f(u(t, 0^-))$ as the interface condition for the relaxation system in the left domain, and then uses the value $u(t, 0^-)$ as the boundary condition for the right region (See 1.7d).

An important property of the domain coupling method is that the two domains are now completely decoupled, and it can be solved numerically in one domain first and then the second domain, using any high resolution shock capturing method. The method can be easily extended to more complicated cases such as dynamic interfaces. (See [15]).

For the *linear* case when $f(u) = \lambda u$ with $|\lambda| < 1$, the authors in [15] showed that the solution of the original system (1.1) and (1.2) is strictly well-posed in the sense that the L^2 norm of solution is bounded by the L^2 norm of the initial and boundary data. Then they proved the asymptotic convergence of (1.1) and (1.2) to the decoupled system (1.5)–(1.6) or (1.7)–(1.8) as $\varepsilon \to 0$ and obtained the optimal convergence rate.

This paper studies the case of *nonlinear* flux $f(u)$. We first establish the well-posedness of the or relaxation system (1.1) with a discontinuous relaxation rate (1.2), and then prove asymptotic limit as $\varepsilon \to 0$. To deal with possible strong oscillations developed at the interface $x = 0$ in the limit $\varepsilon \to 0$, for technical reasons, we assume that the interface is not characteristic; that is, wave velocities do not vanish: namely for some real constant $C_0 > 0$, either

$$0 < C_0 \leqq f'(u), \quad \text{for all } u \in \mathbf{R}, \tag{1.9}$$

or

$$f'(u) \leqq -C_0 < 0, \quad \text{for all } u \in \mathbf{R}. \tag{1.10}$$

Under this assumption, no shock wave can stick to the interface. Without real restriction, we will assume throughout the paper that the flux function f belongs to $C^1(\mathbf{R})$ with $f(0) = 0$. The initial data u_0 is assumed to be compactly supported with

$$u_0(x) \in L^\infty(\mathbf{R}) \cap \mathrm{BV}(\mathbf{R}). \tag{1.11}$$

Then, v_0 is chosen well-prepared (the initial data are in the equilibrium state)

$$v_0(x) = f(u_0(x)), \tag{1.12}$$

so that $v_0 \in L^\infty(\mathbf{R}) \cap \mathrm{BV}(\mathbf{R})$, and is also compactly supported since $f(0) = 0$. This condition is not mandatory on the left part of the domain, here we make this assumption for simplicity. Introduce

$$\begin{aligned} N_0 &= \max(\| u_0 \|_{L^\infty(\mathbf{R})}, \| f(u_0) \|_{L^\infty(\mathbf{R})}), \\ F(N_0) &= \sup_{|\xi|<N_0} |f(\xi)|, \quad B(N_0) = 2N_0 + F(2N_0) \\ M(N_0) &= \sup_{|\xi| \leqq B(N_0)} |f'(\xi)|, \end{aligned} \tag{1.13}$$

the sub-characteristic condition is specified as follows [19]

$$a > M(N_0). \tag{1.14}$$

Under these assumptions, we also obtain the well-posedness of the coupling problem, as well as the regularity of its solution. The results are summarized by the following theorem.

Theorem 1. *Given initial data u_0 satisfying (1.11) and v_0 well-prepared according to (1.12). Assume (1.9) or (1.10) and let $T > 0$ be any real number.*

1. *For any given $\varepsilon > 0$, there exists a unique global solution $(u^\varepsilon, v^\varepsilon)$ to the original two-scale problem (1.1) and (1.2).*
2. *There exists a unique solution (u, v) in $L^1 \cap L^\infty \cap BV((0, T) \times \mathbf{R}_x)^2$ of the coupling problem (1.16)–(1.17). Moreover, two such solutions (u^1, v^1) and (u^2, v^2) verify for almost all t in $(0, T)$ the L^1-contraction property:*

$$\begin{aligned} &\frac{1}{2a} \| r_+^1(t,\cdot) - r_+^2(t,\cdot) \|_{L^1(\mathbf{R}_x^-)} \\ &\quad + \frac{1}{2a} \| r_-^1(t,\cdot) - r_-^2(t,\cdot) \|_{L^1(\mathbf{R}_x^-)} + \| u^1(t,\cdot) - u^2(t,\cdot) \|_{L^1(\mathbf{R}_x^+)} \\ &\leqq \| u_0^1 - u_0^2 \|_{L^1(\mathbf{R}_x)}, \end{aligned} \tag{1.15}$$

   where $r_\pm^i = (au^i \pm v^i)(t,x)$, $i = 1, 2$.
3. *As $\varepsilon \to 0$, the family of two-scale solutions $\{(u^\varepsilon, v^\varepsilon)\}_{\varepsilon>0}$ converges in the $L^1((0,T) \times \mathbf{R}_x)^2$ topology to the unique solution (u, v) of the initial value problem (1.16)–(1.17) with initial data $(u_0, f(u_0))$.*

Let us briefly report on the technical difficulties and the tools we have developed to overcome them. As expected, the multiscale nature of the PDEs (1.1)–(1.2) is responsible for the difficulties: the relaxation rate $\lambda(., \varepsilon)$ in (1.2) is non-homogeneous in the space variable x and furthermore discontinuous. These two original features actually make it challenging to derive a uniform BV estimate for the sequences $\{u^\varepsilon\}_{\varepsilon>0}$ and $\{v^\varepsilon\}_{\varepsilon>0}$. First, the lack of smoothness in the space variable of the coefficients of the PDEs (1.1) obviously makes the standard approach based on a direct differentiation of the equations with respect to x (see [24] for instance) impossible. In the case of a constant relaxation rate, such a procedure allows us to readily infer L^1 estimates of the space derivatives of the unknowns u^ε and v^ε from the quasi-monotone property of the differentiated equations. Another classical approach directly makes use of the L^1-contraction principle underlying the Jin–Xin model with a single relaxation scale ε [17,20]. This approach heavily relies on the invariance of those PDEs with respect to shifts in the space variable; shifting any given solution (u, v) in x gives birth to a new solution while shifting correspondingly the initial data (u_0, v_0). Uniform BV estimates for u^ε and v^ε are then easily inferred. However in the present setting, the non-homogeneity of the relaxation rate λ in x clearly prevents the multiscale equations from the reported invariance property.

To bypass these obstacles, we propose to take advantage of the fact that the coefficients in these equations are independent of the time variable. The leading idea is to derive L^1 estimates for the time derivatives of the unknowns in order to infer corresponding estimates for the space derivatives from the governing PDEs. Let us immediately stress that the resulting estimate fails to be uniform in ε regarding the sequence $\{u^\varepsilon\}_{\varepsilon>0}$. Nevertheless, we will be a position to get the strong convergence of this sequence in a relevant topology using the property that the interface $x = 0$ is not characteristic thanks to either assumption (1.9) or (1.10).

To implement this program, a first suitable regularization of the coefficients and data for the multiscale system (1.1)–(1.2) yields a Cauchy problem for which the standard theory (see Protter and Weinberger [22]) ensures the existence and uniqueness of smooth solutions $u^{\varepsilon,\delta}$ and $u^{\varepsilon,\delta}$ for any given regularizing parameter $\delta > 0$ while by essence, the scale $\varepsilon > 0$ is kept fixed. Differentiating the regularized equations in the time variable gives rise to a system for governing the time derivatives of $u^{\varepsilon,\delta}$ and $u^{\varepsilon,\delta}$ which is quasi-monotone under a natural sub-characteristic condition. Such a property allows us to derive uniform L^1 estimates for the time derivatives under consideration since the well-prepared initial data prevents us from any boundary layer at $t = 0$. The governing equations in the original PDEs (1.1) then provides natural L^1 estimates of the space derivatives of the unknown. Letting the regularizing parameter δ go to zero, we establish that uniform L^1 in time continuity holds for both u^ε and v^ε. We further obtain a uniform BV estimate for the sequence $\{v^\varepsilon\}_{\varepsilon>0}$. But a corresponding uniform BV estimate for $\{u^\varepsilon\}_{\varepsilon>0}$ stays unknown. Nevertheless, we prove that the entropy dissipation rate coming with the multiscale relaxation procedure allows us to recover the strong convergence of $\{u^\varepsilon\}_{\varepsilon>0}$ to a limit u in $L^1([0, T] \times \mathbf{R})$, for any given $T > 0$, from the strong convergence of the sequence $\{v^\varepsilon\}_{\varepsilon>0}$ in this topology. Here, the property that the interface $x = 0$ is not characteristic in view of either (1.9) or (1.10) plays a central role. Again thanks to the entropy dissipation estimate and the non-characteristic property, the limit u is proved to be bounded in the BV semi-norm. Both limits u and v are identified in each half-plane $\mathbf{R}_t^+ \times \mathbf{R}_x^-$ and $\mathbf{R}_t^+ \times \mathbf{R}_x^+$. The structure of the interface relaxation layer at $x = 0$ is revealed using a blow-up technique in order to perform a matched asymptotic analysis. The profile solutions $(\mathcal{U}^{\varepsilon,\delta}, \mathcal{V}^{\varepsilon,\delta})$ are proved to be bounded in sup-norm with bounded local total variation. It is nevertheless enough to pass to the limit thanks to the ODE dynamics underlying the problem with a stable attractive critical state at infinity. The natural monotonicity property of the relaxation profile in $\mathcal{U}$ and underlying Kruzkov like entropy inequalities show a perfect match to the outer solution (u, v).

Other than its value for the understanding of a multiscale and multiphysics coupling computational method, the problem under study also provides a mathematical foundation for the following interesting mathematical problem: given a 2×2 hyperbolic system (1.1) (with $\lambda = O(1)$ for $x < 0$ and the scalar conservation law (1.4) for $x > 0$, how does one couple these two sets of equations through an interface condition at $x = 0$ in a mathematically well-posed way, assuming that the flux v is continuous across $x = 0$? Our study provides such a mathematical framework: first one lifts (or regularizes) the problem in the domain $x > 0$ via the relaxation (1.1) with relaxation rate given by (1.2), and then takes the limit $\varepsilon \to 0$. One then ends up with the coupling system (1.5)–(1.8).

Finally, we also propose a coupling for *general nonlinear* flux f (regardless of the sign of $f'(u(t, 0^+))$):

$$\begin{cases} \partial_t u + \partial_x v = 0, \ x < 0, \ t > 0 & (1.16a) \\ \partial_t v + a^2 \partial_x u = f(u) - v, & (1.16b) \\ v(t, 0^-) = f(u(t, 0^+)), & (1.16c) \\ u(0, x) = u_0(x), \ v(0, x) = v_0(x), & (1.16d) \end{cases}$$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \ x > 0, \ t > 0 & (1.17a) \\ v(t, x) = f(u(t, x)), & (1.17b) \\ u(t, 0^+) \overset{BLN}{:=} u(t, 0^-) & (1.17c) \\ u(0, x) = u_0(x), & (1.17d) \end{cases}$$

where (1.17d) holds in the sense of the well-known Bardos–Leroux–Nédélec condition (BLN) [2]:

$$\begin{aligned} &\operatorname{sgn}(u(t, 0^+) - u(t, 0^-))(f(u(t, 0^+)) - f(k)) \\ &\qquad \leqq 0, \quad \text{for all } k \text{ between } u(t, 0^-) \text{ and } u(t, 0^+). \end{aligned} \tag{1.18}$$

Then notice that in the limit equations (1.16)–(1.17), the equality of the flux at the interface is verified

$$v(t, 0^-) = v(t, 0^+) = f(u(t, 0^+)). \tag{1.19}$$

We point out that the solution u on $x > 0$ obeys by construction and in the usual weak sense the Kruzkov's entropy inequalities

$$\partial_t |u - k| + \partial_x \operatorname{sgn}(u - k)(f(u) - f(k)) \leqq 0, \quad x > 0, \ t > 0, \tag{1.20}$$

for all $k \in \mathbf{R}$. Our coupling (1.16)–(1.17) is *more general* than (1.5)–(1.8) since we do not assume the sign of $f'(u)$ at the interface.

To prove the ε-convergence to the coupled problem (1.16)–(1.17) and to infer BLN, one needs to match the left and right solutions through an interface layer of thickness $O(\varepsilon)$ and blow up the transition profile by introducing a fast variable $y = x/\varepsilon$ to get in limit $\varepsilon \to 0^+$ the following ODE that describes that interface layer connecting the left trace $u(t, 0^-)$ to the flux $v(t, 0^+) = f(u(t, 0^+))$:

$$\begin{cases} a^2 \dfrac{\mathrm{d}}{\mathrm{d}y}\mathcal{U}(t, y) = f(\mathcal{U}(t, y)) - v(t, 0^+), \;\; y > 0, & (1.21a) \\ \mathcal{U}(t, 0) = u(t, 0^-). & (1.21b) \end{cases}$$

We shall see that by construction :

$$\mathcal{U}(t, y) \equiv \mathcal{U}(t, 0), \quad y < 0, \; t > 0 \quad \text{so that}\, \mathcal{U}(t, -\infty) = u(t, 0^-), \quad t > 0. \tag{1.22}$$

It is important to point out that there is no need to solve the inner problem (1.21) to close the coupling problem under consideration: namely the left and right IBVP problems (1.16)–(1.17) form a closed and well posed Cauchy problem. This is in sharp contrast with the kinetic-fluid coupling problem where the knowledge of the solution of the Milne's problem [5,6,25], governing the interface layer, is needed to close the coupling problem.

Vasseur [25] studied the coupling of the Perthame–Tadmor kinetic model [21] with the Burgers equation. As has already been emphasized, a kinetic layer has to be solved in this framework to determine the coupling conditions between the left and right solutions. In addition, Vasseur's analysis makes use of a comparison principle for *special* initial data while we rely on a space-time BV framework for *general* well-prepared initial data.

Let us highlight the reason why the coupling problem (1.16)–(1.17) is an extension of (1.5)–(1.8) to a more general flux. Namely,

(i) if $f'(u) > 0$ for u in between $u(t, 0^-)$ and $u(t, 0^+)$, then BLN gives $u(t, 0^-) = u(t, 0^+)$. In this case $\mathcal{U}(t, y) \equiv u(t, 0^-) = u(t, 0^+)$, so there is no interface layer.

(ii) if $f'(u) < 0$ in between $u(t, 0^-)$ and $u(t, 0^+)$, then BLN always holds and $f'(u(t, 0^-)) \neq f'(u(t, 0^+))$. In this case, (1.21) gives a monotone interface profile starting from $u(t, 0^-)$ and exponentially converges to $u(t, 0^+)$ as $y \to \infty$. This property is crucially used in the proof of Lemma 10.
These two cases correspond to (1.5)–(1.8).

(iii) if $f'(u)$ changes signs between $u(t, 0^-)$ and $u(t, 0^+)$, BLN can give rise to complicated behaviors, specifically, there is a possible steady shock stuck to the interface layer, separating $\mathcal{U}(t, +\infty)$ from $u(t, 0^+)$. Observe that such a shock has to be entropy satisfying since every value along the jump is subject to BLN (1.18). Such a standing shock (having zero shock speed) comes with the property that $f(\mathcal{U}(t, +\infty)) = f(u(t, 0^+))$. As an example, let us consider the setting of the Burger's equation $f(u) = u^2/2$ with the initial data

$$u_0(x) = \begin{cases} +1, & x < 0, \\ -1, & x > 0, \end{cases} \quad v_0(x) = f(u_0(x)), \; x \in \mathbf{R}. \tag{1.23}$$

Choosing $a = 1$, then the (unique) solution of the coupling problem (1.16)–(1.17) is given by:

$$u(t,x) = u_0(x), \quad v(t,x) = f(u_0(x)), \quad x \in \mathbf{R}, \; t > 0 \quad \text{while } \mathcal{U}(t,y) \equiv 1, \quad y > 0, \; t > 0. \tag{1.24}$$

This solution is nothing but a steady shock stuck at the coupling interface layer $\mathcal{U}(t,y) \equiv 1$. In general, the situation where $u(t,0^-) \neq \mathcal{U}(t,+\infty)$, a non-constant interface layer $\mathcal{U}$ exists and in addition a standing shock can be attached to it : $u(t,0^+) \neq \mathcal{U}(t,+\infty)$. This results in a non trivial compound inner structure to connect the outer values $u(t,0^-)$ with the inner one $u(t,0^+)$.

While cases (i) and (ii) are fully studied in this paper, we leave case (iii) to the future.

The rest of the paper is organized as follows. Section 2 proves well-posedness of the original two scale equations (1.1) and provides all the necessary *a priori* estimates used in Section 3 to pass to the limit in $L^1((0,T) \times \mathbf{R})$, $T > 0$, as ε goes to 0. It is also proved in Section 3 that the limit function (u, v) satisfies the coupling equations (1.16)–(1.17) in the left and right domains $\mathbf{R}_t^+ \times \mathbf{R}_x^-$ and $\mathbf{R}_t^+ \times \mathbf{R}_x^+$. Along the justification of this limit we are also able to establish the regularity of the solution (u, v). Section 4 provides the matching interface layer analysis, proving the BNL connection for the outer solution. The proof of the main Theorem 1 is given at the end of Section 4. We conclude the paper in section 5.

2. Well-posedness of the Original Two-scale Hyperbolic System

2.1. A regularized system

Existence and stability of the solution to the Cauchy problem (1.1), (1.2)–(1.12) will rely on a suitable regularization of the equations and the data. Let $\rho(x)$ be a non-negative symmetric kernel with

$$\rho \in C_c^\infty(\mathbf{R}), \quad \rho \geqq 0, \quad \operatorname{supp}(\rho) \subset [-1,1], \quad \int_{\mathbf{R}} \rho(x)\, \mathrm{d}x = 1, \tag{2.1}$$

and consider the sequence of mollifiers $\{\rho_\delta\}_{\delta>0}$ generated by ρ

$$\rho_\delta(x) = \frac{1}{\delta}\rho\left(\frac{x}{\delta}\right), \quad x \in \mathbf{R}. \tag{2.2}$$

The discontinuous relaxation coefficient in (1.2) is given by the following classical regularization

$$\Lambda(x,\varepsilon,\delta) = (\rho_\delta * \lambda(\cdot,\varepsilon))(x) \tag{2.3}$$

with the property $\min(1, \frac{1}{\varepsilon}) \leqq \Lambda(x, \varepsilon, \delta) \leqq \max(1, \frac{1}{\varepsilon})$. More precisely

$$\Lambda(x, \varepsilon, \delta) = \begin{cases} \frac{1}{\varepsilon}, \ x \geqq \delta, \\ \text{smooth transition}, \ -\delta < x < \delta, \\ 1, \ x \leqq -\delta, \end{cases} \tag{2.4}$$

so that for any given fixed $\varepsilon > 0$, $\| \Lambda(\cdot, \varepsilon, \delta) - \lambda(\cdot, \varepsilon) \|_{L^1(\mathbf{R})}$ stays uniformly bounded with respect to $\delta > 0$ with the following well-known property

$$\lim_{\delta \to 0} \| \Lambda(\cdot, \varepsilon, \delta) - \lambda(\cdot, \varepsilon) \|_{L^1(\mathbf{R})} = 0, \tag{2.5}$$

which will be heavily used in the forthcoming analysis. The main strategy is to recover uniform estimates in ε when sending the regularization parameter δ to 0 while keeping ε fixed.

The initial data u_0 in (1.11) is smoothed according to

$$u_0^\delta(x) = \rho_\delta * u_0(x) \tag{2.6}$$

so that $u_0^\delta \in C_c^\infty(\mathbf{R})$ for all $\delta > 0$. Recall that (see Guisti [12] for a proof)

$$\begin{aligned} &\rho_\delta * u_0 \to u_0 \text{ in } L^1(\mathbf{R}) \text{ as } \delta \to 0 \text{ with } \mathrm{TV}(\rho_\delta * u_0) \\ &\leqq \mathrm{TV}(u_0), \quad \| u_0^\delta \|_{L^\infty(\mathbf{R})} \leqq \| u_0 \|_{L^\infty(\mathbf{R})} . \end{aligned} \tag{2.7}$$

Hence, the sequence $\{u_0^\delta\}_{\delta > 0}$ has uniformly bounded total variation and sup norm. The initial data v_0 is also regularized in a well-prepared manner by setting

$$v_0^\delta = f(u_0^\delta), \quad \text{with } v_0^\delta \text{ uniformly bounded in } L^\infty(\mathbf{R}) \cap \mathrm{BV}(\mathbf{R}). \tag{2.8}$$

In view of (2.7), we clearly have that $v_0^\delta \to v_0 = f(u_0)$ in $L^1(\mathbf{R})$ as $\delta \to 0$. Equipped with these regularizations, we propose the following regularized Cauchy problem

$$\begin{cases} \partial_t u^{\varepsilon,\delta} + \partial_x v^{\varepsilon,\delta} = 0, & \text{(2.9a)} \\ \partial_t v^{\varepsilon,\delta} + a^2 \partial_x u^{\varepsilon,\delta} = -\Lambda(x, \varepsilon, \delta)(v^{\varepsilon,\delta} - f(u^{\varepsilon,\delta})), & \text{(2.9b)} \end{cases}$$

with initial data

$$u^{\varepsilon,\delta}(0, x) = u_0^\delta(x), \quad v^{\varepsilon,\delta}(0, x) = f(u_0^\delta(x)). \tag{2.10}$$

The existence theory for the smooth solution to Cauchy problem (2.9)–(2.10) is classical, and we refer the reader to Protter and Weinberger [22], for, instance for the following result. To simplify the notations, we omit the superscripts ε and δ when they are fixed.

Theorem 2. *Given initial data u_0 and v_0 in $C^1(\mathbf{R})$ that vanish outside the interval $[-M, M]$ for $M > 0$, there exists a unique classical solution (u, v) to the Cauchy problem (2.9) defined on a maximal time interval $[0, T_c)$. If the maximal time T_c is finite, then necessarily:*

$$\lim_{t \to T_c} \sup_{x \in \mathbf{R}} (|u(t, x)| + |v(t, x)|) = +\infty. \tag{2.11}$$

The solution (u, v) belongs to $C^1([0, T_c) \times \mathbf{R})$ and vanishes outside $\bigcup_{t \in [0, T_c)} [-(M + at), M + at]$.

Under the sub-characteristic condition (1.14), the global in time existence of the classical solution $(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ of (2.9)–(2.10) for any given $\delta > 0$ and $\varepsilon > 0$ is guaranteed by the following result due to Natalini [19].

Proposition 3. *Under the sub-characteristic condition* (1.14)*, the classical solution* $(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ *of* (2.9)–(2.10) *is bounded in sup norm for all time, uniformly with respect to* ε *and* δ *with*

$$|u^{\varepsilon,\delta}(t,x)| \leqq B(N_0), \quad |v^{\varepsilon,\delta}(t,x)| \leqq aB(N_0), \ \ \textit{for } (t,x) \in (0,\infty) \times \mathbf{R}. \tag{2.12}$$

The smooth non-homogeneity in the space variable x in the relaxation parameter $\Lambda(x, \varepsilon, \delta) > 0$ does not affect the proof given by Natalini [19] which was for constant Λ. In fact, exactly the same steps apply.

2.2. Existence and stability of the solution

The main results of this section ensure existence and stability for the two scale Cauchy problem (1.1), (1.2)–(1.12).

Theorem 4. *Given well-prepared initial data* u_0 *and* v_0 *satisfying the assumption* (1.11)–(1.12)*. Assume the sub-characteristic condition* (1.14)*. Then for any given fixed parameter* $\varepsilon > 0$ *and time* $T > 0$*, the sequence* $\{(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})\}_{\delta>0}$ *of classical solutions of the regularized problem* (2.9)–(2.10) *converges as* $\delta \to 0$ *to a unique weak solution* $(u^{\varepsilon}, v^{\varepsilon})$ *of the Cauchy problem* (1.1)*,* (1.2)–(1.12) *in* $L^1((0,T)\times\mathbf{R})$*. This weak solution satisfies the following a priori estimates, for a real constant* $C > 0$ *independent of* ε*:*

$$\text{(i)} \quad \| u^{\varepsilon}(t,\cdot) \|_{L^\infty(\mathbf{R})} \leqq B(N_0), \quad \| v^{\varepsilon}(t,\cdot) \|_{L^\infty(\mathbf{R})} \leqq aB(N_0); \tag{2.13}$$

$$\text{(ii)} \quad TV(v^{\varepsilon}(t,\cdot)) \leqq C; \tag{2.14}$$

$$\text{(iii)} \quad TV_{\{x<0\}}(u^{\varepsilon}(t,\cdot)) \leqq C(1+T), \quad TV_{\{x>0\}}(u^{\varepsilon}(t,\cdot)) \leqq C\left(1+\frac{T}{\varepsilon}\right); \tag{2.15}$$

$$\text{(iv)} \quad ||u^{\varepsilon} - f^{-1}(v^{\varepsilon})||_{L^1((0,T);L^2(\mathbf{R}_x^+))} \leqq C_T\sqrt{\varepsilon}; \tag{2.16}$$

for all time $T > 0$ *while*

$$\text{(v)} \int_{\mathbf{R}} |u^{\varepsilon}(t_2,x) - u^{\varepsilon}(t_1,x)| \,\mathrm{d} \leqq C|t_2 - t_1|, \quad 0 \leqq t_1 \leqq t_2 \leqq T; \tag{2.17}$$

$$\int_{\mathbf{R}} |v^{\varepsilon}(t_2,x) - v^{\varepsilon}(t_1,x)| \,\mathrm{d} \leqq C|t_2 - t_1|. \tag{2.18}$$

Let us stress that the first estimate in (2.13) ensures from the definition (1.14) of a that

$$|f'(u)| < a. \tag{2.19}$$

It is important to point out that the BV estimate (2.15) for $\{u^\varepsilon\}_{\varepsilon>0}$ in the right half line $\{x > 0\}$ is non uniform with respect to ε. The additional estimate (2.16) will be shown as a consequence of the entropy dissipation estimation. The time and space uniform BV properties satisfied by the sequence $\{v^\varepsilon\}_\varepsilon$ will imply on the first hand that $\{v^\varepsilon\}_\varepsilon$ converges to a limit function v in the strong $L^1((0,R)\times(0,T))$ topology for any given $R > 0$ and $T > 0$, while on the second hand the limit $u = f^{-1}(v)$ is in $BV([0,T]\times \mathbf{R}_x)$ as a consequence of (iv).

The proof of this statement relies on the use of the characteristic variables

$$r_\pm(t,x) = a\, u(t,x) \pm v(t,x), \tag{2.20}$$

where we temporarily skip the small parameters in the notations for simplicity. Then the relaxation system (1.1) or (2.9) can be written in the convenient diagonal form:

$$\begin{cases} \partial_t r_- - a\partial_x r_- = -\Lambda(x,\varepsilon,\delta)G(r_-,r_+), & (2.21a)\\ \partial_t r_+ + a\partial_x r_+ = \ \Lambda(x,\varepsilon,\delta)G(r_-,r_+), & (2.21b)\end{cases}$$

where

$$G(r_-,r_+) = f\left(\frac{r_- + r_+}{2a}\right) - \frac{r_+ - r_-}{2}. \tag{2.22}$$

A key point of the analysis is that the sub-characteristic condition (1.14) (see Katsoulakis and Tzavaras [17], Natalini [19] for instance) makes the mapping G quasi-monotone in the sense that

$$\partial_{r_-} G(r_-,r_+) > 0, \quad \partial_{r_+} G(r_-,r_+) < 0 \tag{2.23}$$

for any given pair (r_-,r_+) with $|r_- + r_+|/2a < B(N_0)$ and thus satisfying $|f'((r_- + r_+)/2a))| < a$ with a prescribed according to (1.14). In other words and equivalently, as soon as the solution $(u^\varepsilon, v^\varepsilon)$ belongs to $\mathcal{D}(u_0)$ in (1.1) then the quasi-monotone property (2.23) is satisfied. In addition, there exists a unique C^1 curve $(r_+, h(r_+))$ of equilibria locally which satisfies

$$G(h(r_+), r_+) = 0 \tag{2.24}$$

for all r_+ in $\mathbf{R}$ such that $|r_+ + h(r_+)|/2a < B(N_0)$ and hence (1.14) holds. For all the r_+ under consideration,

$$h(r_+) \text{ is strictly increasing,} \tag{2.25}$$

and we have $h(0) = 0$. It is worth observing that the (well-defined) unique solution ℓ of $\ell + h(\ell) = 2ak$ for any given k with $|k| < B(N_0)$ satisfies the identity $\ell - h(\ell) = 2f(k)$.

Let us also recall the L^1 contraction property which will be heavily used later on. Given two classical solutions of the equations (2.21), the differences $r_- - \bar{r}_-$ and $r_+ - \bar{r}_+$ satisfy, once respectively multiplied by $\text{sgn}(r_- - \bar{r}_-)$ and $\text{sgn}(r_+ - \bar{r}_+)$:

$$\begin{aligned}&\partial_t|r_- - \bar{r}_-| - a\partial_x|r_- - \bar{r}_-|\\ &\quad = -\Lambda(x,\varepsilon,\delta)(G(r_-,r_+) - G(\bar{r}_-,\bar{r}_+))\text{sgn}(r_- - \bar{r}_-),\end{aligned} \tag{2.26}$$

$$\partial_t|r_+ - \bar{r}_+| + a\partial_x|r_+ - \bar{r}_+|$$
$$= +\Lambda(x, \varepsilon, \delta)(G(r_-, r_+) - G(\bar{r}_-, \bar{r}_+))\text{sgn}(r_+ - \bar{r}_+). \quad (2.27)$$

Thus adding these two identities gives

$$\begin{aligned}\partial_t(|r_+ - \bar{r}_+| + |r_- - \bar{r}_-|) + a\partial_x(|r_+ - \bar{r}_+| - |r_- - \bar{r}_-|)\\ = \Lambda(x, \varepsilon, \delta)(G(r_-, r_+) - G(\bar{r}_-, \bar{r}_+))(\text{sgn}(r_+ - \bar{r}_+)\\ -\text{sgn}(r_- - \bar{r}_-)) \leqq 0,\end{aligned} \quad (2.28)$$

thanks to the quasi-monotonicity property (2.23). For any given $\ell \in \mathbf{R}$ with $|\ell + h(\ell)|/2a < B(N_0)$, choosing $\bar{r}_+ = \ell$ and $\bar{r}_- = h(\ell)$ in (2.28) yields

$$\partial_t\left(|r_+ - \ell| + |r_- - h(\ell)|\right) + a\partial_x\left(|r_+ - \ell| - |r_- - h(\ell)|\right) \leqq 0. \quad (2.29)$$

A first consequence of these inequalities is

Proposition 5. *Under the assumption of Theorem 4, the pair $(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ satisfies for all $\varepsilon, \delta > 0$ the following entropy like inequalities expressed in terms of the characteristic variables*

$$\begin{aligned}\int\int_{\mathbf{R}_t^+\times\mathbf{R}_x}[(|r_+^{\varepsilon,\delta} - \ell| + |r_-^{\varepsilon,\delta} - h(\ell)|)\partial_t\varphi\\ + a(|r_+^{\varepsilon,\delta} - \ell| - |r_-^{\varepsilon,\delta} - h(\ell)|)\partial_x\varphi]\,\mathrm{d}t\,\mathrm{d}x \geqq 0,\end{aligned} \quad (2.30)$$

for any given non negative test function φ in $\mathcal{C}_c^1(\mathbf{R}^+ \times \mathbf{R})$ and all $\ell \in \mathbf{R}$ such that $|\ell + h(\ell)|/2a < B(N_0)$.

A second consequence is the following L^1 contraction property (see also [17, 19]).

Proposition 6. *Assume the sub-characteristic condition (1.14). Let $(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ and $(\bar{u}^{\varepsilon,\delta}, \bar{v}^{\varepsilon,\delta})$ be two classical solutions of the equations (2.9) with initial data (u_0^δ, v_0^δ) and $(\bar{u}_0^\delta, \bar{v}_0^\delta)$ respectively that vanish outside the cone $\bigcup_{t\geqq 0}[-(M+at), (M+at)]$. Then for all time $t > 0$, the associated characteristic variables satisfy the following inequality*

$$\begin{aligned}\int_{-M}^{M}\left(\,|r_+^{\varepsilon,\delta} - \bar{r}_+^{\varepsilon,\delta}| + |r_-^{\varepsilon,\delta} - \bar{r}_-^{\varepsilon,\delta}|\,\right)(t, x)\,\mathrm{d}x\\ \leqq \int_{-(M+at)}^{M+at}\left(\,|(r_+)_0^\delta - (\bar{r}_+)_0^\delta| + |(r_-)_0^\delta - (\bar{r}_-)_0^\delta|\,\right)(x)\,\mathrm{d}x.\end{aligned} \quad (2.31)$$

For a problem invariant by translation in x, the above L^1 contraction principle is known to imply a uniform BV estimate for the solution to (2.9) as long as the initial data (u_0, v_0) is chosen in BV (see [19] for instance). However, the dependence of the relaxation coefficient $\Lambda(x, \varepsilon, \delta)$ in the space variable obviously prevents the classical solution of the regularized equation (2.9) from being translation invariant in x. Thus the expected uniform BV estimate can no longer be inferred from (2.31), neither can it be derived from the direct differentiation with respect to x

of the governing equations (2.9) (see [24]) since in the limit $\delta \to 0$, $\partial_x \Lambda(x, \varepsilon, \delta)$ concentrates in a Dirac mass at the interface $x = 0$. Instead, we can take advantage of the invariance with respect to time variable of the classical solutions of (2.9) and we prove hereafter that some uniform BV estimates can be inferred from it. The key estimates are gathered in the following statement.

Proposition 7. *Under the assumption of Theorem 4, the classical solution* $(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ *of the regularized Cauchy problem* (2.9)–(2.10) *satisfies, for any given* $\varepsilon > 0$ *and* $\delta > 0$,

$$(rmi) \quad \| u^{\varepsilon,\delta}(t,\cdot) \|_{L^\infty(\mathbf{R})} \leqq B(N_0), \quad \| v^{\varepsilon,\delta}(t,\cdot) \|_{L^\infty(\mathbf{R})} \leqq aB(N_0); \tag{2.32}$$

$$(rmii) \quad \| \partial_t u^{\varepsilon,\delta}(t,\cdot) \|_{L^1(\mathbf{R})} \leqq C, \quad \| \partial_t v^{\varepsilon,\delta}(t,\cdot) \|_{L^1(\mathbf{R})} \leqq C; \tag{2.33}$$

$$(rmiii) \quad TV(v^{\varepsilon,\delta}(t,\cdot)) \leqq C; \tag{2.34}$$

$$(rmiv) \quad TV_{\{x \leqq 0\}}(u^{\varepsilon,\delta}(t,\cdot)) \leqq C_T(1 + \| \lambda(\cdot,\varepsilon) - \Lambda(\cdot,\varepsilon,\delta) \|_{L^1(\mathbf{R})}); \tag{2.35}$$

$$TV_{\{x \geqq 0\}}(u^{\varepsilon,\delta}(t,\cdot)) \leqq C_T(1 + \| \lambda(\cdot,\varepsilon) - \Lambda(\cdot,\varepsilon,\delta) \|_{L^1(\mathbf{R})}) \left(1 + \frac{1}{\varepsilon}\right); \tag{2.36}$$

for $t \in (0, T)$, *for any given time* $T > 0$ *and some constant* $C_T > 0$ *independent of* ε *and* δ. *Moreover, under the assumption* (1.9) *or* (1.10), *introducing the inverse function* f^{-1} *of* f, *the following estimate holds*

$$(rmv) \; ||u^{\varepsilon,\delta} - f^{-1}(v^{\varepsilon,\delta})||_{L^1([0,T],L^2(\mathbf{R}^+))} \leqq C_T\sqrt{\varepsilon + \delta}. \tag{2.37}$$

Notice that the BV estimate for $u^{\varepsilon,\delta}$ in the right half line $\{x > 0\}$ is uniform only with respect to δ for fixed $\varepsilon > 0$.

Proof. (i) (2.32) is nothing but the estimate (2.12) stated in Proposition 3.

(ii) Deriving the uniform L^1 estimate for the time derivative of the classical solutions $(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ relies on differentiating system (2.9) with respect to time. Define

$$s_-^{\varepsilon,\delta}(t,x) = a\partial_t u^{\varepsilon,\delta}(t,x) - \partial_t v^{\varepsilon,\delta}(t,x), \tag{2.38}$$

$$s_+^{\varepsilon,\delta}(t,x) = a\partial_t u^{\varepsilon,\delta}(t,x) + \partial_t v^{\varepsilon,\delta}(t,x), \tag{2.39}$$

then they solve the system

$$\begin{cases} \partial_t s_-^{\varepsilon,\delta} - a\partial_x s_-^{\varepsilon,\delta} = -\Lambda(x,\varepsilon,\delta)\mathcal{R}(s_-^{\varepsilon,\delta}, s_+^{\varepsilon,\delta}), & \text{(2.40a)} \\ \partial_t s_+^{\varepsilon,\delta} + a\partial_x s_+^{\varepsilon,\delta} = \Lambda(x,\varepsilon,\delta)\mathcal{R}(s_-^{\varepsilon,\delta}, s_+^{\varepsilon,\delta}), & \text{(2.40b)} \end{cases}$$

with

$$\mathcal{R}(s_-, s_+) = f'(u)\frac{s_- + s_+}{2a} - \frac{s_+ - s_-}{2}, \tag{2.41}$$

where we again omit the superscripts ε, δ for simplicity. Under the sub-characteristic condition (1.14), R obeys the quasi-monotonicity property

$$\partial_{s_-}\mathcal{R}(s_-, s_+) > 0, \quad \partial_{s_+}\mathcal{R}(s_-, s_+) < 0 \tag{2.42}$$

for any given $\varepsilon > 0$ and $\delta > 0$. Similar steps to those used in the derivation of the L^1 contraction principle from quasi-monotonicity yields

$$\int_{|x|<M} (|s_+^{\varepsilon,\delta}(t,x)| + |s_-^{\varepsilon,\delta}(t,x)|)\,\mathrm{d}x \leqq \int_{|x|<M+at} (|s_+^{\varepsilon,\delta}(0,x)| + |s_-^{\varepsilon,\delta}(0,x)|)\,\mathrm{d}x \tag{2.43}$$

for any given real number $M > 0$ such that the initial data $s_+^{\varepsilon,\delta}(0,x), s_-^{\varepsilon,\delta}(0,x)$ vanish for all x with $|x| \geqq M$. Next observe that the governing equations (2.9) expressed at time $t = 0$ reads

$$\begin{cases} \partial_t u^{\varepsilon,\delta}(0,x) = -\dfrac{\mathrm{d}}{\mathrm{d}x} v_0^\delta(x), & \text{(2.44a)} \\ \partial_t v^{\varepsilon,\delta}(0,x) = -a^2 \dfrac{\mathrm{d}}{\mathrm{d}x} u_0^\delta(x) + \Lambda(x,\varepsilon,\delta)(f(u_0^\delta)(x) - v_0^\delta(x)), & \text{(2.44b)} \end{cases}$$

then by the choice of the well-prepared initial data $v_0^\delta = f(u_0^\delta)$, we get

$$\| \partial_t u^{\varepsilon,\delta}(0,\cdot) \|_{L^1(\mathbf{R})} \leqq \| f'(u_0^\delta) \|_{L^\infty(\mathbf{R})} \mathrm{TV}(u_0^\delta), \tag{2.45}$$

$$\| \partial_t v^{\varepsilon,\delta}(0,\cdot) \|_{L^1(\mathbf{R})} \leqq a^2 \mathrm{TV}(u_0^\delta) \tag{2.46}$$

which implies the bound

$$\max(\| s_+^{\varepsilon,\delta}(0,\cdot) \|_{L^1(\mathbf{R})}, \| s_-^{\varepsilon,\delta}(0,\cdot) \|_{L^1(\mathbf{R})}) \leqq a(\| f'(u_0^\delta) \|_{L^\infty(\mathbf{R})} + a)\mathrm{TV}(u_0^\delta) \leqq C, \tag{2.47}$$

where $C > 0$ is independent of δ. We therefore deduce from the L^1 contraction property (2.43) that

$$\| s_-^{\varepsilon,\delta}(t,\cdot) \|_{L^1(\mathbf{R})} \leqq C, \quad \| s_+^{\varepsilon,\delta}(t,\cdot) \|_{L^1(\mathbf{R})} \leqq C. \tag{2.48}$$

These uniform estimates clearly imply (2.33).

(iii) The expected BV estimate (2.34) of $v^{\varepsilon,\delta}$ immediately follows from the identity

$$\partial_x v^{\varepsilon,\delta}(t,x) = -\partial_t u^{\varepsilon,\delta}(t,x).$$

(iv) Let us now establish the uniform BV estimate (2.35). Rewrite the second PDE as

$$a^2 \partial_x u^{\varepsilon,\delta} = -\partial_t v^{\varepsilon,\delta} + \lambda(x,\varepsilon)(f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta}) + \Theta^{\varepsilon,\delta}, \tag{2.49}$$

where we have set

$$\Theta^{\varepsilon,\delta}(t,x) = (\Lambda(x,\varepsilon,\delta) - \lambda(x,\varepsilon))(f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta}). \tag{2.50}$$

Observe that

$$\int_R |\Theta^{\varepsilon,\delta}(t,x)|\,\mathrm{d}x \leqq \| (f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta})(t,.) \|_{L^\infty(\mathbf{R})} \| \lambda(.,\varepsilon) - \Lambda(.,\varepsilon,\delta) \|_{L^1(\mathbf{R})} \leqq C \| \lambda(.,\varepsilon) - \Lambda(.,\varepsilon,\delta) \|_{L^1(\mathbf{R})} \tag{2.51}$$

for some constant $C > 0$ independent of ε and δ. First focusing on negative values of x so that $\lambda(x, \varepsilon) = 1$, we give the easy estimate:

$$a^2 TV_{\{x<0\}}(u^{\varepsilon,\delta}(t,.)) \leqq ||\partial_t v^{\varepsilon,\delta}||_{L^1(\mathbf{R})} + ||f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta}||_{L^1(\mathbf{R})} \\ + C \parallel \lambda(.,\varepsilon) - \Lambda(.,\varepsilon,\delta) \parallel_{L^1(\mathbf{R})} . \tag{2.52}$$

That is, for all $T > 0$ and almost everywhere $t \in [0, T]$

$$TV_{\{x<0\}}(u^{\varepsilon,\delta}(t,.)) \leqq C_T(1+ \parallel \lambda(.,\varepsilon) - \Lambda(.,\varepsilon,\delta) \parallel_{L^1(\mathbf{R})}), \tag{2.53}$$

for some constant $C_T > 0$ independent of ε and δ since $u^{\varepsilon,\delta}(t,.)$ and $v^{\varepsilon,\delta}(t,.)$ have compact support for all finite time $t > 0$. Considering positive values of x for which $\lambda(x,\varepsilon) = 1/\varepsilon$ and following identical steps, one gets, for all $T > 0$ and almost everywhere $t \in [0, T]$,

$$TV_{\{x>0\}}(u^{\varepsilon,\delta}(t,.)) \leqq C_T(1+ \parallel \lambda(.,\varepsilon) - \Lambda(.,\varepsilon,\delta) \parallel_{L^1(\mathbf{R})}) \left(1 + \frac{1}{\varepsilon}\right), \tag{2.54}$$

again for some constant $C_T > 0$ independent of ε and δ.

(v) The derivation of the last estimate (2.37) relies on estimating the entropy dissipation rate taking place on the positive half line, for some convenient smooth entropy pairs first proposed by Chen et al. [7] (see also Natalini [19]). These are smooth functions Φ, Ψ such that Φ is strictly convex with

$$\partial_{vv}\Phi(u,v) \geqq \eta_\Phi > 0, \quad \text{for all } (u,v),\ |u| \leqq B(N_0), |v| \leqq aB(N_0), \tag{2.55}$$

for some strictly positive real number $\eta_\phi > 0$. One also has

$$\partial_v \Phi(u, f(u)) = 0, \quad \text{for all } u \in \mathbf{R}. \tag{2.56}$$

Without loss of generality, we assume that $\Phi(0,0) = 0$. Those properties will suffice to our purpose and readers are referred to [7,19] for additional ones. By construction, the smooth solutions $(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ obey the following entropy differential equation:

$$\partial_t \Phi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta}) + \partial_x \Psi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta}) \\ = -\Lambda(x,\varepsilon,\delta)\partial_v\Phi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta}) \cdot (v^{\varepsilon,\delta} - f(u^{\varepsilon,\delta})). \tag{2.57}$$

Observe from (2.56) that

$$\partial_v\Phi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta}) = \partial_v\Phi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta}) - \partial_v\Phi(u^{\varepsilon,\delta}, f(u^{\varepsilon,\delta})) \\ = \partial_{vv}\Phi(u^{\varepsilon,\delta}, \theta_1^{\varepsilon,\delta})(v^{\varepsilon,\delta} - f(u^{\varepsilon,\delta})), \tag{2.58}$$

for some uniformly bounded value $\theta_1^{\varepsilon,\delta}$ in between $v^{\varepsilon,\delta}$ and $f(u^{\varepsilon,\delta})$. Hence the entropy balance law (2.57) reads

$$\partial_t \Phi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta}) + \partial_x \Psi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta}) \\ = -\Lambda(x,\varepsilon,\delta)\partial_{vv}\Phi(u^{\varepsilon,\delta}, \theta_1^{\varepsilon,\delta}) \{f'(\theta_2^{\varepsilon,\delta})\}^2 |f^{-1}(v^{\varepsilon,\delta}) - u^{\varepsilon,\delta}|^2, \tag{2.59}$$

for some $\theta_2^{\varepsilon,\delta}$ in between $u^{\varepsilon,\delta}$ and $f^{-1}(v^{\varepsilon,\delta})$. Given $T > 0$, integrating the above identity over the time space domain $(0, T) \times \mathbf{R}_+$ clearly yields

$$\int_{\mathbf{R}_+} \Phi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})(T, x)\, \mathrm{d}x + C_0^2\, \eta_\Phi \int_0^T \int_{\mathbf{R}_+} \Lambda(x, \varepsilon, \delta)|f^{-1}(v^{\varepsilon,\delta}) - u^{\varepsilon,\delta}|^2\, \mathrm{d}t\, \mathrm{d}x \leqq \int_{\mathbf{R}_+} \Phi(u_0^{\varepsilon,\delta}, f(u_0^{\varepsilon,\delta}))(x)\, \mathrm{d}x + \int_0^T \Psi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})(t, 0)\, \mathrm{d}t, \tag{2.60}$$

where C_0 comes from non-vanishing wave velocity assumption (1.9) or (1.10), $\eta_\Phi > 0$ is the convexity modulus of Φ introduced in (2.55). Uniform bounded sup-norm for $u^{\varepsilon,\delta}$ and $v^{\varepsilon,\delta}$ then ensures the following upper bound for the right hand side

$$\int_{\mathbf{R}_+} \Phi(u_0^{\varepsilon,\delta}, f(u_0^{\varepsilon,\delta}))(x) dx + \int_0^T \Psi(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})(t, 0)\, \mathrm{d}t \leqq C\,(1 + T), \tag{2.61}$$

for some constant $C > 0$ independent of ε and δ. We therefore infer from (2.60) the estimate:

$$\int_0^T \int_\delta^{+\infty} |f^{-1}(v^{\varepsilon,\delta}) - u^{\varepsilon,\delta}|^2\, \mathrm{d}t\, \mathrm{d}x \leqq C(1 + T)\varepsilon, \tag{2.62}$$

for some uniform constant $C > 0$, since for all $x > \delta$, $\varepsilon\Lambda(x, \varepsilon, \delta) = 1$. Arguing again from the uniform sup norm estimates, we arrive at:

$$\int_0^T \int_{\mathbf{R}_+} |f^{-1}(v^{\varepsilon,\delta}) - u^{\varepsilon,\delta}|^2\, \mathrm{d}t\, \mathrm{d}x \leqq C(1 + T)(\varepsilon + \delta). \tag{2.63}$$

This ends the proof. □

Now we are ready to prove the main theorem in this section.

Proof the Theorem 4. First we show the existence of the limit $(u^\varepsilon, v^\varepsilon)$. Since $u^{\varepsilon,\delta}(t, 0^-) = u^{\varepsilon,\delta}(t, 0^+)$, first observe from the estimates (iv) in Proposition 7 that, for fixed $\varepsilon > 0$ and any given $\delta > 0$

$$\begin{aligned} \mathrm{TV}_R(u^{\varepsilon,\delta}(t, \cdot)) &= \mathrm{TV}_{\{x<0\}}(u^{\varepsilon,\delta}(t, \cdot)) + \mathrm{TV}_{\{x>0\}}(u^{\varepsilon,\delta}(t, \cdot)) \\ &\quad + |u^{\varepsilon,\delta}(t, 0^+) - u^{\varepsilon,\delta}(t, 0^-)| \\ &\leqq C(1 + \| \lambda(\cdot, \varepsilon) - \Lambda(\cdot, \varepsilon, \delta) \|_{L^1(\mathbf{R})}) \left(1 + \frac{1}{\varepsilon}\right), \end{aligned} \tag{2.64}$$

and also

$$\| \partial_t u^{\varepsilon,\delta}(t, \cdot) \|_{L^1(\mathbf{R})} \leqq C, \quad \| \partial_t v^{\varepsilon,\delta}(t, \cdot) \|_{L^1(\mathbf{R})} \leqq C, \tag{2.65}$$

for some positive constant C independent of ε and δ. Now let ε be fixed, these estimates ensure that $\{u^{\varepsilon,\delta}\}_{\delta>0}$ and $\{v^{\varepsilon,\delta}\}_{\delta>0}$ stay uniformly with respect to δ in $BV([0,T]\times R_x)$ for all time $t\in(0,T)$ with $T>0$ given, while being uniformly bounded in sup-norm. The well-known Helly's theorem asserts that for any given compact K in $[0,T]\times R_x$, the canonical embedding of $L^1(K)\cap BV(K)$ is compact in $L^1(K)$. A classical diagonal extraction procedure gives the existence of an extracted subsequence, still labeled by $\{(u^{\varepsilon,\delta},v^{\varepsilon,\delta})\}_{\delta>0}$ which converges in $L^1([0,T],L^1(R_x))$ to some limit $(u^\varepsilon,v^\varepsilon)$ with bounded sup-norm as δ goes to 0.

Let us conclude by showing that all extracted subsequences actually converge to the same limit which proves in turn the uniqueness. Start from the L^1 contraction principle (2.31) which is valid in view of (2.19), one has for all time $t\in(0,T)$ and $M>0$

$$\int_{-M}^{M}(\,|r_+^{\varepsilon,\delta}-\bar r_+^{\varepsilon,\delta}+|r_-^{\varepsilon,\delta}-\bar r_-^{\varepsilon,\delta}|\,)(t,x)\,\mathrm{d}x \leqq \int_{-(M+at)}^{M+at}(\,|(r_+)_0^\delta-(\bar r_+)_0^\delta|+|(r_+)_0^\delta-(\bar r_+)_0^\delta|\,)(x)\,\mathrm{d}x. \tag{2.66}$$

The above convergence results assert that there exists an extracted subsequence $(r_+^{\varepsilon,\delta},r_-^{\varepsilon,\delta})$ (resp. $(\bar r_+^{\varepsilon,\delta},\bar r_-^{\varepsilon,\delta})$) which converges to some limit $(r_+^\varepsilon,r_-^\varepsilon)$ (resp. $(\bar r_+^\varepsilon,\bar r_-^\varepsilon)$) in $L^1((0,T),L^1_{\mathrm{loc}}(R_x))$ as δ goes to 0. These limits are seen to satisfy

$$\int_{-M}^{M}(\,|r_+^\varepsilon-\bar r_+^\varepsilon|+|r_-^\varepsilon-\bar r_-^\varepsilon|\,)(t,x)\,\mathrm{d}x \leqq \int_{-(M+at)}^{M+at}(\,|r_+^0-\bar r_+^0|+|r_-^0-\bar r_-^0|\,)(x)\,\mathrm{d}x, \tag{2.67}$$

which gives the expected uniqueness property. Let us now characterize the limit $(u^\varepsilon,v^\varepsilon)$. First observe from the inequality (2.28) that for any given non-negative test function, $\varphi\in C_c^1((0,T]\times R)$

$$\int\int_{[0,T]\times\mathbf{R}}[(|r_+^{\varepsilon,\delta}-\bar r_+^{\varepsilon,\delta}|+|r_-^{\varepsilon,\delta}-\bar r_-^{\varepsilon,\delta}|)\partial_t\varphi + a(|r_+^{\varepsilon,\delta}-\bar r_+^{\varepsilon,\delta}|-|r_-^{\varepsilon,\delta}-\bar r_-^{\varepsilon,\delta}|)\partial_x\varphi]\,\mathrm{d}t\,\mathrm{d}x\geqq 0. \tag{2.68}$$

Now choose well-prepared initial data $\bar u_0$ and $\bar v_0$ with compact support such that $(\bar r_+)_0^\delta=\psi_\delta(x)\ell$ and $(\bar r_-)_0^\delta=\psi_\delta(x)h(\ell)$ for any given $\ell\in R$ with $|\ell+h(\ell)|/2a<B(N_0)$. Observe that $u^\varepsilon(t,x)=\frac{\ell+h(\ell)}{2a}$, $v^\varepsilon(t,x)=\frac{\ell-h(\ell)}{2}$ trivially solve the Cauchy problem (1.1) so that by uniqueness we have in the limit $\delta\to0$: $\bar r_+^\varepsilon(t,x)=\ell$ and $\bar r_-^\varepsilon(t,x)=h(\ell)$. Therefore we have proved (2.30) for all the ℓ under consideration. An additional characterization of the limit $(u^\varepsilon,v^\varepsilon)$ comes as follows. Let us consider test function $\varphi\in C_c^1((0,T]\times R)$, then the weak form of (2.9) reads

$$\begin{cases} \displaystyle\iint_{[0,T]\times\mathbf{R}_x} u^{\varepsilon,\delta}\varphi_t + v^{\varepsilon,\delta}\varphi_x \,\mathrm{d}x\,\mathrm{d}t = 0, & \text{(2.69a)} \\ \displaystyle\iint_{[0,T]\times\mathbf{R}_x} v^{\varepsilon,\delta}\varphi_t + a^2 u^{\varepsilon,\delta}\varphi_x + \Lambda(x,\varepsilon,\delta)(f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta})\varphi \,\mathrm{d}x\,\mathrm{d}t = 0. & \text{(2.69b)} \end{cases}$$

Notice that since f is smooth with $f(0) = 0$ and $\{u^{\varepsilon,\delta}\}$ stays uniformly bounded in sup norm, one has $f(u^{\varepsilon,\delta}(t,x))| \leqq C|u^{\varepsilon,\delta}|$. Moreover, $f(u^{\varepsilon,\delta}(t,x)) \to f(u^{\varepsilon}(t,x))$ almost everywhere in t and x. Therefore by Lebesgue's dominated convergence theorem,
$f(u^{\varepsilon,\delta}(t,x)) \to f(u^{\varepsilon}(t,x))$ in $L^1([0,T] \times R_x)$. Now passing to the limit in (2.69), notice that

$$\begin{aligned} &\iint_{[0,T]\times\mathbf{R}_x} \Lambda(x,\varepsilon,\delta)(f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta})\varphi \,\mathrm{d}t\,\mathrm{d}x \\ &= \iint_{[0,T]\times\mathbf{R}_x} (\Lambda(x,\varepsilon,\delta) - \lambda(x,\varepsilon))(f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta})\varphi \,\mathrm{d}t\,\mathrm{d}x \\ &\quad + \iint_{[0,T]\times\mathbf{R}_x} \lambda(x,\varepsilon)(f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta})\varphi \,\mathrm{d}t\,\mathrm{d}x, \end{aligned} \tag{2.70}$$

and by (2.5) and the sup norm estimate (2.32), the limit $(u^{\varepsilon}, v^{\varepsilon})$ indeed solves

$$\begin{cases} \displaystyle\iint u^{\varepsilon}\varphi_t + v^{\varepsilon}\varphi_x \,\mathrm{d}x\,\mathrm{d}t = 0, & \text{(2.71a)} \\ \displaystyle\iint v^{\varepsilon}\varphi_t + a^2 u^{\varepsilon}\varphi_x + \lambda(x,\varepsilon)(f(u^{\varepsilon}) - v^{\varepsilon})\varphi \,\mathrm{d}x\,\mathrm{d}t = 0, & \text{(2.71b)} \end{cases}$$

for any given test function φ in $C_c^1((0,T] \times \mathbf{R})$.

Now let us conclude this section by proving the expected a priori estimates stated in Theorem 4.

(i) It has been proved when we showed the existence of the limit.
(ii) Concerning the uniform BV estimate for v^{ε}, first observe from the estimate (2.34) that for any given $\varphi \in C_c^1(\mathbf{R}_x)$ with $\| \varphi \|_{L^\infty(\mathbf{R})\leqq 1}$,

$$\int_{\mathbf{R}} v^{\varepsilon,\delta}(t,x)\partial_x\varphi \,\mathrm{d}x \leqq \sup_{\varphi\in C_c^1(R_x)} \int_{\mathbf{R}_x} v^{\varepsilon,\delta}\partial_x\varphi \,\mathrm{d}x = \mathrm{TV}_R(v^{\varepsilon,\delta}) \leqq C. \tag{2.72}$$

Sending δ to 0 with $\varepsilon > 0$ kept fixed yields

$$\int_{\mathbf{R}_x} v^{\varepsilon}(t,x)\partial_x\varphi \,\mathrm{d}x \leqq C \tag{2.73}$$

with the same uniform constant C above which does not depend on ε. We therefore conclude that

$$\mathrm{TV}_{\mathbf{R}}(v^{\varepsilon}(t,\cdot)) = \sup_{\varphi\in C_c^1(\mathbf{R}_x), \|\varphi\|_{L^\infty(\mathbf{R})}\leqq 1} \int_{\mathbf{R}_x} v^{\varepsilon}(t,x)\partial_x\varphi \,\mathrm{d}x \leqq C. \tag{2.74}$$

(iii) In view of the uniform in $\delta > 0$ BV estimate (2.35) satisfied by $u^{\varepsilon,\delta}$ in the left half-line $\{x < 0\}$, ε being fixed, similar steps apply to get

$$\mathrm{TV}_{\{x<0\}}(u^{\varepsilon}(t,\cdot)) = \sup_{\varphi\in C_c^1(\mathbf{R}_x^-),||\varphi||_{L^\infty(\mathbf{R})}\leqq 1} \int_{\mathbf{R}_x^-} u^{\varepsilon}(t,x)\partial_x\varphi\, \mathrm{d}x \leqq C(1+T), \tag{2.75}$$

for some constant $C > 0$ independent of ε. Identical steps equally apply to infer the following non-uniform in ε BV estimate for $u^{\varepsilon,\delta}$ in the right half-line

$$\mathrm{TV}_{\{x>0\}}(u^{\varepsilon}(t,\cdot)) = \sup_{\varphi\in C_c^1(\mathbf{R}_x^+),||\varphi||_{L^\infty(\mathbf{R})}\leqq 1} \int_{\mathbf{R}_x^+} u^{\varepsilon}\partial_x\varphi\, \mathrm{d}x \leqq C\left(1+\frac{T}{\varepsilon}\right). \tag{2.76}$$

(iv) For any given fixed $\varepsilon > 0$, the expected last estimate

$$||f^{-1}(v^{\varepsilon}) - u^{\varepsilon}||_{L^1((0,T);L^2(\mathbf{R}_x^+))} \leqq C_T\sqrt{\varepsilon}, \tag{2.77}$$

follows by passing to the limit $\delta \to 0$ in (2.37), from the L^1 convergence of $\{u^{\varepsilon,\delta}\}_{\delta>0}$ to u^{ε} and strong L^1 convergence of $\{f^{-1}(v^{\varepsilon,\delta})\}_{\delta>0}$ to $f^{-1}(v^{\varepsilon})$ derived from Lebesgue's dominated convergence theorem, using uniform sup norm boundedness for the sequence $\{v^{\varepsilon,\delta}\}_{\delta>0}$.

(v) At last, let us write that for all times $t_1,\ t_2$ with $0 < t_1 < t_2 < T$, for some $T > 0$

$$\begin{aligned}&\int_{\mathbf{R}_x} |u^{\varepsilon,\delta}(t_2,x) - u^{\varepsilon,\delta}(t_1,x)|\,\mathrm{d}x\\ &\quad\leqq \int_{t_1}^{t_2}\int_{\mathbf{R}_x} |\partial_t u^{\varepsilon,\delta}(t,x)|\,\mathrm{d}x\,\mathrm{d}t \leqq \parallel \partial_t u^{\varepsilon,\delta}(t,.)\parallel_{L^1(\mathbf{R})}(t_2-t_1)\end{aligned} \tag{2.78}$$

which yields the required L^1 Lipschitz continuity property (2.17) for u^{ε} by sending $\delta \to 0$. The estimate (2.18) for v^{ε} is derived the same way. This concludes the proof of Theorem 4. □

3. Strong convergence in $L^1((0,T)\times\mathbf{R})$

In this section, we show the limit behavior when sending the relaxation parameter ε to 0. The following theorem is an extension of Theorem 1.

Theorem 8. *Being given any initial condition u_0, v_0 satisfying (1.11)-(1.12) and assume the sub characteristic condition (1.14).*

i There is a subsequence $(u^{\varepsilon}, v^{\varepsilon})$ that converges to a limit (u, v) in $L^1((0,T)\times\mathbf{R})^2$, for all $T > 0$ with for all $t \in (0,T)$:

$$||u(t,.)||_{L^\infty(\mathbf{R})} \leqq B(N_0), \quad ||v(t,.)||_{L^\infty(\mathbf{R})} \leqq aB(N_0), \tag{3.1}$$

so that (2.19) is again valid.

ii *On the left half line* $\{x < 0\}$*, the limit is a weak solution in the sense that it verifies for any test function* $\varphi \in C_c^1((0,T) \times R_x^-)$ *:*

$$\int_{[0,T]\times R_x^-} u\varphi_t + v\varphi_x \; \mathrm{d}x\,\mathrm{d}t = 0, \tag{3.2}$$

$$\int_{[0,T]\times R_x^-} v\varphi_t + a^2 u\varphi_x - (f(u) - v)\varphi \; \mathrm{d}x\,\mathrm{d}t = 0. \tag{3.3}$$

iii *In the right half line* $\{x > 0\}$*, the limit obeys*

$$v(t,x) = f(u(t,x)), \quad \textit{almost everywhere} t > 0, x > 0, \tag{3.4}$$

and for any given non-negative test function $\varphi \in C_c^1((0,T) \times R_x^+)$*, the limit function* u *solves*

$$\int_{[0,T]\times R_x^+} |u-k|\varphi_t + sgn(u-k)(f(u) - f(k))\varphi_x \, \mathrm{d}x\,\mathrm{d}t \geqq 0, \;\; \varphi \geqq 0, \tag{3.5}$$

for all $k \in R$ *with* $|k| \leqq B(N_0)$.

iv *The following conservation law holds in the weak sense over the whole real line*

$$\int_{[0,T]\times R_x} u\varphi_t + v\varphi_x \, \mathrm{d}x\,\mathrm{d}t = 0, \tag{3.6}$$

for all $\varphi \in C_c^1((0,T) \times R_x)$.

v u *and* v *are in* $L^\infty \cap BV((0,T)\times\mathbf{R}_x)$*. For almost everywhere* $t > 0$*, there exist left and right traces at the coupling interface, denoted by* $u(t,0^-)$*,* $u(t,0^+)$*, and* $v(t,0^-)$*,* $v(t,0^+)$ *with the property* $v(t,0^-) = v(t,0^+)$.

vi *The* L^1 *contraction principle* (1.15) *holds and the estimates* (i)–(iii) *in Theorem* 4 *are satisfied for* (u,v) *as well.*

Proof. For any given $T > 0$ and sufficiently large $R_T > 0$ with the support of the solutions included in $(-R_T, +R_T)$ at time T, define the time–space domain $\mathcal{Q} = (0,T) \times (-R_T, R_T)$. First we consider the convergence of the sequence $\{v^\varepsilon\}_{\varepsilon>0}$ to a limit function v in $L^1(\mathcal{Q})$. Theorem 4 asserts that v^ε is L^1 Lipschitz continuous in time uniformly in ε while $TV_{(-R,R)}(v^\varepsilon(t,\cdot))$ remains uniformly bounded in ε and time $t \in [0,T]$, so that v^ε is in $BV(\mathcal{Q})$. Helly's theorem then ensures the existence of an extracted subsequence, still labelled by $\{v^\varepsilon\}_\varepsilon$ which converges to a limit function v in $L^1(\mathcal{Q})$ as ε goes to zero, and almost everywhere up to another extracted subsequence. Since $\{v^\varepsilon\}_\varepsilon$ has uniformly bounded $L^\infty(\mathcal{Q})$-norm, the Lebesgue dominated convergence theorem ensures that the family $\{f^{-1}(v^\varepsilon)\}_\varepsilon$ converges as ε goes to zero to the limit $f^{-1}(v)$ in $L^1(\mathcal{Q})$.

Let us now introduce the time–space domain $\mathcal{Q}^- = (0,T) \times (-R_T, 0)$ and prove the strong convergence of $\{u^\varepsilon\}_{\varepsilon>0}$ in the limit $\varepsilon \to 0$ to a limit u in $L^1(\mathcal{Q}^-)$. Again the uniform boundedness with respect to ε and time $t \in (0,T)$ of $TV_{(-R_T,0)}(u^\varepsilon(t,\cdot))$ and the uniform in ε L^1 Lipschitz continuity of u^ε imply a uniform estimate in ε for $TV_{\mathcal{Q}^-}(u^\varepsilon)$, and hence yield the existence of a subsequence, denoted by $\{u^\varepsilon\}_{\varepsilon>0}$, which converges in $L^1(\mathcal{Q}^-)$ and almost everywhere to some limit function u.

Then introduce $\mathcal{Q}^+ = (0,T) \times (0, R_T)$, and consider the following triangle inequality

$$\begin{aligned}||u^\varepsilon - f^{-1}(v)||_{L^1((0,T),L^2(\mathbf{R}_+))} &\leqq ||u^\varepsilon - f^{-1}(v^\varepsilon)||_{L^1((0,T);L^2(\mathbf{R}_+))}\\ &+||f^{-1}(v^\varepsilon) - f^{-1}(v)||_{L^1((0,T);L^2(\mathbf{R}_+))},\end{aligned} \tag{3.7}$$

we easily deduce from the estimate $||u^\varepsilon - f^{-1}(v^\varepsilon)||_{L^1((0,T);L^2(\mathbf{R}_+))} \leqq C_T\sqrt{\varepsilon}$ and the property that $f^{-1}(v^\varepsilon) \to f^{-1}(v)$ in the strong $L^1(\mathcal{Q}^+)$ topology while being compactly supported that $\{u^\varepsilon\}_{\varepsilon>0}$ actually converges to $f^{-1}(v)$. As a conclusion u^ε converges to its limit u, strongly and almost everywhere in $\mathcal{Q}^+$. Gathering the above two results, we have proved that u^ε actually converges strongly in $L^1(\mathcal{Q}) = L^1(\mathcal{Q}^-) \cup L^1(\mathcal{Q}^+)$ and almost everywhere to a limit u. Let us then recall that for any given test function $\varphi \in C^1_c((0,T] \times R_x)$, one has:

$$\int_{[0,T]\times\mathbf{R}_x} u^\varepsilon \varphi_t + v^\varepsilon \varphi_x \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{3.8}$$

so that passing to the limit $\varepsilon \to 0$ gives the expected conservation law (3.6):

$$\int_{[0,T]\times\mathbf{R}_x} u \varphi_t + v \varphi_x \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{3.9}$$

As an immediate consequence, the well-defined left and right traces of v at the interface $x = 0$ verify $v(t,0^-) = v(t,0^+)$ for almost everywhere $t > 0$. In addition, u is seen to inherit bounded total variation from v from:

$$TV(u) = TV(f^{-1}(v)) \leqq \frac{1}{C_0} TV(v).$$

Let us now prove that the limit function (u, v) under consideration solves (3.2), (3.3), (3.5) and (3.6) in the usual sense of distribution. Namely choosing any test function $\varphi \in C^1_c(\mathcal{Q}^-)$ in

$$\begin{cases} \displaystyle\int u^\varepsilon \varphi_t + v^\varepsilon \varphi_x \,\mathrm{d}x\,\mathrm{d}t = 0, & \text{(3.10a)}\\ \displaystyle\int v^\varepsilon \varphi_t + a^2 u^\varepsilon \varphi_x + (f(u^\varepsilon) - v^\varepsilon)\varphi \,\mathrm{d}x\,\mathrm{d}t = 0, & \text{(3.10b)}\end{cases}$$

a direct application of Lebesgue's dominated convergence theorem proves that the sequence $\{f(u^\varepsilon)\}_{\varepsilon>0}$ converges to $f(u)$ in $L^1(\mathcal{Q}^-)$ and yields the expected conclusion.

In order to prove that in the time–space domain $\mathcal{Q}^+$, the limit u is an entropy weak solution of the underlying scalar conservation law, let us first derive the expected L^1 contraction principle from the one established in (2.31). Keeping unchanged the notations, arguing of the strong convergence of $(r^\varepsilon_+, r^\varepsilon_-)$ to the limit

$(r_+ = au + v, r_- = au - v)$ in $L^1(\mathcal{Q})$, we pass to the limit $\varepsilon \to 0$ in the reported inequality to get:

$$\int_{-M}^{0} (|r_+ - \bar r_+| + |r_- - \bar r_-|)(t,x)\,\mathrm{d}x + \int_0^M (|r_+ - \bar r_+| + |h(r_+) - h(\bar r_+)|)(t,x)\,\mathrm{d}x \leqq \int_{|x|<M+at} (|r_+^0 - \bar r_+^0| + |h(r_+^0) - h(\bar r_+^0)|)(x)\mathrm{d}x \tag{3.11}$$

since the initial data is well-prepared. Now by the increasing monotonicity property of h, one has

$$(r_+ - \bar r_+)(h(r_+) - h(\bar r_+)) \geqq 0 \tag{3.12}$$

so that

$$|r_+ - \bar r_+| + |h(r_+) - h(\bar r_+)| = |r_+ - \bar r_+ + h(r_+) - h(\bar r_+)| = 2a|u - \bar u|. \tag{3.13}$$

Therefore we can write

$$\int_0^M (|r_+ - \bar r_+| + |h(r_+) - h(\bar r_+)|)(t,x)\,\mathrm{d}x = 2a \int_0^M |u - \bar u|(t,x)\,\mathrm{d}x, \tag{3.14}$$

and correspondingly

$$\int_{|x|\leqq M+at} (|r_+^0 - \bar r_+^0| + |h(r_+^0) - h(\bar r_+^0)|)(x)\mathrm{d}x = 2a \int_{|x|\leqq M+at} |u_0 - \bar u_0|(x)\,\mathrm{d}x. \tag{3.15}$$

We thus have for any given $M > 0$ and time $t > 0$

$$\int_{-M}^{0} (|r_+ - \bar r_+| + |r_- - \bar r_-|)(t,x)\,\mathrm{d}x + 2a \int_0^M |u - \bar u|(t,x)\,\mathrm{d}x \leqq 2a \int_{|x|\leqq M+at} |u_0 - \bar u_0|(x)\mathrm{d}x, \tag{3.16}$$

which proves the L^1 contraction principle (1.15). This principle implies uniqueness of the limit (u, v). Let us at last derive (3.5) from inequality (2.30) focusing on the non-negative test function φ with compact support in R_x^+. Taking the limit $\varepsilon \to 0$ provides

$$\int_{[0,T]\times R_x^+} (|r_+ - \ell| + |h(r_+) - h(\ell)|)\varphi_t + a(|r_+ - \ell| - |h(r_+) - h(\ell)|)\varphi_x \,\mathrm{d}t\,\mathrm{d}x \geqq 0 \tag{3.17}$$

for any given $\ell \in R$ such that $k = \frac{\ell + h(\ell)}{2a}$ verifies $|k| < B(N_0)$. Observe that $\frac{\ell - h(\ell)}{2} = f(k)$. The increasing property met by h ensures

$$|r_+ - \ell| + |h(r_+) - h(\ell)| = |r_+ + h(r_+) - (\ell + h(\ell))| = 2a|u - k| \tag{3.18}$$

as well as

$$|r_+ - \ell| - |h(r_+) - h(\ell)| = \operatorname{sgn}(r_+ - \ell)(r_+ - \ell - h(r_+) + h(\ell))$$
$$= 2\operatorname{sgn}(u - k)(f(u) - f(k)) \qquad (3.19)$$

since $(u - k)(r_+ - \ell) \geqq 0$. This yields the expected Kružkov entropy inequalities. □

4. Matched asymptotic analysis

So far we only showed that the solution of the original system (1.1) converges to the weak solution of (3.2)–(3.5) in $\mathcal{Q}^- \cup \mathcal{Q}^+$. However, since the test function φ vanishes in a neighborhood of the interface, we missed the information at $x = 0$. In this section, we want to derive the interface condition by matched asymptotic analysis in a rigorous way, which is the generalization of the one used in the domain decomposition system (1.5)–(1.8).

Since an interface layer may develop at the interface, we propose to reveal its structure in the limit $\varepsilon \to 0$ using a blow-up technique (see [25] in a related setting). Fix $\varepsilon > 0$ and $\delta > 0$, define the fast variable $y = \frac{x}{\varepsilon}$ so that $x = \varepsilon y$ and let

$$\mathcal{U}^{\varepsilon,\delta}(t, y) = u^{\varepsilon,\delta}(t, \varepsilon y), \quad \mathcal{V}^{\varepsilon,\delta}(t, y) = v^{\varepsilon,\delta}(t, \varepsilon y), \quad y \in \mathbf{R}. \qquad (4.1)$$

Observe that $\mathcal{U}^{\varepsilon,\delta}(t, .)$ and $u^{\varepsilon,\delta}(t, .)$ (resp. $\mathcal{V}^{\varepsilon,\delta}(t, .)$ and $v^{\varepsilon,\delta}(t, .)$) have the same sup norm and total variation, so that Proposition 7 ensures

$$\| \mathcal{U}^{\varepsilon,\delta}(t, \cdot) \|_{L^\infty(\mathbf{R})} \leqq C, \qquad (4.2)$$
$$\mathrm{TV}_{\{y \leqq 0\}}(\mathcal{U}^{\varepsilon,\delta}(t, \cdot)) \leqq C(1 + \| \lambda(\cdot, \varepsilon) - \Lambda(\cdot, \varepsilon, \delta) \|_{L^1(\mathbf{R})})(1 + T), \qquad (4.3)$$
$$\| \mathcal{V}^{\varepsilon,\delta}(t, \cdot) \|_{L^\infty(\mathbf{R})} \leqq C, \quad \mathrm{TV}(\mathcal{V}^{\varepsilon,\delta}(t, \cdot)) \leqq C, \qquad (4.4)$$

for some constant $C > 0$ independent of ε and δ. Since again $\mathcal{U}^{\varepsilon,\delta}$ and $u^{\varepsilon,\delta}$ have identical sup norm, the following sub-characteristic condition holds uniformly in ε and δ :

$$|f'(\mathcal{U}^{\varepsilon,\delta})| < a, \qquad (4.5)$$

with a prescribed according to (1.14). Hence, the quasi-monotone property (2.23) and the monotonicity of h expressed in (2.25) apply for all the values of $\mathcal{U}^{\varepsilon,\delta}$ under consideration. This will play an important role hereafter. Note that

$$\int_0^T \int_{\mathbf{R}_+} |\mathcal{U}^{\varepsilon,\delta}(t, y) - f^{-1}(\mathcal{V}^{\varepsilon,\delta}(t, y))|^2 \,\mathrm{d}y \,\mathrm{d}t$$
$$= \int_0^T \int_{\mathbf{R}_+} |u^{\varepsilon,\delta}(t, \varepsilon y) - f^{-1}(v^{\varepsilon,\delta}(t, \varepsilon y))|^2 \frac{d\varepsilon y}{\varepsilon} \,\mathrm{d}t$$
$$= \frac{1}{\varepsilon} \int_0^T \int_{\mathbf{R}_+} |u^{\varepsilon,\delta}(t, x) - f^{-1}(v^{\varepsilon,\delta}(t, x))|^2 \,\mathrm{d}x \,\mathrm{d}t \leqq C. \qquad (4.6)$$

The entropy dissipation rate estimate thus no longer implies the strong convergence of sequence $\{\mathcal{U}^{\varepsilon,\delta}\}_{\varepsilon,\delta}$ in $L^1((0,T), L^2(\mathbf{R}_y^+))$, but a direct analysis allows to infer a uniform $BV(R_y^+)$ estimate for $\{\mathcal{U}^{\varepsilon,\delta}(t,\cdot)\}_{\varepsilon,\delta}$ for almost everywhere $t \in [0,T]$, $T > 0$. Such an estimate is derived from the governing equations for the rescaled profiles $\mathcal{U}^{\varepsilon,\delta}$ and $\mathcal{V}^{\varepsilon,\delta}$ that are easily seen to be C^1 solutions of

$$\begin{cases} \varepsilon \partial_t \mathcal{U}^{\varepsilon,\delta} + \partial_y \mathcal{V}^{\varepsilon,\delta} = 0, \quad t > 0,\ y \in \mathbf{R}, & \text{(4.7a)} \\ \varepsilon \partial_t \mathcal{V}^{\varepsilon,\delta} + a^2 \partial_y \mathcal{U}^{\varepsilon,\delta} = \varepsilon \Lambda(\varepsilon y, \varepsilon, \delta)(f(\mathcal{U}^{\varepsilon,\delta}) - \mathcal{V}^{\varepsilon,\delta}), & \text{(4.7b)} \end{cases}$$

for any given positive ε and δ.

Lemma 9. *For all $T > 0$ and almost every time $t \in [0,T]$, for all $R > 0$, the following BV estimate holds*

$$TV_{(-R,R)}(\mathcal{U}^{\varepsilon,\delta}(t,\cdot)) \leqq CR \tag{4.8}$$

for some constant $C > 0$ independent of ε and δ.

Proof. Let us recast the second equation in (4.7):

$$a^2 \partial_y \mathcal{U}^{\varepsilon,\delta} = \varepsilon \Lambda(\varepsilon y, \varepsilon, \delta)(f(\mathcal{U}^{\varepsilon,\delta}) - \mathcal{V}^{\varepsilon,\delta}) - \varepsilon \partial_t \mathcal{V}^{\varepsilon,\delta}, \tag{4.9}$$

to infer the crude upper-bound

$$a^2 |\partial_y \mathcal{U}^{\varepsilon,\delta}| \leqq |(f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta})(t, \varepsilon y)| + \varepsilon |\partial_t \mathcal{V}^{\varepsilon,\delta}|, \tag{4.10}$$

since $\varepsilon \Lambda(\varepsilon y, \varepsilon, \delta) \leqq 1$. Integrating the above inequality for $y \in (0, R)$ clearly yields the expected conclusion

$$a^2 TV_{(-R,R)}(\mathcal{U}^{\varepsilon,\delta}(t,\cdot)) \leqq 2||(f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta})(t,\cdot)||_{L^\infty(\mathbf{R})}\, R + ||\partial_t v^{\varepsilon,\delta}(t,\cdot)||_{L^1(\mathbf{R}_x)}, \tag{4.11}$$

since $u^{\varepsilon,\delta}$ and $v^{\varepsilon,\delta}$ are uniformly in ε and δ bounded in sup norm. □

In view of the estimates (4.3)–(4.4)–(4.8), sending δ to 0 with ε fixed and then letting ε go to 0, the sequence $\{(\mathcal{U}^{\varepsilon,\delta}(t,\cdot), \mathcal{V}^{\varepsilon,\delta}(t,\cdot))\}_{\varepsilon,\delta>0}$ can be shown to converge to some limit $(\mathcal{U}(t,\cdot), \mathcal{V}(t,\cdot))$ in $L^1_{\text{loc}}(\mathbf{R}_y)$ for any given $t > 0$. Clearly $(\mathcal{U}(t,\cdot), \mathcal{V}(t,\cdot))$ have bounded sup norm and locally bounded total variation. They will be referred hereafter to as the inner interface layer or the inner solution for short, while $(u(t,\cdot),\ v(t,\cdot))$ will be called the outer solution.

Let us first establish the following results only concerned with the inner solutions.

Lemma 10. *For almost everywhere $t > 0$, the inner solution $(\mathcal{U}(t,\cdot), \mathcal{V}(t,\cdot))$ is Lipschitz continuous in y and admits bounded asymptotic limits $(\mathcal{U}(t,\pm\infty), \mathcal{V}(t,\pm\infty))$. It verifies*

$$\mathcal{V}(t,y) \equiv \mathcal{V}(t,+\infty) = \mathcal{V}(t,-\infty),\ y \in \mathbf{R}, \tag{4.12}$$

and

$$\mathcal{U}(t, y) \equiv \mathcal{U}(t, -\infty), \quad y < 0, \tag{4.13}$$

$$\begin{cases} a^2 d_y \mathcal{U}(t, y) = f(\mathcal{U}(t, y)) - \mathcal{V}(t, +\infty), \quad y > 0, \\ \mathcal{U}(t, 0) = \mathcal{U}(t, -\infty). \end{cases} \tag{4.14}$$

Moreover, we have :

$$f(\mathcal{U}(t, +\infty)) = \mathcal{V}(t, +\infty) = \mathcal{V}(t, -\infty). \tag{4.15}$$

If at time $t > 0$*, the solution* $\mathcal{U}(t, \cdot)$ *to* (4.14) *is not locally constant in the half line* $\{y > 0\}$*, then it must be strictly monotone for* $y > 0$ *with*

$$f'(\mathcal{U}(t, +\infty)) < 0. \tag{4.16}$$

Observe that the time t acts as a parameter for the inner solution. The asymptotic limits $\mathcal{U}(t, \pm\infty)$ and $\mathcal{V}(t, \pm\infty)$ will be determined in the forthcoming matching analysis (see Proposition 11) with the left and right traces at $x = 0$ of the outer solution $u(t, x)$ and $v(t, x)$.

Proof. The weak form of (4.7) reads

$$\begin{cases} \int\int_{[0,T]\times[-R,R]} \varepsilon\, \mathcal{U}^{\varepsilon,\delta} \partial_t \varphi + \mathcal{V}^{\varepsilon,\delta} \partial_y \varphi \ \mathrm{d}t\, \mathrm{d}y = 0, & (4.17a) \\ \int\int_{[0,T]\times[-R,R]} \varepsilon\, \mathcal{V}^{\varepsilon,\delta} \partial_t \varphi + a^2 \mathcal{U}^{\varepsilon,\delta} \partial_y \varphi + \varepsilon \Lambda(\varepsilon y, \varepsilon, \delta)(f(\mathcal{U}^{\varepsilon,\delta}) - \mathcal{V}^{\varepsilon,\delta})\varphi \ \mathrm{d}t\, \mathrm{d}y = 0, & (4.17b) \end{cases}$$

for any given test function $\varphi \in C_c^1((0, T) \times (-R, R))$. We clearly have

$$\lim_{\delta \to 0} \varepsilon \Lambda(\varepsilon y, \varepsilon, \delta) := \alpha(y, \varepsilon) = \begin{cases} \varepsilon, & y < 0, \\ 1, & y > 0, \end{cases} \tag{4.18}$$

so that for any given time $t > 0$ the uniform sup norm and local total variation estimates (4.3)–(4.8) clearly ensure, up to some diagonal extraction procedure, that in the limit $\delta \to 0$, ε fixed, and then $\varepsilon \to 0$, there exists a limit $(\mathcal{U}, \mathcal{V})$ which is bounded in sup norm and has locally bounded total variation. This limit verifies

$$\begin{cases} \int\int_{[0,T]\times[-R,R]} \mathcal{V}(t, y) \partial_y \varphi(t, y) \ \mathrm{d}t\, \mathrm{d}y = 0, & (4.19a) \\ \int\int_{[0,T]\times[-R,R]} a^2 \mathcal{U}(t, y) \partial_y \varphi(y) + \alpha(y, 0)(f(\mathcal{U}(t, y)) - \mathcal{V}(t, y))\varphi(t, y) \ \mathrm{d}t\, \mathrm{d}y = 0. & (4.19b) \end{cases}$$

Choosing test function $\varphi(t, y) = \phi(t)\psi(y)$ for $\phi \in C_c^1([0, T])$ and $\psi \in C_c^1([-R, R])$ yields for almost everywhere $t > 0$

$$\begin{cases} \int_{[-R,R]} \mathcal{V}(t, y)\psi'(y)\, \mathrm{d}y = 0, & (4.20a) \\ \int_{[-R,R]} a^2 \mathcal{U}(t, y)\psi'(y) + \alpha(y, 0)(f(\mathcal{U}(t, y)) - \mathcal{V}(t, y))\psi(y)\, \mathrm{d}y = 0, & (4.20b) \end{cases}$$

since ϕ can be chosen arbitrarily. Obviously $\mathcal{V}(t, y)$ is constant in y for almost everywhere $t > 0$ and thus (4.12) holds. Then choosing ψ with compact support in $[-R, 0]$, (4.18) immediately gives that $\mathcal{U}(t, .)$ also stays constant in the half line $\{y < 0\}$

$$\mathcal{U}(t, y) \equiv \mathcal{U}(t, -\infty) \qquad y < 0. \tag{4.21}$$

Then choosing ψ with compact support in $[0, R]$, one easily infers that $\mathcal{U}(t, .)$ is a classical solution of the ordinary differential equation

$$a^2 \frac{\mathrm{d}}{\mathrm{d}y}\mathcal{U}(t, y) = f(\mathcal{U}(t, y)) - \mathcal{V}(t, +\infty), \quad y > 0. \tag{4.22}$$

To derive the required interface data $\mathcal{U}(t, 0^+)$, we choose at last $\psi \in C_c^1(R_y)$ and argue the property that $\mathcal{U}(t, y)$ is constant with $\alpha(y, 0) = 0$ for $y < 0$ while being a smooth solution of (4.22) for $y > 0$. Integrations by part in (4.19) which we recast as follows

$$\begin{aligned} \int_{[-R,0]} a^2\mathcal{U}(t, y)\psi'(y)\,\mathrm{d}y + \int_{[0,R]} a^2\mathcal{U}(t, y)\psi'(y) \\ +\alpha(y, 0)(f(\mathcal{U}(t, y)) - \mathcal{V}(t, y))\psi(y)\,\mathrm{d}y = 0, \end{aligned} \tag{4.23}$$

gives rise to

$$a^2(\mathcal{U}(t, 0^+) - \mathcal{U}(t, 0^-))\ \psi(0) = 0, \tag{4.24}$$

which is, in view of (4.21)

$$\mathcal{U}(t, 0^+) = \mathcal{U}(t, 0^-) = \mathcal{U}(t, -\infty). \tag{4.25}$$

This identifies the initial data of the ODE Cauchy problem (4.22) and proves by the way the Lipshitz continuity property of $\mathcal{U}(t, \cdot)$ in the fast variable y.

Let us prove that the solution $\mathcal{U}(t, \cdot)$ of the ODE Cauchy problem (4.22)–(4.25) is either trivial, that is $\mathcal{U}(t, y) \equiv \mathcal{U}(t, -\infty)$ for all $y > 0$ (and thus all y in R), or strictly monotone in the half line $\{y > 0\}$. Indeed assume that $d_y\mathcal{U}(t, y)$ vanishes for some $y^\star > 0$ so that $\mathcal{U}(t, y^\star)$ is a critical point of (4.19), that is $f(\mathcal{U}(t, y^\star)) - \mathcal{V}(t, +\infty) = 0$. But classical properties of scalar autonomous ODE problem ensure that a critical point cannot be achieved for finite $y > 0$, so that if $y^\star$ is finite, necessarily $\mathcal{U}(t, y)$ stays constant for all y. Conversely assume the inner solution to be non-trivial, then it is necessarily strictly monotone for all finite $y > 0$ with

$$\lim_{y\to+\infty} d_y\mathcal{U}(t, y) = 0. \tag{4.26}$$

By the Hartman–Grobman's Theorem [14], we observe that the critical point $\mathcal{U}(t, +\infty)$ cannot be unstable, namely $f'(\mathcal{U}(t, +\infty)) > 0$ cannot hold, so that the last claim of Lemma 10

$$f'(\mathcal{U}(t, +\infty)) < 0,$$

must be valid in the present setting. □

We now derive the following matching conditions to link the inner solution $(\mathcal{U}, \mathcal{V})$ with the outer solution (u, v).

Proposition 11. *For almost everywhere $t > 0$, $\mathcal{V}$ and v perfectly match*

$$\mathcal{V}(t, y) \equiv v(t, 0^-) = v(t, 0^+), \quad \textit{for all } y \in \mathbf{R}. \tag{4.27}$$

$\mathcal{U}$ and u are linked according to

$$\mathcal{U}(t, y) \equiv u(t, 0^-), \quad y < 0. \tag{4.28}$$

Defining $R_\pm(t, y) = a\mathcal{U}(t, y) \pm \mathcal{V}(t, y)$, the following inequalities hold

$$\begin{aligned}&\frac{1}{2}(|R_+(t, y) - \ell| - |R_-(t, y) - h(\ell)|)\\ &\geqq sgn(u(t, 0^+) - k)(f(u(t, 0^+)) - f(k)), \quad y > 0,\end{aligned} \tag{4.29}$$

for any given $\ell \in R$ such that $k = (\ell + h(\ell))/2a$ verifies $|k| < B(N_0)$.

Proof. Let $(r_-^{\varepsilon,\delta}, r_+^{\varepsilon,\delta})$ denote the solution of the Cauchy problem (2.21) in diagonal form, with initial data $(r_\pm)_0^\delta = au_0^\delta \pm v_0^\delta$. Then for any given $\ell \in R$ with $|(\ell + h(\ell))/2a| \leqq B(N_0)$, let us consider the following entropy like inequality (2.29)

$$\partial_t(|r_+^{\varepsilon,\delta} - \ell| + |r_-^{\varepsilon,\delta} - h(\ell)|) + a\partial_x(|r_+^{\varepsilon,\delta} - \ell| - |r_-^{\varepsilon,\delta} - h(\ell)|) \leqq 0. \tag{4.30}$$

To condense the notations, let $p_\ell^{\varepsilon,\delta} = |r_+^{\varepsilon,\delta} - \ell| + |r_-^{\varepsilon,\delta} - h(\ell)|$ and $q_\ell^{\varepsilon,\delta} = a(|r_+^{\varepsilon,\delta} - \ell| - |r_-^{\varepsilon,\delta} - h(\ell)|)$, so that (4.30) reads

$$\partial_t p_\ell^{\varepsilon,\delta} + \partial_x q_\ell^{\varepsilon,\delta} \leqq 0. \tag{4.31}$$

Given $\varepsilon > 0$ and fix $y > 0$, consider any $b > 0$ satisfying

$$0 < \varepsilon y < b. \tag{4.32}$$

For any given time $T > 0$, multiply (4.31) by any given non-negative test function $\varphi(t) \in C_c^1((0, T))$ and integrate (t, x) over $[0, T] \times [\varepsilon y, b]$ to obtain:

$$\int_0^T \int_{\varepsilon y}^b -p_\ell^{\varepsilon,\delta}(t, x)\varphi'(t)\, \mathrm{d}t\, \mathrm{d}x + \int_0^T (q_\ell^{\varepsilon,\delta}(t, b) - q_\ell^{\varepsilon,\delta}(t, \varepsilon y))\varphi(t)\, \mathrm{d}t \leqq 0. \tag{4.33}$$

Define $R_\pm^{\varepsilon,\delta}(t, y) = a\mathcal{U}^{\varepsilon,\delta}(t, y) \pm \mathcal{V}^{\varepsilon,\delta}(t, y) = r_\pm^{\varepsilon,\delta}(t, \varepsilon y)$ and set

$$\mathcal{Q}_\ell^{\varepsilon,\delta}(t, y) = a(|R_+^{\varepsilon,\delta} - R_{+,\ell}^{\varepsilon,\delta}| - |R_-^{\varepsilon,\delta} - R_{-,\ell}^{\varepsilon,\delta}|)(t, y),$$

we have the identity $\mathcal{Q}_\ell^{\varepsilon,\delta}(t, y) = q_\ell^{\varepsilon,\delta}(t, \varepsilon y)$. Hence changing the sign in (4.33) gives:

$$\int_0^T \int_{\varepsilon y}^b p_\ell^{\varepsilon,\delta}(t, x)\varphi'(t)\, \mathrm{d}t\, \mathrm{d}x + \int_0^T (\mathcal{Q}_\ell^{\varepsilon,\delta}(t, y) - q_\ell^{\varepsilon,\delta}(t, b))\varphi(t)\, \mathrm{d}t \geqq 0. \tag{4.34}$$

Choose $\eta > \varepsilon y$ and average the above inequality (valid for all $b > \varepsilon y$) for $b \in (\eta, 2\eta)$ to get

$$\frac{1}{\eta}\int_0^T \int_\eta^{2\eta} \int_{\varepsilon y}^b p_\ell^{\varepsilon,\delta}(t,x)\varphi'(t)\,\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}x + \int_0^T \left(\mathcal{Q}_\ell^{\varepsilon,\delta}(t,y) - \frac{1}{\eta}\int_\eta^{2\eta} q_\ell^{\varepsilon,\delta}(t,b)db\right)\varphi(t)\,\mathrm{d}t \geqq 0. \quad (4.35)$$

Observe that

$$\left|\frac{1}{\eta}\int_0^T \int_\eta^{2\eta} \int_{\varepsilon y}^b p_\ell^{\varepsilon,\delta}(t,x)\varphi'(t)\,\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}x\right| \leqq \sup_{0\leqq t\leqq T} \| p_\ell^{\varepsilon,\delta}(t,\cdot) \|_{L^\infty(\mathbf{R})} \| \varphi' \|_{L^1(0,T)} \frac{1}{\eta}\int_\eta^{2\eta}(b-\varepsilon y)db \leqq C\left(\frac{3}{2}\eta - \varepsilon y\right) \quad (4.36)$$

for some uniform constant $C > 0$ in ε and δ thanks to the corresponding uniform sup norm estimates satisfied by the solutions $(u^{\varepsilon,\delta}, v^{\varepsilon,\delta})$ and $(u_\ell^{\varepsilon,\delta}, v_\ell^{\varepsilon,\delta})$ of the regularized problem (2.9). Recall that the sequence $\{\mathcal{Q}^{\varepsilon,\delta}\}_{\varepsilon,\delta>0}$ has uniformly in ε and δ bounded local total variation, it converges by sending first $\delta \to 0$, then $\varepsilon \to 0$ in $L^1_{loc}(\mathbf{R}_y)$ to a limit $\mathcal{Q}_\ell$ and almost everywhere. For fixed time t, the limit is smooth in the fast variable y and thus left and right traces coincide. For fixed $\varepsilon > 0$, passing to the limit $\delta \to 0$ and then to the limit $\varepsilon \to 0$ in (4.35) thus yield

$$\int_0^T \varphi(t)\left(\mathcal{Q}_\ell(t,y) - \frac{1}{\eta}\int_\eta^{2\eta} q_\ell(t,b)\,\mathrm{d}b\right)\mathrm{d}t \geqq -\frac{3}{2}C\eta. \quad (4.37)$$

Observe that this inequality now holds for any $\eta > 0$ since the ordering condition (4.32) implies $b > 0$. In (4.37), we have

$$\mathcal{Q}_\ell(t,y) = a\left(|R_+(t,y) - \ell| - |R_-(t,y) - h(\ell)|\right) \quad (4.38)$$

with $R_\pm(t,y) = a\mathcal{U}(t,y) \pm \mathcal{V}(t,y)$ while

$$q_\ell(t,b) = a\left(|r_+(t,b) - \ell| - |h(r_+(t,b)) - h(\ell)|\right) \quad (4.39)$$

since all b under consideration are positive, that is we deal with the equilibrium half line $\{b > 0\}$ with the property that $r_-(t,b) = h(r_+(t,b))$. Arguments based on the monotonicity of h and equation (3.19) that have already been developed in the proof of Theorem 8 ensure the identity

$$q_\ell(t,b) = 2a\,\mathrm{sgn}(u(t,b) - k)(f(u(t,b)) - f(k)), \quad k = \frac{\ell + h(\ell)}{2a}. \quad (4.40)$$

Passing to the limit $\eta \to 0$ in (4.37) yields (recall that $u(t,b)$ has bounded total variation by Theorem 8.iv, and thus admits a right trace at $b = 0^+$)

$$\int_0^T \varphi(t)\left(\mathcal{Q}_\ell(t,y) - q_\ell(t,0^+)\right)\mathrm{d}t \geqq 0. \quad (4.41)$$

This inequality is valid for any given non-negative test function φ in $C_c^1((0,T))$, so that we deduce the inequality (4.29). Now let us prove that the next matching condition $\mathcal{V}(t,y) \equiv v(t,0^+)$ for $y>0$ and almost everywhere $t>0$. To this aim, we start from the equation $\partial_t u^{\varepsilon,\delta} + \partial_x v^{\varepsilon,\delta} = 0$ and repeat the same arguments as previously to get

$$\int_0^T \varphi(t)\left(\mathcal{V}(t,y) - \frac{1}{\eta}\int_\eta^{2\eta} v(t,b)db\right) = 0, \quad y>0, \text{ almost everywhere } t>0 \tag{4.42}$$

for all $\eta>0$ and all test functions $\varphi \in C_c^1((0,T))$. Sending $\eta \to 0$ yields $\mathcal{V}(t,y) = v(t,0^+)$. To derive the condition $\mathcal{U}(t,y) \equiv u(t,0^-)$ when $y<0$, we proceed *mutatis mutandis* the same way, choosing $\varepsilon>0$, some fixed $y<0$ and negative real number b satisfying the ordering condition $b<\varepsilon y<0$ and apply the above steps to the equation $\partial_t v^{\varepsilon,\delta} + a^2\partial_x u^{\varepsilon,\delta} = f(u^{\varepsilon,\delta}) - v^{\varepsilon,\delta}$ to get

$$\begin{aligned}&\int_0^T \left(\mathcal{U}^{\varepsilon,\delta}(t,y) - \frac{1}{|\eta|}\int_{-2|\eta|}^{-|\eta|} u^{\varepsilon,\delta}(t,b)db\right)\varphi(t)\,\mathrm{d}t\\ &\quad+\frac{1}{|\eta|a^2}\int_0^T\int_{-2|\eta|}^{-|\eta|}\int_{\varepsilon y}^{b}(v^{\varepsilon,\delta}(t,x)\varphi'(t)\\ &\quad+(f(u^{\varepsilon,\delta}(t,x)) - v^{\varepsilon,\delta}(t,x))\varphi(t))\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}b = 0\end{aligned} \tag{4.43}$$

for any given test function $\varphi \in C_c^1((0,T))$. Uniform sup norm estimates for $u^{\varepsilon,\delta}$ and $v^{\varepsilon,\delta}$ again apply to prove that the second term vanishes in the limit $\delta \to 0$ then $\varepsilon \to 0$ and $\eta \to 0$, while the first term gives rise to

$$\int_0^T (\mathcal{U}(t,y) - u(t,0^-))\varphi(t)\,\mathrm{d}t = 0, \quad y<0. \tag{4.44}$$

This implies $\mathcal{U}(t,y) = u(t,0^-)$. Proving that $\mathcal{V}(t,y) = v(t,0^-)$, $y<0,\ t>0$, follows similar steps. Hence the identities (4.27) are readily inferred from the property that $\mathcal{V}(t,y)$ stays constant for all y. □

Observe that the matching condition (4.27) together with the identity $\mathcal{V}(t,-\infty) = f(\mathcal{U}(t,+\infty))$ stated in (4.15) actually ensures

$$f(\mathcal{U}(t,+\infty)) = v(t,0^+) = v(t,0^-). \tag{4.45}$$

In the sequel and for any given real numbers a and b, $\lfloor a,b\rceil$ denotes the interval $(\min(a,b),\max(a,b))$.

Corollary 12. *For almost everywhere $t>0$, the following Kružkov inequalities hold*

$$\begin{aligned}&\operatorname{sgn}(\mathcal{U}(t,+\infty)-k)(f(\mathcal{U}(t,+\infty)) - f(k))\\ &\quad\geqq sgn(u(t,0^+)-k)(f(u(t,0^+)) - f(k)),\end{aligned} \tag{4.46}$$

for all $k \in \lfloor \mathcal{U}(t,+\infty), u(t,0^+)\rceil$. *In particular, we have:*

$$f(\mathcal{U}(t,+\infty)) = f(u(t,0^+)) = v(t,0^-). \tag{4.47}$$

At last $\mathcal{U}(t,+\infty)$ *and* $u(t,0^+)$ *necessarily match*

$$\mathcal{U}(t,+\infty) = u(t,0^+). \tag{4.48}$$

Proof. Sending y to $+\infty$ in (4.29), one has for any given $l \in R$ such that $k = (\ell + h(\ell))/2a$ verifies $|k| < B(N_0)$:

$$\begin{aligned}&\frac{1}{2}(|R_+(t,+\infty) - \ell| - |R_-(t,+\infty) - h(\ell)|)\\ &\quad \geqq \operatorname{sgn}(u(t,0^+) - k)(f(u(t,0^+)) - f(k)).\end{aligned} \tag{4.49}$$

In particular, these inequalities are valid for specially chosen ℓ such that k belongs to $\lfloor \mathcal{U}(y,+\infty), u(t,0^+)\rceil$. This will suffice for our purpose. Note from (4.15) that $R_\pm(t,+\infty) = a\mathcal{U}(t,+\infty) \pm f(\mathcal{U}(t,+\infty))$, thus the identity $R_-(t,+\infty) = h(R_+(t,+\infty))$ holds. Rephrasing arguments developed in the course of Theorem 8 (see (3.19)), the left hand side of the inequality (4.49) is seen to boil down to $\operatorname{sgn}(\mathcal{U}(t,+\infty)-k)(f(\mathcal{U}(t,+\infty)) - f(k))$ for all the k under consideration, which is nothing but (4.46). It then suffices to choose successively $k = \mathcal{U}(t,+\infty)$ and $k = u(t,0^+)$ to get (4.47). From the first equality stated in (4.47), the assumption (1.9) or (1.10) immediately implies $\mathcal{U}(t,+\infty) = u(t,0^+)$. □

To further explore the matching conditions in between $\mathcal{U}$ and u in the setting of a general flux function f, let us state another consequence of the entropy inequalities (4.29) :

Lemma 13. *For almost everywhere* $t > 0$, *the following inequalities are met :*

$$sgn(u(t,0^+) - k)(f(u(t,0^+)) - f(k)) \leqq 0, \tag{4.50}$$

for all k *in* $\lfloor u(t,0^-), u(t,0^+)\rceil$.

Proof. Let us first observe from the reported matching properties that $R_\pm(t,y) = a\mathcal{U}(t,y) \pm \mathcal{V}(t,y) = a\mathcal{U}(t,y) \pm f(\mathcal{U}(t,+\infty))$ for almost everywhere $t > 0$, from the matching condition (4.27). Now given $y > 0$ fixed, choose $\ell_y = a\mathcal{U}(t,y) + f(\mathcal{U}(t,y))$ in (4.29) with the properties that $h(\ell_y) = a\mathcal{U}(t,y) - f(\mathcal{U}(t,y))$, $k_y = (\ell_y + h(\ell_y))/2a$
$= \mathcal{U}(t,y)$ with $|k_y| < B(N_0)$. Observe that :

$$\begin{aligned}|R_+(t,y) - \ell_y| &= |f(\mathcal{U}(t,y)) - f(\mathcal{U}(t,+\infty))|,\\ |R_-(t,y) - h(\ell_y)| &= |f(\mathcal{U}(t,y)) - f(\mathcal{U}(t,+\infty))|,\end{aligned}$$

so that (4.29) ensures

$$0 \geqq \operatorname{sgn}(u(t,0^+) - \mathcal{U}(t,y))(f(u(t,0^+)) - f(\mathcal{U}(t,y))), \quad y > 0,\ t > 0, \tag{4.51}$$

since again $k_y = \mathcal{U}(t, y)$. But by construction, $\mathcal{U}(t, y)$ covers monotonically the range $\lfloor u(t, 0^-), u(t, 0^+) \rceil$ as y runs over $\mathbf{R}_+$ since $\mathcal{U}(t, y)$ is monotone and continuous with $\mathcal{U}(t, 0) = u(t, 0^-)$ and $\mathcal{U}(t, +\infty) = u(t, 0^+)$. This readily implies that for all $y > 0$, $\text{sgn}(u(t, 0^+) - \mathcal{U}(t, y)) = \text{sgn}(u(t, 0^+) - k)$ for all $k \in \lfloor u(t, 0^-), u(t, 0^+) \rceil$. □

As a direct consequence of Lemma 13, the most important outcome of the matching analysis is the last result of this section:

Proposition 14. *For almost everywhere $t > 0$, left and right traces of the outer solution $u(t, x)$ at $x = 0$ obey :*

$$\text{sgn}(u(t, 0^+) - u(t, 0^-))(f(u(t, 0^+)) - f(k)) \leqq 0, \quad k \in \lfloor u(t, 0^-), u(t, 0^+) \rceil, \tag{4.52}$$

while

$$v(t, 0^-) = f(u(t, 0^+)). \tag{4.53}$$

Proof. It just follows from the fact that $\text{sgn}(u(t, 0^+) - u(t, 0^-)) = \text{sgn}(u(t, 0^+) - k)$ for all $k \in \lfloor u(t, 0^-), u(t, 0^+) \rceil$. □

Rephrasing this statement, the traces of the outer solution $u(t, x)$ at $x = 0$ are linked by the so-called Bardos–Leroux–Nédélec boundary condition (1.18). With this noted and as already emphasized along the introduction, the detailed knowledge of the inner solution $\mathcal{U}$ can be bypassed.

To conclude, let us deduce Theorem 1 as a consequence of the main results of the present and previous sections.

Proof of Theorem 1. Theorem 4 ensures the existence and uniqueness of global solution $(u^\varepsilon, v^\varepsilon)$ of the original two-scale problem. This is part 1. Then Theorem 8.i gives the existence of a limit (u, v) in $L^1((0, T) \times \mathbf{R})$ for a subsequence $(u^\varepsilon, v^\varepsilon)$ as ε goes to 0. From part (iv) of the above Theorem, the limit (u, v) belongs to $L^\infty \cap BV((0, T) \times \mathbf{R})$. In addition, the lower semi-continuity of the L^1-norm implies for any given $t \in (0, T)$, $T > 0$:

$$||u(t, .) - u_0||_{L^1(\mathbf{R})} \leqq \liminf_{\varepsilon \to 0} ||u^\varepsilon(t, .) - u_0||_{L^1(\mathbf{R})}, \quad ||v(t, .) - v_0||_{L^1(\mathbf{R})} \leqq \liminf_{\varepsilon \to 0} ||v^\varepsilon(t, .) - v_0||_{L^1(\mathbf{R})}. \tag{4.54}$$

Then the estimates (2.17)–(2.18) yield, choosing $t_2 = t$ and $t_1 = 0$:

$$||u(t, .) - u_0||_{L^1(\mathbf{R})} \leqq C\, t, \quad ||v(t, .) - v_0||_{L^1(\mathbf{R})} \leqq C\, t, \tag{4.55}$$

so that (1.5c)–(1.6c) or (1.7c)–(1.8c) holds.

By Theorem 8.ii, (1.6a)–(1.6b) or (1.7a)–(1.7b) holds in the sense of the distributions. Then (3.4) is nothing but (1.5b) or (1.8b). To prove that $u(t, x)$ for $x > 0$ solves (1.5a) or (1.8a) in the sense of the distributions and in view of (3.1), it suffices to choose successively $k = B(N_0)$ and $k = -B(N_0)$ in (3.5). The identity (4.53) reads as the transmission condition stated in (1.6d) or (1.7d). Turning to consideration of the transmission condition (1.8d), it directly follows from (4.52) under the assumption (1.9). To summarize, all the conditions stated from (1.5a) to (1.8d) are valid. Theorem 8.vi gives the L^1-contraction property (1.15) and implies uniqueness. This concludes the proof of parts (2) and (3) of Theorem 1. □

5. Conclusion

The present work has been devoted to the analysis of the Jin–Xin relaxation system in a two- scale setting. We have established its well-posedness and singular limit as the smaller relaxation time goes to zero. The limit is a multiscale coupling problem which couples the original Jin–Xin system on the domain when the relaxation time is O(1) with its relaxation limit in the other domain through interface conditions which can be derived by matched interface layer analysis. This sets up a theoretical foundation of multiscale computation for hyperbolic systems with drastically different relaxation time scales.

Our analysis is based on the assumption that the interface is non-characteristic. This allows us to prevent possible strong oscillations from developing at the interface. It has also ruled out any standing shock sticking to the interface. An interesting problem for the future is to analyze the coupling system found in (1.16) and (1.17), which allows the intricate interplay between an interface layer and a standing shock, and to more general hyperbolic systems.

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